PHYSICAL REVIEW A **81**, 033819 (2010)

Effects of photon losses on phase estimation near the Heisenberg limit using coherent light and squeezed vacuum

Takafumi Ono[1,*] and Holger F. Hofmann[1,2,†]

[1]*Graduate School of Advanced Sciences of Matter, Hiroshima University, Kagamiyama 1-3-1, Higashi Hiroshima 739-8530, Japan*

[2]*JST, CREST, Sanbancho 5, Chiyoda-ku, Tokyo 102-0075, Japan*

(Received 19 October 2009; published 11 March 2010)

Two-path interferometry with coherent states and a squeezed vacuum can achieve phase sensitivities close to the Heisenberg limit when the average photon number of the squeezed vacuum is close to the average photon number of the coherent light. Here, we investigate the phase sensitivity of such states in the presence of photon losses. It is shown that the Cramer-Rao bound of phase sensitivity can be achieved experimentally by using a weak local oscillator and photon counting in the output. The phase sensitivity is then given by the Fisher information F of the state. In the limit of high squeezing, the ratio $(F-N)/N^2$ of Fisher information above shot noise to the square of the average photon number N depends only on the average number of photons lost, n_{loss}, and the fraction of squeezed vacuum photons μ. For $\mu = 1/2$, the effect of losses is given by $(F-N)/N^2 = 1/(1+2n_{\text{loss}})$. The possibility of increasing the robustness against losses by lowering the squeezing fraction μ is considered, and an optimized result is derived. However, the improvements are rather small, with a maximal improvement by a factor of 2 at high losses.

DOI: 10.1103/PhysRevA.81.033819 PACS number(s): 42.50.St, 03.65.Ta, 06.20.Dk, 42.50.Dv

I. INTRODUCTION

Quantum states of light can improve the sensitivity of phase measurements beyond the limits that apply to classical light sources. The phase sensitivity of coherent light (and hence of all classical light) is limited by the shot noise of independent photon detection events to the standard quantum limit of $\delta\phi^2 = 1/N$. This limit can be overcome by using the multiphoton coherences of nonclassical light [1–13]. For two-mode N-photon systems, the highest possible phase sensitivity is achieved by maximally path-entangled states, which are superposition states $(|N;0\rangle + |0;N\rangle)/\sqrt{2}$ where all photons are either in one path or in the other path of a two-path interferometer [14–22]. The phase sensitivity of these states defines the Heisenberg limit of $\delta\phi^2 = 1/N^2$. Since no N-photon states achieve a higher phase sensitivity, this is the absolute limit of phase estimation for a fixed photon number N [11].

Unfortunately, it is rather difficult to generate maximally path-entangled states using the available sources of nonclassical light [23–25]. It was therefore a significant discovery that the interference of a coherent state and a squeezed vacuum produces a high fraction of maximal path-entangled states when the average photon number from the squeezed vacuum is about equal to the average photon number of the coherent light [26,27]. In particular, Pezze and Smerzi showed that conventional two-path interferometry can achieve phase sensitivities close to the Heisenberg limit even in the presence of fluctuating total photon number [27]. These results seem to put Heisenberg-limited phase estimation within the reach of well-established quantum technologies. However, maximally path-entangled states are very sensitive to photon losses, since the loss of a single photon can completely randomize the multiphoton coherence between the paths [28–30]. We can therefore expect that the Heisenberg-limited phase sensitivity achieved by coherent light and squeezed vacuum will rapidly decline as photon losses increase.

In the following, we investigate the effect of photon losses on the phase sensitivity of the two-mode states generated by interference of coherent light and squeezed vacuum in detail. Assuming equal losses in both optical modes, we derive the mixed state after losses and find an optimal phase estimator based on the general analysis for quantum metrology using mixed states [31]. We find that the Cramer-Rao bound giving the maximal phase sensitivity of the state can be achieved by a simple experimental setup using a weak local oscillator field and photon detection. It is therefore possible to obtain a phase sensitivity equal to the Fisher information F in an experimentally feasible setup using only linear optics and photon detection. In the limit of high squeezing, the ratio $(F-N)/N^2$ of Fisher information above shot noise to the square of the average photon number N depends only on the fraction of squeezed vacuum photons μ and the average number of photons lost n_{loss}. Thus the effect of photon losses on phase sensitivities close to the Heisenberg limit has the same dependence on the average number of photons lost, regardless of the total average photon number N. In particular, photon losses reduce the phase sensitivity for Heisenberg-limited estimation at $\mu = 1/2$ by a factor of $1/(1+2n_{\text{loss}})$. Finally, we investigate the possibility of improving the Fisher information by optimizing the squeezing fraction μ for a given number of photons lost. However, the result shows only small improvements, approaching a maximal increase of Fisher information by a factor of 2 in the limit of high photon losses.

