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Analytical proof of Gisin's theorem for three qubits

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Gisin's theorem assures that for any pure bipartite entangled state, there is violation of the inequality of Bell and of Clauser, Horne, Shimony, and Holt, revealing its contradiction with local realistic model. Whether a similar result holds for three-qubit pure entangled states remained unresolved. We show analytically that all three-qubit pure entangled states violate a Bell-type inequality, derived on the basis of local realism, by exploiting the Hardy's nonlocality argument.

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I. INTRODUCTION

Not all measurement correlations in some state of a composite quantum system can be described by local hidden variable theory (LHVT) [1], a fact which is said to be "the most profound discovery of science" [2]. Experimental verification of this fact (i.e., whether measurement correlations in nature obey quantum rules or LHVT) goes also in favor of quantum theory, modulo some loopholes [3]. Every LHVT description [4] of the measurement correlations of a composite system (assumed to be finite dimensional in the present article) gives rise to one (or more than one) linear inequality (or, inequalities) involving these correlations [5]. There are states of a composite quantum system which violate some or all these inequalities for suitable choices of the subsystem observables.

Gisin's theorem assures that for any pure entangled state of two-qudits, the above-mentioned violation is generic for two settings per site (i.e., for the choice of one between two noncommuting observables per qudit) [6]. In other words, *all* pure entangled states of two d-dimensional quantum systems violate a *single* Bell-type inequality with two settings per site, where the choice of the observables depends on that of the state. The question of extending Gisin's work for multipartite pure entangled states was first addressed by Popescu and Rohrlich [7], although their approach was essentially confined to bipartite pure entanglement. However, the validity of Gisin's theorem for multipartite systems is not guaranteed still today. For example, for odd N, there is a family of entangled pure states of N qubits, each of which satisfies *all* Bell-type inequalities involving correlation functions, arising out of measurement of one between two noncommuting dichotomic observables per qubit [8]. Later, Chen *et al.* [9] provided a Bell-type inequality involving joint probabilities, associated to measurement of one between two noncommuting dichotomic observables per qubit, which is violated by all the states of the above-mentioned family [10]. But a single Bell-type inequality is not guaranteed to be violated by *all* pure entangled states of three-qubits, although there are claims of having numerical evidences in favor of this violation [9,11,12].

Quantum theory also shows contradiction with LHVT via "nonlocality without inequalities" (NLWI) [13]. In this case, a set of values of joint probabilities of outcomes of measurements of one between two noncommuting observables per site contradicts LHVT but can be realized in quantum theory. Unfortunately, NLWI is weaker than Bell-type inequalities, as *no* maximally entangled state of two-qudits seems to show NLWI (in the case of Hardy-type NLWI, this has been shown in Ref. [14]) even though each of them violates a Bell-type inequality. This situation changes drastically when we consider Hardy-type NLWI for three two-level systems [15,16], where all but one of the joint probabilities in the above-mentioned set are zero. *Every* maximally entangled state of three qubits [17] satisfies Hardy-type NLWI for suitably chosen pairs of noncommuting dichotomic observables per qubit [18]. An attempt was made in Ref. [18] toward achieving the result that *every* pure state of three qubits, having genuine tripartite entanglement, satisfies Hardy-type NLWI. But in absence of the discovery of canonical form for three-qubit pure states (which was done later in Ref. [19]), it did not yield a complete proof. A definite process to exclude the product states was not given in Ref. [18], rather a somewhat iterative process was described for each of those product states.

Now, from the set of joint probabilities in any NLWI argument, one can, in principle, construct a linear inequality involving these joint probabilities by using local realistic assumption. This inequality is automatically violated by every quantum state which satisfies the corresponding NLWI argument. In the case of Hardy-type NLWI argument for two two-level systems, this inequality (given in Eq. (11) of Ref. [20], Eq. (11) of Ref. [21], and Eq. (26) of Ref. [22]) is nothing but the corresponding CH inequality [23]. So, by Gisin's theorem, *every* two-qubit pure entangled state (irrespective of its amount of entanglement) will violate the former inequality. In this article, we show *analytically* that *every* three-qubit pure entangled state violates a linear inequality of the above-mentioned type [see Eq. (3) below] involving joint probabilities associated with the Hardy-type NLWI, irrespective of whether the state has genuine tripartite entanglement or pure bipartite entanglement.

