Exactly solvable relativistic model with the anomalous interaction

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A special class of Dirac-Pauli equations with time-like vector potentials of an external field is investigated. An exactly solvable relativistic model describing the anomalous interaction of a neutral Dirac fermion with a cylindrically symmetric external electromagnetic field is presented. The related external field is a superposition of the electric field generated by a charged infinite filament and the magnetic field generated by a straight line current. In the nonrelativistic approximation the considered model is reduced to the integrable Pron'ko-Stroganov model.

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I. INTRODUCTION

Exact solutions of relativistic wave equations are both very rare and important. First, they provide explicit solutions to concrete physical problems free of the inaccuracies and inconveniences of approximate methods. Second, such exact solutions can serve as convenient basis sets for expanding the solutions of other physical problems that are not necessarily exactly solvable.

A good survey of the exact solutions of relativistic wave equations can be found in [1]. Notwithstanding the fact that this book was published in 1990, it continues to be a good information resource on exactly solvable relativistic systems for particles with spins 0 and $1/2$. Surely this collection is not exhaustive: Many new results have been obtained during the last two decades, including the problems for the Dirac equation in lower dimension space and the problems for neutral Dirac particles.

Exact solutions of the Dirac equation describing electrically neutral particles with nonminimal interaction with an external electromagnetic field are noteworthy. Physically, such solutions have a very big application value since they can be used to model the motion of a neutron in realistic situations. In particular, they have relations to the nuclear reactors security problems. Moreover, magnetic trapping of neutrons is a subject of direct experimental studies (refer, e.g., to [2]). Mathematically, the anomalous interaction terms depending on tensor fields dramatically reduce the number of problems that can be solved using the complete separation of variables. In addition, just neutral particles anomalously coupled to the external magnetic field give rise to the Aharonov-Casher effect [3] with its interesting physical and mathematical aspects.

The very possibility of solving a problem exactly stems from the existence of a dynamical symmetry, which is more extended than the geometric symmetry of the problem. Famous examples of such exactly solvable systems are the Kepler problems and isotropic oscillator whose dynamical symmetries are defined by groups SO(4) and U(3), respectively. One more well-known example is the Pron'ko-Stroganov (PS) problem [4], which describes the anomalous interaction of a nonrelativistic electrically neutral particle of spin $1/2$ with the field of a straight line current. The related dynamical symmetry of negative energy states is described by group SO(3) while the geometrical symmetry of the system is reduced to the rotation group in two dimensions [i.e., to SO(2)]. Let us stress that the PS problem was formulated for the Schrödinger-Pauli equation for neutral particles (i.e., it is essentially nonrelativistic).

Paper [4] was followed by a number of publications devoted to exactly solvable problems for neutral particles. In particular, the supersymmetric aspects of the PS model were investigated in [5–7], more realistic models based on the magnetic field produced by a current of a thin filament were discussed in [6,8] following the nonrelativistic approach. A rather completed study of the Dirac-Pauli equation for neutral particles can be found in [9], the case of a purely electric time-independent external field was studied in [10]. However, an exactly solvable relativistic analog of the PS problem was not known until now.

We can add that searching for exact solutions of the Dirac equation belongs to evergreen problems, apparently the most recent result in this field can be found in [11]. Exactly solvable two-particle Dirac equations are discussed in [12]. For exact solutions of relativistic wave equations for particles with higher and arbitrary spins, see [13,14].

In the present paper we discuss a certain class of relativistic problems describing the anomalous interaction of the Dirac fermion with an external electromagnetic field. The considered equations admit an effective reduction to equations invariant with respect to the $1+2$-dimensional Galilei group, which can be made by using the light cone coordinates.

Light cone coordinates were introduced by Dirac [15] in an attempt to formulate relativistic dynamics with direct interaction. Then it was recognized that these coordinates present powerful tools for the solution of relativistic wave equations, which include plane-wave potentials, or more generally, if the potentials do not depend on scalar products of the coordinate vector with a constant time-like vector [16,17].

We show that the considered class of equations includes exactly solvable relativistic systems and we study in detail one of them, namely the one closely related to both the

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Pron'ko-Stroganov problem and relativistic problems for the neutron interacting with an external field. The corresponding external field is a superposition of the magnetic field generated by the straight line constant current and the electric field of a charged infinite filament. In spite of that our main goal is to present an exactly solvable problem for neutral fermions, for the sake of generality we also consider more general problems with both minimal and anomalous interactions. We also indicate an exactly solvable model of this general type, see Sec. VII.

II. DIRAC-PAULI EQUATIONS AND REDUCTION SO(1,3) → HG(1,2)

Consider the Dirac-Pauli equation for a charged particle which interacts anomalously with an external electromagnetic field

$$(\gamma^{\mu}\pi_{\mu} - m - \lambda S^{\mu\nu}F_{\mu\nu})\psi = 0. \tag{2.1}$$

Here $\pi_\mu = p_\mu - eA_\mu$, $p_\mu = \mathrm{i}\frac{\partial}{\partial x^\mu}$, A_μ are components of the vector-potential of the external electromagnetic field, γ^μ are Dirac matrices satisfying the Clifford algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{2.2}$$

$g^{\mu\nu}$ is the metric tensor whose nonzero elements are $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$, $S^{\mu\nu} = \mathrm{i}\,(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)/4$ is the spin tensor, and $F_{\mu\nu} = \mathrm{i}[\pi_\mu, \pi_\nu]$ is the tensor of the electromagnetic field such that $F_{0a} = -E_a$, $F_{ab} = \varepsilon_{abc}B_c$, where E_a and B_c are components of vectors of the electric- and magnetic-field strengths. In addition, e and λ denote particle charge and the constant of anomalous coupling. The latter is usually represented as

$$\lambda = g\mu_0, \tag{2.3}$$

where μ_0 is the Bohr or nuclear magneton and g is the Landé factor. We use Heaviside units with $\hbar = c = 1$.