II. EFFECTS OF LOSSES ON THE TWO-MODE SQUEEZED-COHERENT STATE

Figure 1 shows a possible experimental setup realizing Heisenberg-limited phase estimation with coherent light and

*d075972@hiroshima-u.ac.jp

†hofmann@hiroshima-u.ac.jp

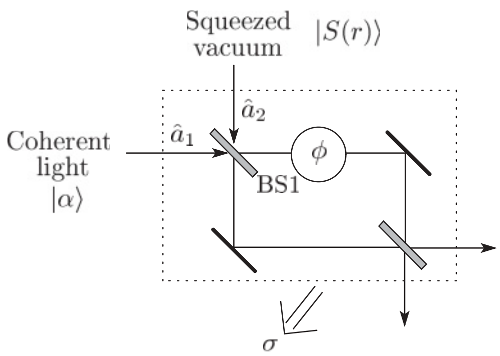

FIG. 1. Illustration of Heisenberg-limited estimation of a small phase shift ϕ with coherent light and squeezed vacuum in the presence of linear losses. The probability that any given photon is lost between generation and detection is given by the loss fraction σ.

squeezed vacuum in the input ports of a two-path interferometer. Initially, mode $\hat{a}_1$ is in a coherent state $|\alpha\rangle$ and mode $\hat{a}_2$ is in a squeezed-vacuum state $|S(r)\rangle = \hat{S}(r)|\text{vac.}\rangle$, where $\hat{S}(r) = \exp[\frac{1}{2}(r\hat{a}_2\hat{a}_2 - r^*\hat{a}_2^\dagger\hat{a}_2^\dagger)]$ is the squeezing operator. Interference at the initial beam splitter then results in multiphoton coherences between the two paths inside the interferometer, as discussed in [26,27]. However, photon losses occurring at any point between the generation and the detection of the light fields will reduce these multiphoton coherences. In the following, we assume linear losses with equal loss rates in the two modes. It is then possible to represent the losses by the loss fraction σ, defined as the probability that any given photon is lost between generation and detection.

The effects of linear losses on the two input modes correspond to interference with a vacuum state, followed by a trace over the modes representing the losses. Since photon losses from orthogonal modes are statistically independent, it is possible to consider the effect of photon losses on the two input modes separately. For the coherent state, the losses simply reduce the amplitude α by a factor of $\sqrt{1-\sigma}$, so that the output amplitude is $\alpha_{\text{red}} = \sqrt{1-\sigma}\alpha$, and the density matrix $\hat{\rho}_1$ of mode $\hat{a}_1$ after losses is

$$\hat{\rho}_1 = |\alpha_{\text{red}}\rangle\langle\alpha_{\text{red}}|. \tag{1}$$

In the case of the squeezed vacuum, losses change the variances of the quadrature components $\hat{x}_2$ and $\hat{y}_2$ of the field mode $\hat{a}_2 = \hat{x}_2 + i\hat{y}_2$. The output is a Gaussian state with quadrature variances of

$$\begin{aligned} 4\Delta x_2^2 &= \sigma + (1-\sigma)e^{-2r}, \\ 4\Delta y_2^2 &= \sigma + (1-\sigma)e^{2r}. \end{aligned} \tag{2}$$

In general, a Gaussian mixed state defined by the variances Δx_2^2 and Δy_2^2 can be described by a squeezed thermal state,

$$\hat{\rho}_2 = \hat{S}(r_{\text{red}})\hat{\rho}_{\text{th}}(\lambda)\hat{S}^\dagger(r_{\text{red}}), \tag{3}$$

where the thermal state is given by

$$\hat{\rho}_{\text{th}}(\lambda) = (1-\lambda)\sum_n^\infty \lambda^n|n\rangle\langle n|, \tag{4}$$

and $r_{\text{red}} < r$ is the reduced squeezing parameter obtained from the ratio of the variances after losses. In terms of the parameters λ and r_{red}, the variances of $\hat{\rho}_2$ are

$$\begin{aligned} 4\Delta x_2^2 &= \frac{1+\lambda}{1-\lambda}e^{-2r_{\text{red}}}, \\ 4\Delta y_2^2 &= \frac{1+\lambda}{1-\lambda}e^{2r_{\text{red}}}. \end{aligned} \tag{5}$$

Thus it is possible to determine the values of λ and r_{red} corresponding to an initial squeezing parameter r and a loss probability σ from Eqs. (3) and (6).

The complete two-mode state after losses is given by the product of the states in mode $\hat{a}_1$ and in mode $\hat{a}_2$,

$$\hat{\rho} = \hat{\rho}_1 \otimes \hat{\rho}_2. \tag{6}$$

The phase sensitivity achieved by this state can be analyzed using the general formalism for mixed states [31]. It is then possible to determine both the quantum Cramer-Rao bound of phase estimation and the measurement procedure that achieves this bound in the presence of photon losses.