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This article is arranged as follows. In Sec. II, we describe the Hardy-type NLWI argument for three two-level systems and describe the corresponding quantum mechanical counterpart. The classification of all three-qubit pure states in product, bipartite nonmaximal, bipartite maximal, and genuine tripartite entangled states, is made in Sec. III. Section IV describes the observable settings for the different subclasses of Sec. III to satisfy the Hardy nonlocality condition (1) quantum mechanically. We derive a Bell-type inequality corresponding to the Hardy's nonlocality condition (1) in Sec. V. In Sec. VI, it has been shown that every three-qubit pure entangled states violates this Bell-type inequality. We conclude the article in Sec. VII with discussion on visibility and some open problems.

II. HARDY-TYPE NLWI ARGUMENT FOR THREE TWO-LEVEL SYSTEMS

Hardy-type NLWI argument starts from the following set of five joint probability conditions for three two-level systems:

$$\begin{aligned} &P(D_1=+1,U_2=+1,U_3=+1)=0,\\ &P(U_1=+1,D_2=+1,U_3=+1)=0,\\ &P(U_1=+1,U_2=+1,D_3=+1)=0,\\ &P(D_1=-1,D_2=-1,D_3=-1)=0,\\ &P(U_1=+1,U_2=+1,U_3=+1)>0,\end{aligned} \tag{1}$$

where each U_j (as well as D_j) is a $\{+1,-1\}$-valued random variable. This set of conditions cannot be satisfied by a local realistic theory, and hence it contradicts LHVT [15,18]. To see this explicitly, let us assume that λ be any local hidden varriable taking values from the set Λ with probability distribution $\rho(\lambda)$, for which all the five conditions in (1) are simultaneously satisfied. Thus, under this assumption, there exist probability densities $f(U_j=+1;\lambda)$, $f(D_j=+1;\lambda)$, $f(U_j=-1;\lambda)$, $f(D_j=-1;\lambda)$ on Λ (for $j=1,2,3$) such that $0=P(D_1=+1,U_2=+1,U_3=+1)=\int_{\lambda\in\Lambda}\rho(\lambda)d\lambda f(D_1=+1;\lambda)f(U_2=+1;\lambda)f(U_3=+1;\lambda)$, and so on. The probability densities $f(U_j=\pm1;\lambda)$, $f(D_j=\pm1;\lambda)$ are such that $f(U_j=+1;\lambda)+f(U_j=-1;\lambda)=1=f(D_j=+1;\lambda)+f(D_j=-1;\lambda)$ for $j=1,2,3$ and for all $\lambda\in\Lambda$. This is so because the LHV λ should also reproduce the marginal experimental probabilities $P(U_j=+1;\lambda)+P(U_j=-1;\lambda)=1$, and so on. So, from the last condition in (1) we see that there exists a value range (Λ', say) of Λ within which $f(U_1=+1;\lambda)$, $f(U_2=+1;\lambda)$, $f(U_3=+1;\lambda)$ and $\rho(\lambda)$ are all nonzero. Now the first condition of (1) provides us $f(D_1=+1;\lambda)=0$ for all λ in Λ'. This immediately implies that $f(D_1=-1;\lambda)=1$ for all $\lambda\in\Lambda'$. Similarly, we get from the second and third conditions of (1) that $f(D_2=-1;\lambda)=1=f(D_3=-1;\lambda)$ for all $\lambda\in\Lambda'$. Therefore, $P(D_1=-1,D_2=-1,D_3=-1)=\int_{\lambda\in\Lambda}\rho(\lambda)d\lambda f(D_1=-1;\lambda)f(D_2=-1;\lambda)f(D_3=-1;\lambda)\geqslant\int_{\lambda\in\Lambda'}\rho(\lambda)d\lambda f(D_1=-1;\lambda)f(D_2=-1;\lambda)f(D_3=-1\lambda)=\int_{\lambda\in\Lambda'}\rho(\lambda)d\lambda>0$, which is in contradiction with the fourth condition of (1). Hence, (1) has contradiction with LHVT.