Equation (2.1) describes both the minimal and anomalous interactions of the Dirac fermion with an external electromagnetic field. Setting in (2.1) $\lambda = 0$ and supposing $e \neq 0$, we come to the equation describing the anomalous interaction only, while for $e = 0$, $\lambda \neq 0$ we obtain the Dirac-Pauli equation describing a neutral fermion. In the latter case the parameter g in (2.3) is just the contribution to the Landé factor arising from the anomalous magnetic moment.

Equation (2.1) is transparently invariant with respect to the Lorentz group SO(1,3) which transforms time and space variables x_0, x_1, x_2, x_3 between themselves. Among the subgroups of this group there is the homogeneous Galilei group HG(1,2), which includes the transformations of variables τ, x_1, x_2 where $\tau = (x_0 - x_3)/2$ (for all nonequivalent subgroups of SO(1,3) and of the Poincaré group, see [18]).

To search for exactly solvable problems based on Eq. (2.1) we restrict ourselves to a special class of external fields, which makes it possible to expand solutions of (2.1) via solutions of reduced equations invariant with respect to group HG(1,2). In other words, we will discuss such external fields for which these reduced equations are integrable.

To this end, we first suppose that the vector potential $A = (A_0, A_1, A_2, A_3)$ be light-like, that is,

$$A_\mu A^\mu = 0. \tag{2.4}$$

This condition can be always satisfied up to gauge transformations and so it does not lead to any loss of generality. Then we restrict ourselves to the vector potentials of the following special form compatible with (2.4)

$$A = (\varphi, 0, 0, \varphi), \tag{2.5}$$

where φ is a function of time and spatial variables. In addition, we suppose that φ depends on three variables only, namely,

$$\varphi = \varphi(\tau, x_1, x_2). \tag{2.6}$$

Vector potentials (2.5) and (2.6) satisfy the Lorentz gauge condition $p_\mu A^\mu = 0$ identically and the invariants of the related external field are both equal to zero

$$F_{\mu\nu}F^{\mu\nu} = 0 \quad \text{and} \quad \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = 0. \tag{2.7}$$

For convenience we fix a nonstandard realization of the Dirac matrices and set

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} \mathbf{0} & I \\ -I & \mathbf{0} \end{pmatrix}, \quad \gamma^\alpha = \mathrm{i}\begin{pmatrix} \sigma_\alpha & \mathbf{0} \\ \mathbf{0} & -\sigma_\alpha \end{pmatrix}, \tag{2.8}$$

where $\alpha = 1,2$, σ_α are Pauli matrices and $\mathbf{0}$ and I are the 2×2 zero and unit matrix, respectively. Then

$$S^{0\alpha} = \frac{1}{2}\begin{pmatrix} \mathbf{0} & \sigma_\alpha \\ -\sigma_\alpha & \mathbf{0} \end{pmatrix}, \quad S^{3\alpha} = \frac{1}{2}\begin{pmatrix} \mathbf{0} & \sigma_\alpha \\ \sigma_\alpha & \mathbf{0} \end{pmatrix}. \tag{2.9}$$

System (2.1) with a particular class of the vector potential given by Eqs. (2.5) and (2.6) is homogeneous with respect to the sum of independent variables $x_0 + x_3$. Thus it is convenient to rewrite it in the light cone variables

$$\tau = \tfrac{1}{2}(x_0 - x_3) \quad \text{and} \quad \xi = \tfrac{1}{2}(x_0 + x_3).$$

As a result we obtain

$$L\psi \equiv (\tilde{\gamma}_\mu\tilde{\pi}^\mu - m - \lambda\eta_\alpha F_\alpha)\psi = 0, \tag{2.10}$$

where $F_\alpha = \frac{\partial\varphi}{\partial x_\alpha}$, $\alpha = 1,2$,

$$\begin{aligned} \tilde{\gamma}_0 = \gamma_0 + \gamma_3, \quad \tilde{\gamma}_3 = \frac{1}{2}(\gamma_0 - \gamma_3), \quad \tilde{\gamma}_\alpha = \gamma_\alpha, \\ \eta_\alpha = \frac{1}{2}(\gamma_0\gamma_\alpha + \gamma_3\gamma_\alpha), \quad \tilde{\pi}_0 = \mathrm{i}\frac{\partial}{\partial\tau} - 2e\varphi, \\ \tilde{\pi}_3 = 2P_3 = 2\mathrm{i}\frac{\partial}{\partial\xi}, \quad \tilde{\pi}_\alpha = p_\alpha = -\mathrm{i}\frac{\partial}{\partial x_\alpha}, \end{aligned} \tag{2.11}$$

and the summation with respect to repeated indices μ and α is imposed over the values $\mu = 0,1,2,3$ and $\alpha = 1,2$, respectively. In addition, we impose on solutions of (2.10) the standard condition of square integrability and ask for $\psi \to 0$ when $x_\alpha \to 0$.

Operator P_3 commutes with L and so is a constant of the motion for Eq. (2.10). Let us expand solutions of this equation via eigenvectors ψ_M of P_3

$$P_3\psi_M = M\psi_M \;\Rightarrow\; \psi_M = \exp(\mathrm{i}M\xi)\psi(\tau, x_1, x_2). \tag{2.12}$$

Let us denote

$$\psi(\tau, x_1, x_2) = \begin{pmatrix} \rho(\tau, x_1, x_2) \\ \chi(\tau, x_1, x_2) \end{pmatrix}, \tag{2.13}$$

where ρ and χ are two-component spinors. Substituting (2.12) and (2.13) into (2.10) and using realization (2.8) of γ matrices we obtain the following system

$$(\mathrm{i}\sigma_\alpha p_\alpha - m)\,\rho + \left(\mathrm{i}\frac{\partial}{\partial\tau} - 2e\varphi - \lambda\sigma_\alpha F_\alpha\right)\chi = 0, \quad (2.14)$$

$$2M\rho - (\mathrm{i}\sigma_\alpha p_\alpha + m)\chi = 0. \quad (2.15)$$

It is easy to convince oneself from Eqs. (2.14) and (2.15) that, without loss of generality, we can assume that M cannot take the zero value. Indeed, setting $M = 0$ in (2.15) we reduce it to the equation for χ which does not have nontrivial normalizable solutions. Then, equating χ to zero in (2.14), we obtain the equation for ρ whose normalizable solutions are trivial also.