III. PHASE ESTIMATION WITH MIXED STATES

Quantum phase estimation is performed by measuring a phase estimator $\hat{A}$. If the average value of $\hat{A}$ is chosen to be zero at $\phi = 0$, the phase derivative of the average gives the ratio of the average of $\hat{A}$ and the small phase shift that quantum estimation seeks to detect. Thus the average phase estimate $\langle\phi_{\text{est}}\rangle = \langle\hat{A}\rangle/(\partial\langle\hat{A}\rangle/\partial\phi)$ converges on the correct value ϕ as the number of measurements increases. However, each individual measurement has a statistical error of $\delta\phi^2$ that determines how quickly the average of the measurement results converges on the correct value of ϕ. In terms of this measurement error, the phase sensitivity obtained with a specific estimator $\hat{A}$ is given by

$$\frac{1}{\delta\phi^2} = \frac{|\partial(\text{Tr}\{\hat{A}\hat{\rho}\})/\partial\phi|^2}{\text{Tr}\{\hat{A}^2\hat{\rho}\}}. \tag{7}$$

An optimal phase estimator $\hat{A}$ maximizes this phase sensitivity and achieves the quantum Cramer-Rao bound of the state $\hat{\rho}$. As was shown in [31], the optimal estimator is given by the symmetric logarithmic derivative $\hat{G}$ of the density matrix $\hat{\rho}$, as defined by the operator relation

$$\frac{\partial}{\partial\phi}\hat{\rho} = \frac{1}{2}(\hat{\rho}\hat{G} + \hat{G}\hat{\rho}). \tag{8}$$

Note that this relation does not uniquely define $\hat{G}$ if $\hat{\rho}$ has eigenvalues of zero. In that case, any operator $\hat{G}$ fulfilling Eq. (8) is an optimal estimator. The maximal phase sensitivity achieved by an optimal estimator $\hat{G}$ is equal to the Fisher information $F = 1/\delta\phi_{\text{opt}}^2$ of the quantum state $\hat{\rho}$. The Fisher information can be evaluated from Eq. (7) by using $\hat{A} = \hat{G}$ and Eq. (8). The result is equal to the variance of the optimal phase estimator $\hat{G}$,

$$F = \text{Tr}\{\hat{G}^2\hat{\rho}\}. \tag{9}$$

Equations (8) and (9) summarize the results for quantum metrology with mixed states obtained in [31] without the explicit expansion into eigenstates of the density matrix used in the original derivation. In general, these results apply to any parameter ϕ that changes the quantum state $\hat{\rho}$. In the specific case of a phase shift, ϕ is the parameter of a unitary transformation $\exp[-i\phi\hat{h}]$ generated by an operator $\hat{h}$. The phase derivative of the density matrix is therefore given by the commutation relation of $\hat{\rho}$ and $\hat{h}$,

$$\frac{\partial}{\partial\phi}\hat{\rho} = -i(\hat{h}\hat{\rho} - \hat{\rho}\hat{h}). \tag{10}$$

To find an optimal estimator $\hat{G}$ for a given generator $\hat{h}$ and a given quantum state $\hat{\rho}$, we have to solve Eq. (8) using the phase derivative given by Eq. (10). This relation can be summarized by

$$\tfrac{1}{2}(\hat{G}\hat{\rho} + \hat{\rho}\hat{G}) = -i(\hat{h}\hat{\rho} - \hat{\rho}\hat{h}). \tag{11}$$

For unitary transforms, an optimal estimator $\hat{G}$ is therefore obtained when the anticommutation of $\hat{G}$ and $\hat{\rho}$ has the same form as the commutation of $\hat{h}$ and $\hat{\rho}$.

IV. DERIVATION OF AN OPTIMAL PHASE ESTIMATOR

We can now derive an optimal estimator for phase estimation with coherent light and squeezed vacuum in the presence of losses. The density matrix was derived in Sec. II, and the generator $\hat{h}$ for a two-path interferometer is given by half the photon number difference between the two paths. In the present context, the two-mode density matrix is a product of the two-input-mode density matrices. It is therefore convenient to express the generator $\hat{h}$ in terms of the input modes $\hat{a}_1$ and $\hat{a}_2$, which are equal superpositions of the modes describing the two paths inside the interferometer. The photon number difference between the two paths is then equal to an interference term of the input modes $\hat{a}_1$ and $\hat{a}_2$. Specifically, it can be written as

$$\hat{h} = -i\tfrac{1}{2}(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1). \tag{12}$$

With this generator, Eq. (11) provides a relation between the optimal estimator $\hat{G}$ and the two-mode state $\hat{\rho}$ in terms of the creation and annihilation operators of the input states,

$$\hat{G}\hat{\rho} + \hat{\rho}\hat{G} = \hat{\rho}(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1) - (\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1)\hat{\rho}. \tag{13}$$

Since the density matrix can be written as a product of states in mode $\hat{a}_1$ and mode $\hat{a}_2$, the effects of the operators $\hat{a}_1$ and $\hat{a}_2$ on the states $\hat{\rho}_1$ and $\hat{\rho}_2$ can be determined separately. It is then possible to derive a particularly simple form of $\hat{G}$ by only considering the relations between single-mode Gaussian states and the creation and annihilation operators of their respective modes.