To show that in quantum theory there are states which exhibit this kind of nonlocality, we replace U_j and D_j by the $\{+1,-1\}$-valued observables $\hat{U}_j$ and $\hat{D}_j$, respectively, with $[\hat{U}_j,\hat{D}_j]\neq 0$. The probabilities appearing in (1) are expectation values of one-dimensional projectors corresponding to the following five product vectors:

$$\begin{aligned} &|\hat{D}_1=+1\rangle|\hat{U}_2=+1\rangle|\hat{U}_3=+1\rangle,\\ &|\hat{U}_1=+1\rangle|\hat{D}_2=+1\rangle|\hat{U}_3=+1\rangle,\\ &|\hat{U}_1=+1\rangle|\hat{U}_2=+1\rangle|\hat{D}_3=+1\rangle,\\ &|\hat{D}_1=-1\rangle|\hat{D}_2=-1\rangle|\hat{D}_3=-1\rangle,\\ &|\hat{U}_1=+1\rangle|\hat{U}_2=+1\rangle|\hat{U}_3=+1\rangle.\end{aligned}$$

One can easily check that these five vectors are linearly independent and hence span five-dimensional subspace of the eight-dimensional Hilbert space associated to the total system. Hence one can choose any one (among infinitely many) vector which is orthogonal to the first four vectors and nonorthogonal to the last one. Actually this result shows that for any choice of observables in the above-mentioned noncommuting fashion, one can always find a quantum state which exhibits contradiction with local realism [24].

III. CLASSIFICATION OF THREE-QUBIT PURE STATES

But, in this article, our purpose is to find the converse. We would like to see whether every three-qubit pure entangled state exhibits contradiction with local realism. For this purpose, one could start from the most general pure state $|\psi\rangle$ satisfying Hardy's nonlocality argument (1) corresponding to given set of observables and calculate the values of the invariants of the corresponding local unitary group [19]. Each of these invariants involves several parameters and hence finding out its range is a nontrivial job, even numerically. For this reason, we are going to take another route via the canonical form for three-qubit pure state [see Eq. (2)]. In this context, it has to be mentioned that Gisin's theorem could provide a single prescription for finding the observables for any bipartite pure state to show violation of the Bell-CHSH inequality, due to the existence of Schmidt decomposition. Schmidt decomposition, in its strict sense [25], is absent for systems comprising of three and more subsystems. This gives rise to complications and one needs to find the observables for each inequivalent case (depending on the values of the parameters describing the state) separately. In this direction, we start with an arbitrary three-qubit pure state $|\psi\rangle$, which can always be taken in the canonical form [19]:

$$|\psi\rangle=\lambda_0|000\rangle+\lambda_1e^{i\phi}|100\rangle+\lambda_2|101\rangle+\lambda_3|110\rangle+\lambda_4|111\rangle, \tag{2}$$

where $0\leqslant\lambda_j$ (for $j=0,1,2,3,4$), $\sum_{j=0}^{4}\lambda_j^2=1$, and $0\leqslant\phi\leqslant\pi$.

We now fully classify the above-mentioned three-qubit state $|\psi\rangle$ into four major classes: (A) $|\psi\rangle$ is a fully product state, (B) $|\psi\rangle$ has pure two-qubit nonmaximal entanglement, (C) $|\psi\rangle$ has pure two-qubit maximal entanglement, and (D) $|\psi\rangle$ has genuine pure three-qubit entanglement. Depending on the values of $\lambda_i's$ and ϕ, in Table I, we further classify each of these four classes into several subclasses: (A) consists of (A.1)–(A.3); (B) consists of (B.1)–(B.5); (C) consists of (C.1)–(C.3); (D) consists of (D.1)–(D.14).

TABLE I. Classification of $|\psi\rangle$.