It is interesting to note that this system is nothing but a $(1+2)$-dimensional version of the Galilei-invariant Lévi-Leblond equation [19] with anomalous interaction, as can be immediately deduced by comparing (2.14) and (2.15) with Eq. (52) for $e = k = 0$ in [20]. Solving Eq. (2.15) for ρ under the condition $M \neq 0$ and substituting it into Eq. (2.14), we obtain the Schrödinger-Pauli equation for the two-component spinor χ

$$\mathrm{i}\frac{\partial\chi}{\partial\tau} = \left(\varepsilon_0 + \frac{p^2}{2M} + 2e\varphi + \lambda\sigma_\alpha F_\alpha\right)\chi, \quad (2.16)$$

where $\varepsilon_0 = \frac{m^2}{2M}$ and $p^2 = p_1^2 + p_2^2$.

Surely Eq. (2.16) is easier to handle then the initial equation (2.1), since it includes smaller numbers of dependent and independent variables. In particular, a number of exactly (and quasi-exactly) solvable Scrödinger-Pauli equations (2.16) is well studied, and many of them can be used to construct solvable relativistic problems using the scheme inverse to the previously proposed.

In Sec. IV we use this idea to generate a relativistic analog of the PS problem.

III. CYLINDRICALLY SYMMETRIC POTENTIALS

Consider in more detail a physically interesting subclass of Eqs. (2.1), (2.5), and (2.6) when the corresponding potential φ depends on the square $x^2 = x_1^2 + x_2^2$ of two vector $\mathbf{x} = (x_1, x_2)$ and is independent on τ. The related reduced Eq. (2.16) takes the form

$$\mathrm{i}\frac{\partial\chi}{\partial\tau} = H\chi, \quad (3.17)$$

where

$$H = \varepsilon_0 + \frac{p^2}{2M} + 2e\varphi + \lambda\frac{\sigma_\alpha x_\alpha}{x}\frac{\partial\varphi}{\partial x}. \quad (3.18)$$

Equation (3.17) has three additional constants of motion, namely,

$$P_0 = \mathrm{i}\frac{\partial}{\partial\tau}, \quad J_{12} = x_1 p_2 - x_2 p_1 + \frac{\mathrm{i}}{2}\sigma_3, \quad Q = \sigma_1 R_1, \quad (3.19)$$

where R_1 is the reflection operator which acts on χ as follows

$$R_1\chi(\tau,x_1,x_2) = \chi(\tau,-x_1,x_2).$$

Operators P_0 and J_{12} are generator of shifts with respect to variable τ and the rotation generator, respectively. They commute with the Hamiltonian (3.18) and between themselves. Expanding solutions of (3.17) via complete sets of eigenfunctions of P_0 and J_{12}, it is possible to separate variables in this equation.

Operator Q represents a discrete symmetry with respect to the reflection of the first coordinate axis. It commutes with P_0 and H (3.18) but *anticommutes* with J_{12}. It follows from the above that eigenvalues of P_0 and H should be degenerated with respect to the sign of eigenvalues of J_{12}.

Notice that Eq. (3.17) admits other discrete symmetries like reflections of x_2 or both x_1 and x_2. But all such additional symmetries are either rotation transformations or products of reflection Q and rotations.

Let us separate the variables in Eq. (3.17). First we define the eigenvectors of P_0 which have the following form

$$\chi_\varepsilon = \exp(-\mathrm{i}\varepsilon\tau)\chi(\mathbf{x}). \quad (3.20)$$

Then, substituting Eq. (3.20) into Eq. (3.17), we obtain the equation

$$\varepsilon\chi = H\chi, \quad (3.21)$$

where H is the Hamiltonian (3.18).

In addition to the coupling constants e and λ, Eq. (3.21) includes two parameters (i.e., ε and M). We suppose that functions ξ are square integrable and tend to zero with $x \to 0$. Then for a fixed nonzero M this equation defines an eigenvalue problem for ε.

Now we can use the symmetry of (3.21) with respect to the rotation group (whose generator is J_{12}) to separate radial and angular variables. To do this we rewrite Eq. (3.21) in terms of angular variables, that is, set $x_1 = x\cos\theta$, $x_2 = x\sin\theta$, and $r = 2M|\tilde{\lambda}|x$ (where $\tilde{\lambda}$ is a normalizing parameter), and expand χ via eigenfunctions of the angular momentum operator J_{12}

$$\chi = C_k\chi_k, \quad \chi_k = \frac{1}{\sqrt{r}}\begin{pmatrix} \exp\left(\mathrm{i}\left(k-\frac{1}{2}\right)\theta\right)\phi_1 \\ \epsilon\exp\left(\mathrm{i}\left(k+\frac{1}{2}\right)\theta\right)\phi_2 \end{pmatrix}, \quad (3.22)$$

where C_k are constants, $\epsilon = \tilde{\lambda}/|\tilde{\lambda}|$, ϕ_1 and ϕ_2 are functions of r, and summation is imposed over the repeated indices $k = 0, \pm 1, \pm 2, \ldots$.

In the following we restrict ourselves to solutions χ_k, which correspond to nonnegative values of k. Then solutions with k negative will be obtained by acting on χ_k by operator Q (3.19).