First, we consider the effects of the annihilation and creation operators of mode $\hat{a}_1$ on the coherent state density matrix $\hat{\rho}_1$. The coherent state $|\alpha_{\text{red}}\rangle$ is a right eigenstate of the annihilation operator $\hat{a}_1$. It is therefore possible to replace the operator $\hat{a}_1$ operating from the left on $\hat{\rho}$ and the operator $\hat{a}_1^\dagger$ operating from the right with the complex number α_{red}. The application of $\hat{a}_1^\dagger$ to the coherent state $|\alpha_{\text{red}}\rangle$ changes that state into a superposition of the original state with an amplitude of α_{red} and an orthogonal state that can be represented by a displaced one-photon state. The creation operator can thus be written as a sum of the coherent amplitude α_{red} and an operator that changes the initial state into an orthogonal state,

$$\hat{a}_1^\dagger = \alpha_{\text{red}} + (\hat{a}_1^\dagger - \alpha_{\text{red}}). \tag{14}$$

It is possible to separate the relation for the estimator $\hat{G}$ given by Eq. (13) into two parts, one that leaves the coherent state unchanged, and one that describes the transition matrix elements between the coherent state and the displaced one-photon state. The separation is achieved by writing the optimal estimator as $\hat{G} = \hat{g}_1 + \hat{g}_2$, where $\hat{g}_1$ is the component of the estimator associated with the transition matrix elements in mode $\hat{a}_1$, and $\hat{g}_2$ is the component of the estimator that commutes with the coherent state $\hat{\rho}_1$. The two relations defining $\hat{g}_1$ and $\hat{g}_2$ then read

$$\hat{g}_1\hat{\rho} + \hat{\rho}\hat{g}_1 = -[(\hat{a}_1^\dagger - \alpha_{\text{red}})\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger(\hat{a}_1 - \alpha_{\text{red}})], \tag{15}$$

$$\hat{g}_2\hat{\rho} + \hat{\rho}\hat{g}_2 = -\alpha_{\text{red}}[(\hat{a}_2 - \hat{a}_2^\dagger)\hat{\rho} - \hat{\rho}(\hat{a}_2 - \hat{a}_2^\dagger)]. \tag{16}$$

To find $\hat{g}_1$, we make use of the fact that $(\hat{a}_1 - \alpha_{\text{red}})\hat{\rho} = 0$. It is therefore possible to add or subtract multiples of this operator and its self-adjoint operator to the right side of Eq. (15) without changing the relation. The solution for the self-adjoint operator $\hat{g}_1$ obtained in this manner is

$$\hat{g}_1 = -[(\hat{a}_1^\dagger - \alpha_{\text{red}})\hat{a}_2 + \hat{a}_2^\dagger(\hat{a}_1 - \alpha_{\text{red}})]. \tag{17}$$

To find $\hat{g}_2$, we make use of the fact that Eq. (16) only includes operators acting on the state in mode $\hat{a}_2$. Therefore, we only need to consider the density matrix $\hat{\rho}_2$ of the squeezed thermal state. Since $(\hat{a}_2 - \hat{a}_2^\dagger) = 2i\hat{y}_2$ corresponds to the antisqueezed quadrature component, the right side of Eq. (16) corresponds to the commutation relation between the quadrature $\hat{y}_2$ and the density matrix $\hat{\rho}_2$. For Gaussian states, this kind of commutation relation can be converted into an anticommutation relation for a different quadrature component. In the case of the squeezed thermal states given by $\hat{\rho}_2$, the conversion is given by

$$-i\left(\hat{y}_2\hat{\rho}_2 - \hat{\rho}_2\hat{y}_2\right) = \frac{1-\lambda}{1+\lambda}e^{2r_{\text{red}}}\left(\hat{x}_2\hat{\rho}_2 + \hat{\rho}_2\hat{x}_2\right). \tag{18}$$

Comparison with Eq. (16) indicates that $\hat{g}_2$ is a multiple of the quadrature component $\hat{x}_2$. Specifically, Eq. (16) can be solved by

$$\hat{g}_2 = \frac{\alpha_{\text{red}}}{2\Delta x_2^2}\,\hat{x}_2, \tag{19}$$

where the coefficients λ and r_{red} have been expressed in terms of the quadrature variance Δx_2^2 using Eq. (6).