Condition	Case
$\lambda_0\lambda_1 \neq 0,\ \lambda_2 = \lambda_3 = \lambda_4 = 0$	(A.1)
$\lambda_0 \neq 0,\ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$	(A.2)
$\lambda_0 = 0,\ \lambda_1\lambda_4 e^{i\phi} = \lambda_2\lambda_3$	(A.3)
$\lambda_0\lambda_1\lambda_2 \neq 0,\ \lambda_3 = \lambda_4 = 0$	(B.1)
$\lambda_0\lambda_1\lambda_3 \neq 0,\ \lambda_2 = \lambda_4 = 0$	(B.2)
$0 < \lambda_0\lambda_2 < 1/2,\ \lambda_1 = \lambda_3 = \lambda_4 = 0$	(B.3)
$0 < \lambda_0\lambda_3 < 1/2,\ \lambda_1 = \lambda_2 = \lambda_4 = 0$	(B.4)
$\lambda_0 = 0$ and $\sqrt{2}\begin{pmatrix}\lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4\end{pmatrix}$ is neither a singular matrix nor a unitary matrix	(B.5)
$\lambda_0\lambda_2 = 1/2,\ \lambda_1 = \lambda_3 = \lambda_4 = 0$	(C.1)
$\lambda_0\lambda_3 = 1/2,\ \lambda_1 = \lambda_2 = \lambda_4 = 0$	(C.2)
$\lambda_0 = 0$ and $\sqrt{2}\begin{pmatrix}\lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4\end{pmatrix}$ is a unitary matrix	(C.3)
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0,\ \phi > 0$	(D.1)
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0,\ \phi = 0,\ \lambda_2\lambda_3 \neq \lambda_1\lambda_4$	(D.2)
$\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0,\ \phi = 0,\ \lambda_2\lambda_3 = \lambda_1\lambda_4$	(D.3)
$\lambda_0\lambda_1\lambda_2\lambda_3 \neq 0,\ \lambda_4 = 0$	(D.4)
$\lambda_0\lambda_1\lambda_2\lambda_4 \neq 0,\ \lambda_3 = 0$	(D.5)
$\lambda_0\lambda_1\lambda_3\lambda_4 \neq 0,\ \lambda_2 = 0,\ \lambda_0 \neq \lambda_4$	(D.6)
$\lambda_0\lambda_1\lambda_3\lambda_4 \neq 0,\ \lambda_2 = 0,\ \lambda_0 = \lambda_4$	(D.7)
$\lambda_0\lambda_1\lambda_4 \neq 0,\ \lambda_2 = \lambda_3 = 0$	(D.8)
$\lambda_0\lambda_3\lambda_4 \neq 0,\ \lambda_1 = \lambda_2 = 0$	(D.9)
$\lambda_0\lambda_2\lambda_3\lambda_4 \neq 0,\ \lambda_1 = 0,\ \lambda_2 \neq \lambda_4$	(D.10)
$\lambda_0\lambda_2\lambda_3\lambda_4 \neq 0,\ \lambda_1 = 0,\ \lambda_2 = \lambda_4$	(D.11)
$\lambda_0\lambda_2\lambda_3 \neq 0,\ \lambda_1 = \lambda_4 = 0$	(D.12)
$\lambda_0\lambda_2\lambda_4 \neq 0,\ \lambda_1 = \lambda_3 = 0$	(D.13)
$\lambda_0\lambda_4 \neq 0,\ \lambda_1 = \lambda_2 = \lambda_3 = 0$	(D.14)

IV. OBSERVABLE SETTINGS

If $|\psi\rangle$ has only bipartite nonmaximal entanglement we then first consider the situation where $|\psi\rangle = |\eta\rangle \otimes |\chi\rangle$, with $|\eta\rangle$ being a two-qubit nonmaximally entangled state of the first and the second qubits, while $|\chi\rangle$ is a state of the third qubit. Hardy [13] has shown that for all two-qubit nonmaximally entangled pure states, one can choose observables for both the qubits in such a way that the condition of nonlocality without inequality holds. Now in our three-qubit case, we first choose $|\hat{U}_3 = +1\rangle = (1/\sqrt{2})(|\chi\rangle + |\chi^{\perp}\rangle)$ and $|\hat{D}_3 = +1\rangle = |\chi^{\perp}\rangle$, where $\langle\chi^{\perp}|\chi\rangle = 0$. We can then choose two pairs of noncommuting dichotomic observables $(\hat{U}_1, \hat{D}_1)$ and $(\hat{U}_2, \hat{D}_2)$ in such a way that the state $|\eta\rangle$ satisfies Hardy's NLWI conditions for two two-level systems corresponding to these observables. This immediately shows that the state $|\psi\rangle$ satisfies the Hardy-type NLWI condition (1). As condition (1) is symmetric with respect to the qubits, we see that for each of the cases (B.1)–(B.5), the state $|\psi\rangle$ will satisfy the Hardy-type NLWI argument (1).