Substituting Eq. (3.22) into Eq. (3.21), we come to the following system

$$H_k\phi \equiv \left[-\frac{\partial^2}{\partial r^2} + k(k-\sigma_3)\frac{1}{r^2} + 2e\varphi + \sigma_1\frac{\lambda}{\tilde{\lambda}}\frac{\partial\varphi}{\partial r}\right]\phi = \tilde{\varepsilon}\phi, \quad (3.23)$$

with $\phi = \mathrm{column}(\phi_1,\ \phi_2)$ and

$$\tilde{\varepsilon} = (\varepsilon - \varepsilon_0)/2M\tilde{\lambda}^2. \quad (3.24)$$

Thus we reduce (3.21) to the system of two ordinary differential equations for radial functions, given by formula (3.23). Its solutions must be normalizable and vanish at $r = 0$. For some types of potential φ (and particular restrictions imposed on the coupling constants e and λ) this system is integrable and its solutions can be expressed via special functions. In the following section we consider an example of integrable equation (3.23), which corresponds to a neutral particle interacting anomalously with an external field.

IV. RELATIVISTIC ANALOG OF PS PROBLEM

Let us set $e = 0$ in (2.14) and choose the following particular realization for the potential φ

$$\varphi = \omega \ln(x), \tag{4.25}$$

where ω is a constant. Then the related Eqs. (3.17) and (3.23) are reduced to the following forms

$$\varepsilon'\chi = \left(\frac{p^2}{2M} + \tilde{\lambda}\frac{\sigma_\alpha x_\alpha}{x^2}\right)\chi, \quad \varepsilon' = \varepsilon - \varepsilon_0, \tag{4.26}$$

and

$$H_k\phi \equiv \left(-\frac{\partial^2}{\partial r^2} + k(k-\sigma_3)\frac{1}{r^2} + \sigma_1\frac{1}{r}\right)\phi = \tilde{\varepsilon}\phi, \tag{4.27}$$

respectively, provided we set $\tilde{\lambda} = \omega\lambda$.

The electromagnetic field whose potential is defined by relations (2.5) and (4.25) has a transparent physical meaning. Namely, it is a superposition of the electric field $\mathbf{E} = (E_1, E_2, E_3)$ whose components are

$$E_1 = \omega\frac{x_1}{x^2}, \quad E_2 = \omega\frac{x_2}{x^2}, \quad E_3 = 0, \tag{4.28}$$

and the magnetic field $\mathbf{B} = (B_1, B_2, B_2)$ with

$$B_1 = -\omega\frac{x_2}{x^2}, \quad B_2 = \omega\frac{x_1}{x^2}, \quad B_3 = 0. \tag{4.29}$$

Such an electric field can be identified as the field of a charged infinite filament coinciding with the third coordinate axis. Let us designate the charge density of this filament by ρ then the coupling constant ω should be equal to 2ρ. On the other hand, the magnetic field $\mathbf{B}$ is nothing but the field of a straight line constant current j directed along the third coordinate axis provided the coupling constant ω be equal to $2j$. Of course the related charge density and current should be equal between themselves

$$j = \rho = \omega/2. \tag{4.30}$$

Let us show that Eq. (4.26) is exactly solvable and find its solutions. The simplest way to prove the integrability of (4.26) is to make the unitary transformation

$$\chi \to \chi' = U\chi,$$
$$\varepsilon' - \left(\frac{p^2}{2M} + \tilde{\lambda}\frac{\sigma_\alpha x_\alpha}{x^2}\right) \to U\left[\varepsilon' - \left(\frac{p^2}{2M} + \tilde{\lambda}\frac{\sigma_\alpha x_\alpha}{x^2}\right)\right]U^\dagger, \tag{4.31}$$

where $U = \frac{1}{\sqrt{2}}(1 - i\sigma_3)$. As a result we reduce (4.26) to the following form

$$\varepsilon'\chi = \left(\frac{p^2}{2M} - 2\tilde{\lambda}\frac{S_1x_2 - S_2x_1}{x^2}\right)\chi, \tag{4.32}$$

where $S_1 = \frac{1}{2}\sigma_1$ and $S_2 = \frac{1}{2}\sigma_2$ are spin matrices and in accordance with Eqs. (2.3), (4.25), and (4.30) $\tilde{\lambda} = \lambda\omega = 2g\mu_0 j$.

For a fixed M and up to the value of the coupling constant Eq. (4.32) coincides with the Schrödinger equation for a neutral particle minimally interacting with the field generated by an infinite thin current filament (in our case the standard coupling constant is multiplied by a factor of 2). This equation was studied in numerous papers starting with [4] and continuing with [5–7] and many others. It has the following nice properties:

- Equation (4.32) admits a hidden dynamical symmetry with respect to group SO(3) for negative eigenvalues $\tilde{\varepsilon}$, group SO(1,2) for $\tilde{\varepsilon}$ positive, and group E(2) for $\tilde{\varepsilon} = 0$ [4];
- It possesses a hidden supersymmetry [5];
- Using any of the above-mentioned properties the equation can be integrated in closed form [4–6].

Since Eq. (4.26) is unitary equivalent to Eq. (4.32) it succeeds the above-mentioned properties. In particular, eigenvalues ε' are the same in both Eqs. (4.26) and (4.32).

V. RELATIVISTIC AND QUASIRELATIVISTIC ENERGY LEVELS

In the next section we will present exact solutions of Eq. (4.26) for coupled states and define the related eigenvalues ε'. In fact these eigenvalues are well known, and using directly the results of paper [4] (or of the papers [5–8]) we can immediately write ε' in the following form

$$\varepsilon' = -\frac{2\tilde{\lambda}^2 M}{N^2}, \tag{5.33}$$

where N is a positive natural number.

Eigenvalues (5.33) are degenerated since they do not depend on eigenvalues k of the angular momentum operator J_{12}. The degeneration factor is equal to $2k+1$, and the quantum number N can be represented as

$$N = 2(n+k) + 1, \tag{5.34}$$

where n is a natural number [4–6].