The optimal estimator $\hat{G}$ is given by the sum of $\hat{g}_1$ and $\hat{g}_2$. Since $\hat{g}_1$ is a quadratic function of the creation and annihilation operators, and $\hat{g}_2$ is a linear function of the operators of mode $\hat{a}_2$, it is possible to express $\hat{G}$ as a quadratic function of the field operators. Specifically, the result can be written as an interference term of $\hat{a}_2$ and a field $\hat{b}$,

$$\hat{G} = \hat{b}^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{b}, \tag{20}$$

where the field $\hat{b}$ is given by

$$\hat{b} = \alpha_{\text{red}}\left(\frac{1}{4\Delta x_2^2} + 1\right) - \hat{a}_1. \tag{21}$$

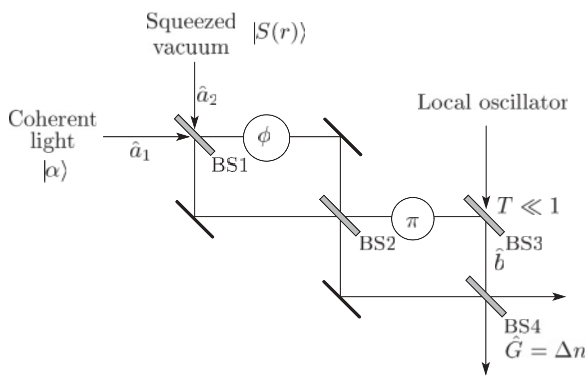

FIG. 2. Schematic setup for the observation of optimal phase sensitivity obtained by coherent light and squeezed vacuum in the presence of losses. The optimal phase estimation is realized by subtracting a coherent amplitude of $\alpha_{\rm red}(\frac{1}{4}\Delta x_2^2+1)$ from the output mode $\hat{a}_1$ using a local oscillator and a beam splitter with low transmittivity T (BS3).

Experimentally, this estimator can be realized by subtracting a coherent amplitude of $\alpha_{\rm red}(\frac{1}{4}\Delta x_2^2+1)$ from the output field in $\hat{a}_1$ using interference with a local oscillator. A possible setup is shown in Fig. 2. At zero phase shift, the output modes correspond to the input modes. A small displacement of the coherent field mode $\hat{a}_1$ is realized by interference with a local oscillator field at a beam splitter of very low transmittivity T (BS3). Finally, the displaced mode $\hat{b}$ and the output mode $\hat{a}_2$ interfere at a fourth beam splitter (BS4). The phase estimator is then equal to the photon number difference in the output.

Interestingly, the estimator is a linear function of the detected photon numbers. This is quite different from the optimal phase estimation for pure states considered in the initial work on Heisenberg-limited phase estimation with coherent and squeezed light, where higher order moments of the detected output photon number distribution were essential [27]. In the context of general approaches to Heisenberg-limited phase estimation, it has also been pointed out that sensitivities near the Heisenberg limit can be achieved when standard homodyne detection is used for two-mode squeezed vacuum inputs [32]. However, the estimator used in that proposal is a quadratic function of the output intensity differences. Although the optimized measurement obtained here could be seen as a hybrid of photon detection and homodyne detection, it seems remarkable that it achieves the Heisenberg limit through a linear relation between output photon number difference and phase estimate, in contrast to the nonlinear estimators of both [27] and [32]. The present setup therefore represents a major simplification of the phase estimation procedure for phase sensitivities close to the Heisenberg limit, even in the pure state case where the phase sensitivity is equal to that obtained from direct photon counting in the output.

V. DEPENDENCE OF PHASE SENSITIVITY ON PHOTON LOSSES

As mentioned in Sec. III, the Fisher information of the quantum state $\hat{\rho}$ is equal to the expectation value of the squared estimator $\hat{G}$. Using the result of Eq. (20), the Fisher information of the squeezed-coherent state $\hat{\rho}$ is found to be

$$\mathrm{Tr}\{\hat{G}^2\hat{\rho}\} = \frac{\alpha_{\rm red}^2}{4\Delta x_2^2} + n_2, \tag{22}$$

where n_2 is the average number of photons in the squeezed mode $\hat{a}_2$ after losses. Here, the effects of losses are expressed indirectly through the values of n_2, $\alpha_{\rm red}$, and Δx_2. The specific effects of a loss probability of σ on the input state are given by $\alpha_{\rm red} = \sqrt{1-\sigma}\alpha$, $n_2 = (1-\sigma)\sinh^2 r$, and Eq. (3). The Fisher information can then be expressed in terms of the input amplitude α, the input squeezing r, and the loss probability σ. The result reads

$$F = (1-\sigma)\left(\frac{\alpha^2}{\sigma+(1-\sigma)e^{-2r}} + \sinh^2 r\right). \tag{23}$$

In this representation of the Fisher information, the most significant effect of the losses is the limitation of squeezing effects represented by e^{-2r}. However, it is difficult to see how this limitation relates to the maximal phase sensitivities achieved at equal intensities of coherent light and squeezed vacuum. It is therefore convenient to express the result in terms of average photon numbers instead.