Again, let $|\psi\rangle = |\eta\rangle \otimes |\chi\rangle$, where $|\eta\rangle$ is a two-qubit maximally entangled state of the first and the second qubits, while $|\chi\rangle$ is a state of the third qubit. If we now demand $|\psi\rangle$ to satisfy (1), it will immediately follow that $|\eta\rangle$ must satisfy Hardy's NLWI conditions for two two-level systems—an impossibility [13]. As above, we see that in none of the cases (C.1)–(C.3), state $|\psi\rangle$ will satisfy the Hardy-type NLWI argument.

If $|\psi\rangle$ have genuine tripartite entanglement then to show that it satisfies the Hardy-type NLWI argument (1), one can choose the three pairs of $\{+1,-1\}$-valued noncommutating observables $(\hat{U}_j, \hat{D}_j)$ (where j is associated with j-th system ($j = 1,2,3$)) as follows:

$$|\hat{U}_j = +1\rangle = k_j(\alpha_j|0\rangle + \beta_j|1\rangle), |\hat{D}_j = +1\rangle = l_j(\gamma_j|0\rangle + \delta_j|1\rangle),$$

where $0 < |k_j l_j(\alpha_j\gamma_j^* + \beta_j\delta_j^*)|, |k_j l_j(\alpha_j\delta_j - \beta_j\gamma_j)| < 1$, $|k_j\alpha_j|^2 + |k_j\beta_j|^2 = |l_j\gamma_j|^2 + |l_j\delta_j|^2 = 1$, and k_j's, l_j's are the normalization constants (for $j = 1,2,3$). The values of $\alpha_j, \beta_j, \gamma_j, \delta_j$ are given in Table II for all the cases (D.1)–(D.14).

The set of values of $\alpha_j, \beta_j, \gamma_j, \delta_j$ in Table II are not the only possible values; one can get different such sets of values of $\alpha_j, \beta_j, \gamma_j, \delta_j$. The method we have adopted here to choose these values is the following. First try to fix some or all of $|\hat{U}_1 = +1\rangle$, $|\hat{U}_2 = +1\rangle$, $|\hat{U}_3 = +1\rangle$ [and that will fix some or all the pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$] so that the last condition in Eq. (1) satisfied corresponding to the chosen state $|\psi\rangle$. And then try to fix rest of the $|\hat{U}_j = +1\rangle$'s and all of $|\hat{D}_j = +1\rangle$'s according to the other conditions in (1) corresponding to the same state $|\psi\rangle$. As an example consider the observable settings in Table II for the subclass (D.14). Here $|\psi\rangle = \lambda_0|000\rangle + \lambda_4|111\rangle$ with $\lambda_0, \lambda_4 \in \mathcal{R}$ and $\lambda_0\lambda_4 \neq 0$. Choosing $|\hat{U}_1 = +1\rangle = |\hat{U}_2 = +1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \equiv |+\rangle$, the last condition $P(U_1 = +1, U_2 = +1, U_3 = +1) > 0$ of (1) will be satisfied provided we chose $|\hat{U}_3 = +1\rangle$ to be nonorthogonal to the state $\lambda_0|0\rangle + \lambda_4|1\rangle$. And in this case, according to the third condition of (1), we need to choose $|\hat{D}_3 = +1\rangle$ to be proportional to $\lambda_4|0\rangle - \lambda_0|1\rangle$. Choosing now $|\hat{U}_3 = +1\rangle = k_3(\alpha_3|0\rangle + \beta_3|1\rangle)$ (so here, we must have: $\alpha_3\lambda_0 + \beta_3\lambda_4 \neq 0$), one can find $|\hat{D}_1 = +1\rangle$ from the first condition and $|\hat{D}_2 = +1\rangle$ from the second condition of (1), which will, in turn fix α_3, β_3 using the fourth condition of (1).