Using (5.33), we already can find energy levels for the initial relativistic problem. Indeed, since $P_0 = p_0 + p_3$ and $P_3 = \frac{1}{2}(p_0 - p_3)$, it is possible to write analogous relations for eigenvalues E of p_0, κ of p_3 and ε, M

$$\varepsilon = E + \kappa, \quad 2M = E - \kappa. \tag{5.35}$$

Then, using definitions (4.26) and (5.35) for ε', E, and M we find from (5.33) the relativistic energy spectrum

$$E = \frac{m}{K + \tilde{\kappa}} + \kappa, \tag{5.36}$$

where

$$K = \sqrt{1 + \tilde{\kappa}^2 + \frac{\tilde{\lambda}^2}{N^2}}, \quad \tilde{\kappa} = \frac{\kappa}{m}, \quad \tilde{\lambda} = 2\mu_0 g j. \tag{5.37}$$

We see that, in spite of the fact that the neutron motion along the third coordinate axis is free, the third component of momentum κ makes a rather nontrivial contribution into the values of energy levels (5.36). In accordance with (5.36) $E > \kappa$, and so the condition $M \neq 0$ is actually satisfied.

The most simple expression for energy levels corresponds to the particular value $\kappa = 0$

$$E = \frac{m}{\sqrt{1 + \frac{\tilde{\lambda}^2}{N^2}}}, \tag{5.38}$$

which for small $\tilde{\lambda}$ becomes

$$E = m\left(1 - \frac{\tilde{\lambda}^2}{2N^2}\right) + \cdots = m - \frac{2m(g\mu_0 j)^2}{N^2} + \cdots. \tag{5.39}$$

Up to the rest energy term m the approximate energy levels (5.39) are exactly the same as in the nonrelativistic PS problem [4,6–8]. In particular, both the approximate and exact levels given by Eqs. (5.36), (5.38), and (5.39) are degenerated with respect to eigenvalues k of the third component of angular momentum, which is a constant of motion for the considered system. As in [4] this degeneration is caused by a hidden dynamical symmetry of the system.

Let both $\tilde{\lambda}$ and $\tilde{\kappa}$ be small. Expanding E (5.39) in a power series of $\tilde{\lambda}$ and $\tilde{\kappa}$ we obtain the quasirelativistic approximation for the energy levels

$$E \approx m + \frac{\kappa^2}{2m} - \frac{\kappa^4}{8m^3} - \frac{m\tilde{\lambda}_\kappa^2}{2N^2}, \tag{5.40}$$

where

$$\tilde{\lambda}_\kappa = (1 - \tilde{\kappa})\tilde{\lambda}. \tag{5.41}$$

The first three terms in (5.40) represent, respectively, the rest energy, the kinetic energy of free motion along the third coordinate axis, and the relativistic correction to this energy. The last (dynamical) term in (5.40) is quite similar to the corresponding nonrelativistic term [compare with (5.39)], but includes the corrected coupling constant $\tilde{\lambda}_\kappa$ instead of $\tilde{\lambda}$.

Consider also the ultrarelativistic situation when $\tilde{\kappa}$ is large but $\tilde{\lambda}$ is still small. Then the energy values (5.39) can be expanded as

$$E = \sqrt{m^2 + \kappa^2} - m\delta\frac{\tilde{\lambda}^2}{2N^2} + \cdots, \tag{5.42}$$

where the dots denote the terms of order $\tilde{\lambda}^4$ and

$$\delta = \left(2\sqrt{1 + \tilde{\kappa}^2} - 2\tilde{\kappa} - \frac{1}{\sqrt{1 + \tilde{\kappa}^2}}\right). \tag{5.43}$$

Comparing (5.42) with (5.39) we recognize that the relativistic binding energy levels include the additional multiplier δ which considerably differs from 1 for ultrarelativistic k.

Notice that since the electromagnetic field defined by relations (2.5) and (2.6) has no components in the x_3 direction, the motion of the particle in this direction is free. This motion can be quantized by imposing the periodic boundary condition. Then

$$\kappa = \frac{2\pi\ \tilde{N}}{L}, \quad \tilde{N} = 0, \pm 1, \pm 2, \ldots, \tag{5.44}$$

and energy levels (5.36) through (5.42) are labeled by the pairs of quantum numbers N and $\tilde{N}$.

VI. EXACT SOLUTIONS FOR BOUND STATES

To find the solutions of Eq. (4.26) we use the fact that the Hamiltonian

$$H_k = \left(-\frac{\partial^2}{\partial r^2} + k(k - \sigma_3)\frac{1}{r^2} + \sigma_1\frac{1}{r}\right), \tag{6.45}$$

can be factorized as

$$H_k = a_k^+ a_k + C_k, \tag{6.46}$$

where

$$a_k = \frac{\partial}{\partial r} + W_k, \quad a_k^+ = -\frac{\partial}{\partial r} + W_k, \quad C_k = -\frac{1}{(2k+1)^2},$$

and W is a *matrix superpotential*

$$W_k = \frac{1}{2r}\sigma_3 - \frac{1}{2k+1}\sigma_1 - \frac{\left(k + \frac{1}{2}\right)}{r}. \tag{6.47}$$

It can be verified by direct calculation that

$$H_k^+ = a_k a_k^+ + C_k = -\frac{\partial^2}{\partial r^2} + (k+1)(k+1-\sigma_3)\frac{1}{r^2} + \sigma_1\frac{1}{r},$$

that is, the superpartner Hamiltonian H_k^+ for H_k is equal to H_{k+1}. Thus the eigenvalue problem (3.23) possesses a supersymmetry with shape invariance and so it can be solved using the standard technique of the supersymmetric quantum mechanics [21]. We will not reproduce the related routine calculations whose details can be found in [6,7] but restrict ourselves to the presentation of the solutions of Eq. (4.26).

The ground-state solutions $\phi(0,k;r) =$ column $[\phi_1(0,k;r), \phi_2(0,k;r)]$ are square integrable and normalizable solutions of equation $a_k\phi(0,k;r) = 0$. They can be expressed in the following form

$$\begin{aligned} \phi_1(0,k;r) &= r^{k+1}K_1\left(\frac{r}{2k+1}\right), \\ \phi_2(0,k;r) &= -r^{k+1}K_0\left(\frac{r}{2k+1}\right), \end{aligned} \tag{6.48}$$

where K_0 and K_1 are the modified Bessel functions. The corresponding eigenvalue $\tilde{\varepsilon}_k$ in (4.26) and (3.23) is equal to $-\frac{1}{(2k+1)^2}$.