The total average photon number after losses is given by $N = (1-\sigma)(\alpha^2 + \sinh^2 r)$. To evaluate the distribution of photons between the coherent light and the squeezed vacuum, we introduce the squeezing fraction $\mu = (1-\sigma)\sinh^2 r/N$, defined as the fraction of photons in the squeezed mode $\hat{a}_2$. Finally, the effects of losses can be given in terms of the average number of photons lost, $n_{\rm loss} = N\sigma/(1-\sigma)$. Since N is the average photon number after losses, the total photon number of the input state is given by the sum of N and $n_{\rm loss}$, as shown in Fig. 3. The Fisher information is then given by

$$F = N^2\frac{4(1-\mu)\mu}{1-e^{-2r}+4\mu n_{\rm loss}} + N. \tag{24}$$

Note that this phase sensitivity can be greater than N^2, since the actual Heisenberg limit for fluctuating photon numbers is given by the average of the squared photon number, not the square of the average photon number [33].

Since we are mainly interested in Heisenberg-limited phase sensitivities with large photon numbers, it is reasonable to assume that the squeezing levels will be high enough to satisfy $e^{-2r} \ll 1$. We can then neglect the r-dependent term in Eq. (24) to obtain a particularly simple relation between phase sensitivity and photon losses. Specifically, the Fisher information above the standard quantum limit, $F - N$, is given by a fraction of N^2 determined only by the squeezing fraction μ and the average number of photons lost. Since this fraction does not depend on N, it provides a photon-number-independent expression of the effects of losses on the phase sensitivity of squeezed-coherent states. In the following, we

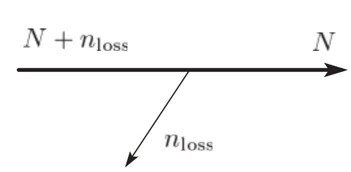

FIG. 3. Illustration of the definition of N and $n_{\rm loss}$. N is the average photon number after losses.

will refer to this expression as the enhancement of sensitivity,

$$\frac{F - N}{N^2} = \frac{4\mu(1-\mu)}{1 + 4\mu n_{\text{loss}}}. \tag{25}$$

As reported in [26,27], equal intensities of squeezed vacuum and coherent light ($\mu = 1/2$) result in maximal multiphoton coherences, including a significant fraction of maximally path-entangled states. In the absence of losses, the enhancement of sensitivity for these pure states is $(F - N)/N^2 = 1$, the maximal value that can be achieved in the limit of high squeezing. However, Eq. (25) also shows that photon losses rapidly reduce this enhancement of phase sensitivity. Specifically,

$$\left.\frac{F - N}{N^2}\right|_{\mu=\frac{1}{2}} = \frac{1}{1 + 2n_{\text{loss}}}. \tag{26}$$

This dependence of phase sensitivity on the average number of photons lost reflects the fact that the loss of a single photon completely randomizes the N-photon coherence of a maximally path-entangled state, irrespective of the total photon number N. Thus, the average loss of just half a photon already reduces the enhancement of sensitivity to half of its original value.

Equation (25) indicates that the effect of photon losses on the enhancement of sensitivity $(F - N)/N^2$ decreases when the squeezing fraction μ is lowered. Specifically, photon losses reduce the enhancement of sensitivity by a factor of $1 + 4\mu n_{\text{loss}}$, defined by the product of squeezing fraction and photon losses. Therefore, states with lower squeezing fraction μ are more robust against photon losses. Figure 4 shows a comparison of the loss-dependent enhancements of sensitivity $(F - N)/N^2$ for different squeezing fractions $\mu \leqslant 1/2$. At low losses, the enhancement of sensitivity is maximal for $\mu = 1/2$ and decreases with decreasing μ. As losses increase, the enhancement of sensitivity for $\mu = 1/2$ drops to values below the corresponding enhancements at lower μ, indicating that states with lower squeezing fraction can have higher Fisher information in the presence of losses. If the average number of photons lost is fixed, the highest enhancement of sensitivity