V. DERIVATION OF BELL-TYPE INEQUALITY

We now derive a linear inequality (mentioned in Eq. (7) of Ref. [26] for n qubits) involving the joint probabilities in Eq. (1), starting from local realistic theory. For this, we first assume that all the experimental probabilities $P(A_k = i_k)$, $P(A_k = i_k, A_l = i_l)$, $P(A_k = i_k, A_l = i_l, A_m = i_m)$ (with $A_k \in \{U_k, D_k\}$, and $i_k \in \{+1,-1\}$ for $k,l,m = 1,2,3$, and $k \neq l \neq m$) can be described by a local hidden variable ω, defined on the probability space Ω with probability density $\rho(\omega)$. For local realistic theory, the probabilities would satisfy the following conditions:

(i) $P_\omega(A_k = i_k)$ (for $i_k \in \{+1,-1\}$ with $k = 1,2,3$) can only take values 1 or 0.

(ii) $P_\omega(A_k = i_k, A_l = i_l) = P_\omega(A_k = i_k)P_\omega(A_l = i_l)$, $P_\omega(A_k = i_k, A_l = i_l, A_m = i_m) = P_\omega(A_k = i_k)P_\omega(A_l = i_l)P_\omega(A_m = i_m)$.

Condition (i) is equivalent to assigning definite values to the observables. In any LHVT, the experimental probabilities would be reproduced in the following way:

$P(A_k = i_k) = \int_\Omega \rho(\omega)d\omega P_\omega(A_k = i_k)$, $P(A_k = i_k, A_l = i_l) = \int_\Omega \rho(\omega)d\omega P_\omega(A_k = i_k, A_l = i_l)$, $P(A_k = i_k, A_l = i_l, A_m = i_m) = \int_\Omega \rho(\omega)d\omega P_\omega(A_k = i_k, A_l = i_l, A_m = i_m)$, where $\int_\Omega \rho(\omega)d\omega = 1$.

TABLE II. Observables for genuine tripartite pure entanglement.