Solutions corresponding to the first excited state (i.e., when $n = 1$) are $\phi(1,k;r) = a_k^+\phi(0,k+1;r)$, or being written componentwise

$$\begin{aligned} \phi_1(1,k;r) &= -\left(\frac{\partial}{\partial r} + \frac{k}{r}\right)\phi_1(0,k+1;r) \\ &\quad - \frac{1}{(2k+1)}\phi_2(0,k+1;r) \\ &= \frac{4(k+1)}{(2k+1)(2k+3)}r^{k+2}K_0\left(\frac{r}{2k+3}\right) \\ &\quad - (2k+1)r^{k+1}K_1\left(\frac{r}{2k+3}\right), \\ \phi_2(1,k;r) &= -\left(\frac{\partial}{\partial r} + \frac{k+1}{r}\right)\phi_2(0,k+1;r) \\ &\quad - \frac{1}{(2k+1)}\phi_1(0,k+1;r) \\ &= (2k+3)r^{k+1}K_0\left(\frac{r}{2k+3}\right) \\ &\quad - \frac{4(k+1)}{(2k+1)(2k+3)}r^{k+2}K_1\left(\frac{r}{2k+3}\right). \end{aligned} \tag{6.49}$$

The corresponding eigenvalue $\tilde{\varepsilon}_k$ is equal to $-\frac{1}{(2(k+1)+1)^2} = -\frac{1}{(2k+3)^2}$.

Finally, solutions which correspond to an arbitrary value of the quantum number $n > 0$ can be represented as

$$\phi(n,k;r) = a_k^+ a_{k+1}^+ \cdots a_{k+n-1}^+ \phi(0,k+n;r),\quad n = 1,2,\ldots,$$

which gives rise to the recurrence relations

$$\begin{aligned}\phi_1(n,k;r) &= -\frac{\partial}{\partial r}\phi_1(n-1,k+1;r) - \frac{k}{r}\phi_1(n-1,k+1;r) + \frac{1}{2k+1}\phi_2(n-1,k+1;r),\\ \phi_2(n,k;r) &= -\frac{\partial}{\partial r}\phi_2(n-1,k+1;r) - \frac{k+1}{r}\phi_2(n-1,k+1;r) + \frac{1}{2k+1}\phi_1(n-1,k+1;r).\end{aligned} \tag{6.50}$$

The related eigenvalue $\tilde{\varepsilon}_k$ is given by relations (5.33) and (5.34).

It is now possible to present exact solutions of the initial Dirac-Pauli equation defined by relations (2.1), (2.5), and (4.25). In accordance with the above-such solutions are labeled by the main quantum number N, which can be expressed by Eq. (5.34), and by eigenvalues κ and k of the third component of momenta and total orbital momentum. Using Eqs. (2.12), (2.13), (2.15), (3.20), (3.22), and (6.48) through (6.50) we find these solutions in the following form

$$\psi_{n,\kappa,k} = \frac{1}{\sqrt{2\pi Lr}}\exp[-\mathrm{i}(Ex_0 - \kappa x_3)] \times \begin{pmatrix} \exp\left(i\left(k-\frac{1}{2}\right)\theta\right)\eta_1(n,k;r)\\ \exp\left(i\left(k+\frac{1}{2}\right)\theta\right)\eta_2(n,k;r)\\ \exp\left(i\left(k-\frac{1}{2}\right)\theta\right)\phi_1(n,k;r)\\ \exp\left(i\left(k+\frac{1}{2}\right)\theta\right)\epsilon\phi_2(n,k;r)\end{pmatrix}. \tag{6.51}$$

Here k and n are nonnegative natural numbers

$$r = \frac{\sqrt{x_1^2+x_2^2}}{r_0},\quad r_0 = \frac{1}{M|\tilde{\lambda}|} = \frac{K+\tilde{\kappa}}{m|\tilde{\lambda}|},\quad \epsilon = \frac{\tilde{\lambda}}{|\tilde{\lambda}|}, \tag{6.52}$$

where $\tilde{\kappa} = \kappa/m$, E, K, and κ are given by Eqs. (5.36), (5.37), and (5.44), $\phi_1(n,k;r)$ and $\phi_2(n,k;r)$ are functions defined by recurrence relations (6.48), (6.50),

$$\begin{aligned}\eta_1(n,k;r) &= \tilde{\lambda}\left(\frac{\partial}{\partial r} + \frac{k}{r}\right)\epsilon\phi_2(n,k;r) + (K+\tilde{\kappa})\phi_1(n,k;r),\\ \eta_2(n,k;r) &= \tilde{\lambda}\left(\frac{\partial}{\partial r} - \frac{k}{r}\right)\phi_1(n,k;r) + (K+\tilde{\kappa})\epsilon\phi_2(n,k;r).\end{aligned} \tag{6.53}$$

Solutions (6.51) are normalizable and tend to zero with $r \to 0$. They are defined for nonnegative eigenvalues k of the total angular momentum while solutions for k negative can be obtained acting on (6.51) by the reflection operator $\hat{Q} = i\gamma_0\gamma_2\gamma_3 R_1$ where R is the reflection of the first spatial variable [i.e., $R_1\psi(x_0,x_1,x_2,x_3) = \psi(x_0,-x_1,x_2,x_3)$ and $R_1\psi(x_0,r,\theta,x_3) = \psi(x_0,r,-\theta,x_3)$]. Using the Dirac matrices (2.8) we find the solutions with negative values of k in the following form

$$\psi_{n,\kappa,k} = \frac{\mathrm{i}}{\sqrt{2\pi Lr}}\exp[\mathrm{i}(Ex_0 - \kappa x_3)] \times \begin{pmatrix} -\exp\left(-i\left(k+\frac{1}{2}\right)\theta\right)\eta_2(n,k;r)\\ -\exp\left(i\left(\frac{1}{2}-k\right)\theta\right)\eta_1(n,k;r)\\ \epsilon\exp\left(-i\left(k+\frac{1}{2}\right)\theta\right)\phi_2(n,k;r)\\ -\exp\left(i\left(\frac{1}{2}-k\right)\theta\right)\phi_1(n,k;r)\end{pmatrix}. \tag{6.54}$$