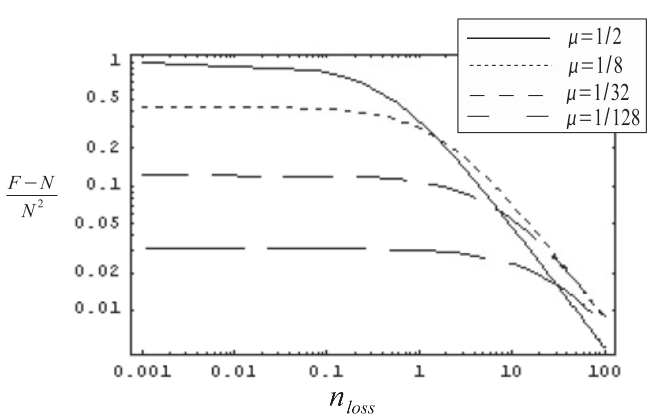

FIG. 4. Effects of photon losses n_{loss} on the enhancement of sensitivity $(F - N)/N^2$ for phase estimation at squeezing fractions of $\mu = 1/2$ and $\mu = 1/8, 1/32, 1/128$. Since photon losses reduce the enhancement of sensitivity by a factor of $1 + 4\mu n_{\text{loss}}$, states with lower squeezing fractions are more robust against photon losses than the states with maximal multiphoton coherence at $\mu = 1/2$.

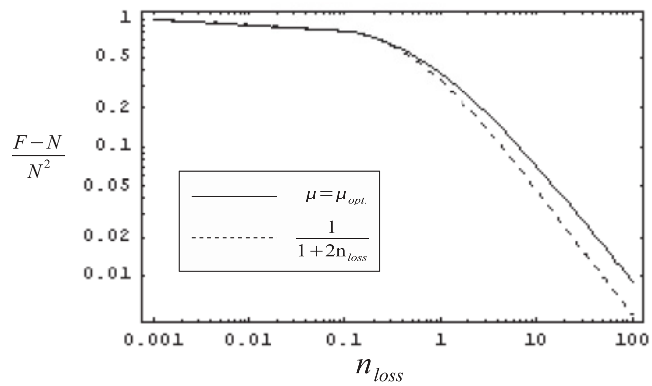

FIG. 5. Comparison of enhancement of sensitivity $(F - N)/N$ obtained at the optimized squeezing fraction $\mu = \mu_{\text{opt}}$ with the enhancement of sensitivity obtained at $\mu = 1/2$.

given by Eq. (25) is found at a squeezing fraction of

$$\mu_{\text{opt}} = \frac{1}{4n_{\text{loss}}}\left(\sqrt{1 + 4n_{\text{loss}}} - 1\right). \tag{27}$$

The enhancement of sensitivity at this optimal squeezing fraction μ_{opt} is given by

$$\left.\frac{F - N}{N^2}\right|_{\mu=\mu_{\text{opt}}} = \left(\frac{\sqrt{1 + 4n_{\text{loss}}} - 1}{2n_{\text{loss}}}\right)^2. \tag{28}$$

Figure 5 shows a comparison of the enhancement of sensitivity at μ_{opt} with the enhancement of sensitivity at $\mu = 1/2$. Although the reduction of squeezing fraction μ results in higher enhancements of phase sensitivity, the relative improvements seem to be rather small. Figure 6 shows the improvement factor given by the ratio of the enhancement of sensitivity at μ_{opt} and the enhancement of sensitivity at $\mu = 1/2$. The improvement factor is negligibly small at low losses, with a value of only 1.072 at average losses of half a photon. Thus, the optimization of the squeezing fraction can do little to compensate for the reduction of the enhancement of phase sensitivity to half its value at $n_{\text{loss}} = 1/2$. As losses increase, the improvement achieved by an optimization of the squeezing fraction becomes more significant. However, the

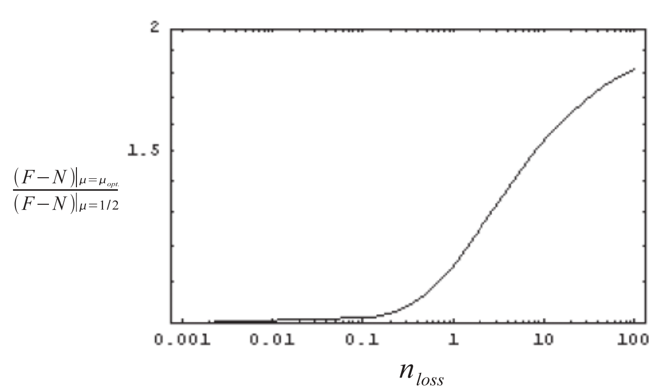

FIG. 6. Ratio of the sensitivity enhancement obtained at optimal squeezing fraction and the sensitivity enhancement obtained at $\mu = 1/2$. The ratio is close to unity at low losses and approaches a maximal value of 2 at high losses.

improvement is limited by its asymptotic value of

$$\lim_{n_{\text{loss}}\to\infty} \frac{(F-N)|_{\mu=\mu_{\text{opt}}}}{(F-N)|_{\mu=1/2}} = 2, \qquad (29)$$

so that the optimization of the squeezing fraction can at most double the enhancement of phase sensitivity achieved at a squeezing fraction of $\mu = 1/2$. The phase sensitivity achieved at $\mu = 1/2$ therefore remains close to the maximal phase sensitivity that can be achieved with any squeezed-coherent state, even in the presence of very high photon losses.