Case	Set of observables for different cases
(D.1), (D.2), (D.4), (D.5)	$\alpha_1 = \lambda_1, \beta_1 = -\lambda_0 e^{i\phi}, \gamma_1 = 0, \delta_1 = 1; \alpha_2 = 1, \beta_2 = 0, \gamma_2 = \lambda_2\lambda_3 e^{i\phi} - \lambda_1\lambda_4, \delta_2 = \lambda_1\lambda_2; \alpha_3 = \lambda_2 e^{i\phi}$, $\beta_3 = -\lambda_1, \gamma_3 = 1, \delta_3 = 0$
(D.3)	$\alpha_1 = 0, \beta_1 = 1, \gamma_1 = \lambda_0\lambda_1, \delta_1 = (1 - \lambda_o^2); \alpha_2 = \lambda_1\tau - \lambda_3\epsilon, \beta_2 = \lambda_3\tau + \lambda_1\epsilon, \gamma_2 = \lambda_3, \delta_2 = -\lambda_1$; $\alpha_3 = \lambda_1 + \lambda_2, \beta_3 = \lambda_2 - \lambda_1, \gamma_3 = \lambda_2, \delta_3 = -\lambda_1$, where, $\tau = \lambda_0^2\lambda_3(\lambda_1 + \lambda_2), \epsilon = \lambda_0^2\lambda_1(\lambda_1 + \lambda_2) + (1 - \lambda_0^2)$
(D.6)	$\alpha_1 = 0, \beta_1 = 1, \gamma_1 = \lambda_1 e^{-i\phi}(\lambda_4^2 - \lambda_0^2), \delta_1 = -\lambda_0(1 - \lambda_0^2); \alpha_2 = \lambda_3(1 - \lambda_0^2), \beta_2 = -\lambda_1 e^{-i\phi}(1 - \lambda_4^2), \gamma_2 = \lambda_3$, $\delta_2 = -\lambda_1 e^{-i\phi}; \alpha_3 = 1, \beta_3 = 0, \gamma_3 = \lambda_4(1 - \lambda_4^2), \delta_3 = \lambda_3(\lambda_4^2 - \lambda_0^2)$
(D.7)	$\alpha_1 = \lambda_1 e^{-i\phi}, \beta_1 = -\lambda_0, \gamma_1 = 0, \delta_1 = 1; \alpha_2 = \lambda_3, \beta_2 = -\lambda_1 e^{-i\phi}, \gamma_2 = 1, \delta_2 = 0; \alpha_3 = 1, \beta_3 = 0$, $\gamma_3 = \lambda_0, \delta_3 = -\lambda_3$
(D.8)	$\alpha_1 = 0, \beta_1 = 1, \gamma_1 = \lambda_1 e^{-i\phi}(\epsilon + \lambda_4), \delta_1 = -\lambda_o\epsilon; \alpha_2 = 1, \beta_2 = 1, \gamma_2 = \lambda_4, \delta_2 = -\epsilon; \alpha_3 = \epsilon$, $\beta_3 = \lambda_1 e^{-i\phi}, \gamma_3 = \lambda_4, \delta_3 = -\lambda_1 e^{-i\phi}$; with ϵ being a solution of $z^2(1 - \lambda_4^2) + z\lambda_4(1 - \lambda_0^2) + \lambda_4^4 = 0$
(D.9), (D.10)	$\alpha_1 = \lambda_2(\lambda_2^2 + \lambda_4^2) + \lambda_4(1 - \lambda_0^2), \beta_1 = -\lambda_0\lambda_3\lambda_4, \gamma_1 = 1, \delta_1 = 0; \alpha_2 = 1, \beta_2 = 1, \gamma_2 = \lambda_4, \delta_2 = -\lambda_2$; $\alpha_3 = 0, \beta_3 = 1, \gamma_3 = \lambda_3\lambda_4, \delta_3 = \lambda_2^2 + \lambda_4^2$
(D.11)	$\alpha_1 = 0, \beta_1 = 1, \gamma_1 = \lambda_2^2\lambda_3, \delta_1 = \lambda_0(\lambda_2^2 + \lambda_3^2); \alpha_2 = \lambda_2^2 + \lambda_3^2, \beta_2 = -\lambda_2^2, \gamma_2 = 1, \delta_2 = 0; \alpha_3 = 1$, $\beta_3 = 0, \gamma_3 = \lambda_3, \delta_3 = \lambda_2$
(D.12)	$\alpha_1 = 0, \beta_1 = 1, \gamma_1 = \delta\lambda_0\lambda_2\lambda_3, \delta_1 = \lambda_2^3\delta + \lambda_3^3; \alpha_2 = 1, \beta_2 = 1, \gamma_2 = \lambda_3, \delta_2 = -\lambda_2\delta; \alpha_3 = 1$, $\beta_3 = \delta, \gamma_3 = \lambda_2, \delta_3 = -\lambda_3$; with δ being a solution of $z^2\lambda_2^4 + z\lambda_2\lambda_3 + \lambda_3^4 = 0$
(D.13)	$\alpha_1 = 1, \beta_1 = 1, \gamma_1 = \lambda_2, \delta_1 = -\lambda_0\epsilon; \alpha_2 = 1, \beta_2 = 0, \gamma_2 = \lambda_4, \delta_2 = -(\lambda_0\epsilon + \lambda_2); \alpha_3 = \epsilon, \beta_3 = 1$, $\gamma_3 = \lambda_2, \delta_3 = -\lambda_0$; with ϵ being a solution of $z^2\lambda_0^4 + z\lambda_0\lambda_2(\lambda_0^2 + \lambda_2^2) + \lambda_2^2(\lambda_2^2 + \lambda_4^2) = 0$
(D.14)	$\alpha_1 = 1, \beta_1 = 1, \gamma_1 = i\lambda_0, \delta_1 = -\lambda_4; \alpha_2 = 1, \beta_2 = 1, \gamma_2 = i\lambda_0, \delta_2 = -\lambda_4 ; \alpha_3 = \lambda_4^2, \beta_3 = i\lambda_0^2$, $\gamma_3 = \lambda_4, \delta_3 = -\lambda_0$

Now consider the following quantity

$$\begin{aligned} B(\omega) = {} & P_\omega(D_1 = -1)P_\omega(D_2 = -1)P_\omega(D_3 = -1) \\ & + P_\omega(D_1 = +1)P_\omega(U_2 = +1)P_\omega(U_3 = +1) \\ & + P_\omega(U_1 = +1)P_\omega(D_2 = +1)P_\omega(U_3 = +1) \\ & + P_\omega(U_1 = +1)P_\omega(U_2 = +1)P_\omega(D_3 = +1) \\ & - P_\omega(U_1 = +1)P_\omega(U_2 = +1)P_\omega(U_3 = +1). \end{aligned}$$