Let us present explicitly the components of solutions (6.51) and (6.54) for $n = 0$ and $n = 1$. If $n = 0$ then the related functions $\phi_1(n,k;r) = \phi_1(0,k;r)$ and $\phi_2(0,k;r)$ are given by Eq. (6.48) while $\eta_1(0,k;r)$ and $\eta_2(0,k;r)$ have the following form

$$\begin{aligned}\eta_1(0,k;r) &= \Lambda_k\phi_1(0,k;r) + \frac{\tilde{\lambda}(2k+1)}{r}\phi_2(0,k;r),\\ \eta_2(0,k;r) &= \Lambda_k\phi_2(0,k;r),\end{aligned} \tag{6.55}$$

where $\Lambda_k = (\frac{\tilde{\lambda}}{2k+1} + K + \tilde{\kappa})$. If $n = 1$ the corresponding functions $\phi_1(1,k;r)$ and $\phi_2(1,k;r)$ are given by Eq. (6.49) and

$$\begin{aligned}\eta_1(1,k;r) &= \Lambda_{k+1}\phi_1(1,k;r) + \frac{\tilde{\lambda}(2k+1)}{r}\phi_2(1,k;r) - \frac{2\tilde{\lambda}}{(2k+3)r}\phi_1(0,k+1;r),\\ \eta_2(1,k;r) &= \Lambda_{k+1}\phi_2(1,k;r) + \frac{2\tilde{\lambda}}{r}\phi_1(1,k;r) + \frac{2\tilde{\lambda}}{(2k+3)r}\phi_2(0,k+1;r) + \frac{2(2k+1)\tilde{\lambda}}{r^2}\phi_1(0,k+1;r),\end{aligned} \tag{6.56}$$

where $\Lambda_{k+1} = (\frac{\tilde{\lambda}}{2k+3} + K + \tilde{\kappa})$.

Functions $\eta_a(a = 1,2)$ in (6.55) and (6.56) include the terms proportional to ϕ_a (the first terms in the right-hand side). The remaining terms are small in comparison with ϕ_a, which results in similarity of probability distributions for neutrons in our model to the distributions in the PS model, see Appendix.

VII. DISCUSSION

In Secs. II and III we study a class of Dirac-Pauli systems which can be effectively reduced to a set of Schrödinger-Pauli equations. The main inspiration for our research was to find an integrable relativistic formulation of the nonrelativistic PS problem [4]. This goal cannot be achieved by a straightforward relativization of the PS problem since the Dirac-Pauli equation for a neutral particle interacting with the magnetic field generated by a filament current is not integrable.

In the present paper we succeed in obtaining an integrable relativistic model which in many aspects can be treated as an analog of the PS model. To this end we introduce a superposition of magnetic and electric fields, which is not equivalent to the field of straight current. Nevertheless, in the nonrelativistic limit our model is reduced to the PS one. To justify this statement we return to Eq. (4.30) and note that in the cgs units it takes the following form

$$\rho = j/c, \tag{7.57}$$

where c denotes the velocity of light.

In accordance with (7.57) the required charge density is small and tends to zero in the nonrelativistic approximation when $c \to \infty$. Thus in the nonrelativistic limit the external field which we consider reduces to the field used in the PS model. In addition, the energy spectrum (5.36) is reduced to the PS form, see Eq. (5.39). That is why we claim that the nonrelativistic limit of the model defined by Eqs. (2.1), (2.5), and (4.25) is exactly the PS model. This property can be directly proved using the Foldy-Wouthuysen transformation [22].

Let us discuss the obtained energy levels for coupled states in the quasirelativistic approximation (5.40). Albeit the motion along the third Cartesian coordinate is free, the third momentum component κ makes a contribution into the effective coupling constant $\tilde{\lambda}_\kappa$. The origin of this contribution is the anomalous interaction of neutron moving along the charged line with the magnetic field generated by this line. In the rest frame this motion is effectively changed by the current which flows in the line in the opposite direction, which is in perfect accordance with Eq. (5.41).

The contribution of κ into the effective coupling constant $\tilde{\lambda}_\kappa$ (5.41) is small. Namely, it is proportional to the inverse speed of light. However, it affects the energy levels (5.40) much more than the relativistic correction to the kinetic energy $-\kappa^4/8m^3$, which is proportional to the squared inverse speed of light.

In the case of ultrarelativistic neutron motion along the charge and current carrying lines the contribution of the related momentum into the coupling energy becomes very essential. As it follows from (5.42) the distances between the energy levels can significantly differ from the nonrelativistic ones since the multiplier δ (5.43) changes continuously from 0.121 (for $\tilde{\kappa} \to -1$) to 4.121 (for $\tilde{\kappa} \to 1$).

In conclusion, we present an exactly solvable problem for the Dirac-Pauli equation describing a neutral particle which interacts anomalously with a rather particular external field given by Eqs. (4.28) through (4.30) having, however, a clear physical meaning. This type of anomalous interaction is the key to expanding solutions of the problem via solutions of the $(1+2)$-dimensional Levi-Leblond equation invariant with respect to the Galilei group. Moreover, the considered problem possesses a hidden symmetry and supersymmetry which causes the $(2n+1)$-fold degeneration of the energy levels given by Eqs. (5.36) and (5.34).

A natural question arises whether the considered relativistic problem with its symmetries is unique or there are other problems which can be effectively solved using reduction SO(1, 3) → HG(1,2). In Secs. II and III we study a certain class of such problems that can be effectively reduced to radial Eq. (3.23), which is exactly solvable when $e = 0$ and $\varphi = \omega \ln(x)$. We believe that there are other exactly solvable equations (3.23) and at least two of them can be immediately written down if we set $\varphi = \alpha/x$ and consider the alternative cases $e = 0$ and $e \neq 0$. The related Dirac-Pauli equations (2.1), (2.5), and (2.6) can be solved explicitly. We plan to study these and probably other integrable models in the future.