VI. CONCLUSIONS

We have shown that photon losses reduce the phase sensitivity of the multiphoton coherences obtained from interferences of equal intensities of squeezed vacuum and coherent light by a factor of $1 + 2n_{\text{loss}}$, where n_{loss} is the average number of photons lost. This result corresponds to the expectation that a single photon loss randomizes the coherence of maximally path-entangled states, regardless of the total photon number N. A small improvement of the robustness against losses can be achieved by reducing the fraction of squeezed vacuum in the total photon number. However, the improvements are rather small and indicate that the robustness against losses of N-photon states depends mainly on their phase sensitivity, regardless of the type of state used.

We have also shown that the Cramer-Rao bound of squeezed-coherent states in the presence of losses can be achieved in experimentally feasible measurements using a weak local oscillator field, linear optics, and photon counting. Interestingly, the use of the local oscillator simplifies the phase estimation procedure to an estimator linear in the detected photon numbers. It may therefore present an interesting alternative to direct photon counting in the output, even if the improvement in phase sensitivity is negligibly small.

In general, our results confirm that phase estimation near the Heisenberg limit can be performed with squeezed-coherent light, but only if the average number of photons lost can be kept low. Therefore, high efficiencies of photon transmission and detection will be essential for quantum metrology close to the Heisenberg limit.

ACKNOWLEDGMENTS

Part of this work was supported by the Grant-in-Aid program of the Japan Society for the Promotion of Science, JSPS.

[1] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
[2] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).
[3] M. Xiao, L.-A. Wu, and H. J. Kimble, Phys. Rev. Lett. **59**, 278 (1987).
[4] M. J. Holland and K. Burnett, Phys. Rev. Lett. **71**, 1355 (1993).
[5] B. C. Sanders and G. J. Milburn, Phys. Rev. Lett. **75**, 2944 (1995).
[6] M. G. A. Paris, Phys. Lett. **201**, 132 (1995).
[7] D. W. Berry and H. M. Wiseman, Phys. Rev. Lett. **85**, 5098 (2000).
[8] H. Wang and T. Kobayashi, Phys. Rev. A **71**, 021802(R) (2005).
[9] L. Pezze and A. Smerzi, Phys. Rev. A **73**, 011801(R) (2006).
[10] H. F. Hofmann, Phys. Rev. A **74**, 013808 (2006).
[11] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).
[12] T. Ono and H. F. Hofmann, J. Phys. B **41**, 095502 (2008).
[13] L. Pezze and A. Smerzi, Phys. Rev. Lett. **102**, 100401 (2009).
[14] H. Lee, P. Kok, N. J. Cerf, and J. P. Dowling, Phys. Rev. A **65**, 030101(R) (2002).
[15] O. Steuernagel, Phys. Rev. A **65**, 033820 (2002).
[16] K. Edamatsu, R. Shimizu, and T. Itoh, Phys. Rev. Lett. **89**, 213601 (2002).
[17] P. Kok, H. Lee, and J. P. Dowling, Phys. Rev. A **65**, 052104 (2002).
[18] J. Fiurasek, Phys. Rev. A **65**, 053818 (2002).
[19] G. J. Pryde and A. G. White, Phys. Rev. A **68**, 052315 (2003).
[20] P. Walther, J.-W. Pan, M. Aspelmeyer, R. Ursin, S. Gasparoni, and A. Zeilinger, Nature (London) **429**, 158 (2004).
[21] M. W. Mitchell, J. S. Lundeen, and A. M. Steinberg, Nature (London) **429**, 161 (2004).
[22] T. Nagata, R. Okamoto, J. L. O'Brien, K. Sasaki, and S. Takeuchi, Science **316**, 726 (2007).
[23] H. F. Hofmann, Phys. Rev. A **70**, 023812 (2004).
[24] H. Cable and J. P. Dowling, Phys. Rev. Lett. **99**, 163604 (2007).
[25] A. E. B. Nielsen and K. Molmer, Phys. Rev. A **75**, 063803 (2007).
[26] H. F. Hofmann and T. Ono, Phys. Rev. A **76**, 031806(R) (2007).
[27] L. Pezze and A. Smerzi, Phys. Rev. Lett. **100**, 073601 (2008).
[28] S. D. Huver, C. F. Wildfeuer, and J. P. Dowling, Phys. Rev. A **78**, 063828 (2008).
[29] U. Dorner, R. Demkowicz-Dobrzanski, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Phys. Rev. Lett. **102**, 040403 (2009).
[30] R. Demkowicz-Dobrzanski, U. Dorner, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Phys. Rev. A **80**, 013825 (2009).
[31] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[32] O. Steuernagel and S. Scheel, J. Opt. B **6**, S66 (2004).
[33] H. F. Hofmann, Phys. Rev. A **79**, 033822 (2009).