One can easily check that $B(\omega) \geqslant 0$ for all $\omega \in \Omega$. Then obviously

$$\int_\Omega \rho(\omega) d\omega B(\omega) \geqslant 0,$$

which, in turn, gives rise to the following Bell-type inequality:

$$\begin{aligned} & P(D_1 = -1, D_2 = -1, D_3 = -1) \\ & \quad + P(D_1 = +1, U_2 = +1, U_3 = +1) \\ & \quad + P(U_1 = +1, D_2 = +1, U_3 = +1) \\ & \quad + P(U_1 = +1, U_2 = +1, D_3 = +1) \\ & \quad - P(U_1 = +1, U_2 = +1, U_3 = +1) \geqslant 0. \end{aligned} \qquad (3)$$

Thus we see that every LHVT satisfies the inequality (3).

VI. VIOLATION OF THE INEQUALITY

From our above-mentioned discussion on Hardy-type NLWI, it follows that *every* three-qubit pure state will violate the inequality (3) unless it is a fully product state or it has pure bipartite maximal entanglement. We now show that this inequality is even violated when $|\psi\rangle$ has pure bipartite maximal entanglement, although, in this case, $|\psi\rangle$ does not satisfy the Hardy-type NLWI condition (1). Without loss of generality, we can take $|\psi\rangle$ in this case as: $|\psi\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle) \otimes |0\rangle$. Choose $|\hat{U}_1 = +1\rangle = (\sqrt{0.96}|0\rangle + 0.2|1\rangle)$, $|\hat{D}_1 = +1\rangle = |0\rangle$, $|\hat{U}_2 = +1\rangle = (0.2|0\rangle + \sqrt{0.96}|1\rangle)$, $|\hat{D}_2 = +1\rangle = |1\rangle$, $|\hat{U}_3 = +1\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ $|\hat{D}_3 = +1\rangle = |1\rangle$. With this choice, $|\psi\rangle$ will violate the inequality (3).

VII. DISCUSSION

We have established the result that *every pure entangled state of three qubits violates the Bell-type inequality* (3).

If a three-qubit state $|\psi\rangle$ violates the inequality (3) maximally corresponding to the set $\mathcal{S}(\psi)$ of three pairs of noncommuting observables $(\hat{U}_1, \hat{D}_1)$, $(\hat{U}_2, \hat{D}_2)$, and $(\hat{U}_3, \hat{D}_3)$, then the minimum value of the coefficient $v \in [0,1]$ for which the state $\rho(\psi, v) \equiv v|\psi\rangle\langle\psi| + [(1 - v)/8]I$ (I being the 8×8 identity matrix) also violates the inequality (3), is called the "threshold visibility" of the state $|\psi\rangle$. The lower the amount of threshold visibility, the higher the amount of noise the inequality can sustain. The maximum negative violation of the inequality (3) by the GHZ state is numerically found to be -0.175459 (approx.), and so the threshold visibility $v_{\rm GHZ}^{\rm thr}$ of this state turns out to be 0.68125 (approx.), which is approximately same as that found in Ref. [12]. On the other hand, the maximum negative violation of the inequality (3) by the W state $(1/\sqrt{3})(|001\rangle + |010\rangle + |100\rangle)$ is numerically found to be -0.192608 (approx.), and so the threshold visibility $v_W^{\rm thr}$ of this state turns out to be 0.6606676 (approx.), which is also approximately same with the value 0.660668 of $v_W^{\rm thr}$, found in Ref. [12]. It is to be noted that so far as the values of $v_{\rm GHZ}^{\rm thr}$, $v_W^{\rm thr}$ are concerned, although the probabilistic Bell-type inequality (18) of Ref. [12] and the above-mentioned inequality (3) provide approximately the same values, unlike inequality (3), neither inequality (18) of Ref. [12] nor any other inequality mentioned in the literature so far (see, for example, Refs. [9,11,12]) is analytically guaranteed to be violated by all pure entangled states of three qubits. By considering a modified form of the inequality (3) (e.g., inequality (11) of Ref. [21]), one may get a lower value of the threshold visibility for the states.

One may also try to find similar feature (i.e., violation of Bell-type inequality, derived from Hardy-type NLWI argument, by all pure entangled states) in the case of n-partite quantum systems.

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