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APPENDIX: COUPLING CONSTANTS AND PROBABILITY DISTRIBUTIONS

In the main text we consider an idealized model with the infinite thin current filament and charged line. To give an idea about its physical realizability let us discuss the probability density which corresponds to found solutions (6.51).

First we present in more transparent form the coupling constant $\tilde{\lambda}$ and scaling interval r_0 (6.52). Going from the Heaviside units to cgs ones we should make the following changes in Eqs. (5.36) and (6.52)

$$\begin{gathered} m \to mc^2, \quad \kappa \to c\kappa, \quad \tilde{\kappa} \to \tilde{\kappa}' = \frac{\kappa}{mc}, \\ \tilde{\lambda} \to \frac{\tilde{\lambda}}{\hbar c} = \tilde{\lambda}', \quad r_0 \to \frac{2C_n}{|\tilde{\lambda}'|}\left(K + \frac{\kappa}{mc}\right), \end{gathered} \tag{A1}$$

where c is the velocity of light and C_n is the Compton wave length for the neutron. The dimensionless constant $\tilde{\lambda}'$ and the r_0 can be represented as

$$\begin{gathered} \tilde{\lambda}' = -\frac{g\alpha C_n N_c \hat{j}}{c} = 7.633 \times 10^{-7}\hat{j}, \\ r_0 = \frac{34.5\ \text{Å}}{\hat{j}}\left(K + \frac{\kappa}{mc}\right), \end{gathered} \tag{A2}$$

where $\hat{j} = j/\text{A}$ is the current measured in amperes, $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$ is the fine-structure constant, $g = -3.82$ is the neutron Landé factor, $N_c = C/e = 6.242 \times 10^{18}$ is the charge equal to 1 Coulomb measured in elementary charges. Surely for realistic current values parameter $\tilde{\lambda}$ is small thus the expansions in a power series of $\tilde{\lambda}$ made in Sec. V was well grounded.

We formulated our problem for neutrons. However, the obtained results can be extended to other neutral particles which have nontrivial magnetic moments. As an example, we consider here the Na atoms in the ground state. Then $|g| \to 5.4$, $m \to 23m$, and the parameters (A2) are transformed to the following ones

$$\tilde{\lambda}' = 4.68 \times 10^{-8}\hat{j}, \quad r_0 = \frac{24.42\ \text{Å}}{\hat{j}}\left(K + \frac{\kappa}{mc}\right). \tag{A3}$$

Consider now solutions (6.51) and evaluate the corresponding probability density

$$\begin{aligned} \bar{\psi}_{n,\kappa,k}\gamma_0 r \psi_{n,\kappa,k} = C^2_{n,\kappa,k}\Bigg(&\phi_1^2 + \phi_2^2 + \tilde{\lambda}'(K+\kappa')\epsilon\delta\frac{\partial(\phi_1\phi_2)}{\partial r} \\ &+ \frac{\delta\tilde{\lambda}'^2}{r}\frac{\partial\left(\phi_2^2 - \phi_1^2\right)}{\partial r}\Bigg), \end{aligned} \tag{A4}$$

where $\delta = \frac{1}{(K+\kappa')^2+1}$ and $C_{n,\kappa,k}$ is a normalization constant. In particular, for $n = 0$,

$$\bar{\psi}_{0,\kappa,k}\gamma_0 r\psi_{0,\kappa,k} = C_k^2\left(\phi_1^2 + \phi_2^2 - \epsilon\delta_1\frac{\phi_1\phi_2}{r} + \delta_2\frac{\phi_2^2}{r^2}\right), \tag{A5}$$

where ϕ_1 and ϕ_2 are functions defined in (6.48) and

$$\delta_1 = 2\tilde{\lambda}'((2k+1)(K+\kappa') + \tilde{\lambda}')\delta, \quad \delta_2 = \tilde{\lambda}'^2(2k+1)^2\delta. \tag{A6}$$

Analyzing (A5) we conclude that the last two terms in the brackets are small. First they include the small multiplier $\tilde{\lambda}'$ (A2). Second, for $k > 1/2$ functions $\frac{\phi_1\phi_2}{r}$ and $\frac{\phi_2^2}{r^2}$ are negligibly small in comparison with $\phi_1^2 + \phi_2^2$. The same statement is correct for Eq. (A4) which can be proven by using the identities

$$\frac{\partial K_0(\lambda r)}{\partial r} = -\lambda K_1(\lambda r), \quad \frac{\partial K_1(\lambda r)}{\partial r} = -\lambda K_0(\lambda r) - \frac{1}{r}K_1(\lambda r).$$

Thus, practically without loss of accuracy, we can write

$$\bar{\psi}_{n,\kappa,k}\gamma_0 r\psi_{n,\kappa,k} \approx C_{n,\kappa,k}^2(\phi_1^2 + \phi_2^2), \tag{A7}$$

and so the probability distribution calculated for our relativistic problem is virtually the same as the one obtained in [6,8] for the nonrelativistic PS problem.

Thus the main statements presented in [6,8] concerning the possibility, in principle, to observe experimentally the neutrons and Na atoms trapped by the current filament can be generalized to our model if we restrict ourselves to small $\tilde{\kappa}$. We will not repeat the reasonings given in the mentioned papers but remind the reader that reasonable current values are $j \approx 50$ mA for trapping neutrons and $j \approx 400$ mA for trapping the Na atoms in the ground state.

The principally new feature of the relativistic model is the essential dependence of the coupling energy and of the scaling parameter r_0 on the third component of momentum. Indeed the coupling energy $-m\delta\frac{\tilde{\lambda}^2}{2N^2}$ in (5.42) and characteristic distance r_0 (A3) include κ-dependent multipliers δ (5.43) and $(K + \frac{\kappa}{mc})$ respectively. Both these multipliers do not appear in the nonrelativistic PS problem.

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