Strong superadditivity and monogamy of the Rényi measure of entanglement

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Employing the quantum Rényi α entropies as a measure of entanglement, we numerically find the violation of the strong superadditivity inequality for a system composed of four qubits and $\alpha > 1$. This violation gets smaller as $\alpha \to 1$ and vanishes for $\alpha = 1$ when the measure corresponds to the entanglement of formation. We show that the Rényi measure aways satisfies the standard monogamy of entanglement for $\alpha = 2$, and only violates a high-order monogamy inequality, in the rare cases in which the strong superadditivity is also violated. The sates numerically found where the violation occurs have special symmetries where both inequalities are equivalent. We also show that every measure satisfying monogamy for high-dimensional systems also satisfies the strong superadditivity inequality. For the case of Rényi measure, we provide strong numerical evidences that these two properties are equivalent.

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I. INTRODUCTION

Quantum resources present several counterintuitive features allowing more efficient realization of classical and quantum communication tasks. Unfortunately, it is hard to predict the way in which those features are distributed or extended. This is the case for several and important additivity problems, such as for the Holevo capacity of a quantum channel, minimal output entropy of a quantum channel, and the additivity of entanglement of formation (EOF) [1], one of the most important entanglement measures. These were shown by Shor [2] to be all equivalent to the strong superadditivity (SS) [3] of the EOF. An entanglement measure E satisfies SS if

$$E_{a_1a_2|b_1b_2} \geqslant E_{a_1|b_1} + E_{a_2|b_2}, \tag{1}$$

meaning that, when Alice holds parties a_1 and a_2 and Bob holds parties b_1 and b_2, the entanglement between Alice and Bob is larger than the entanglement between a_1 and b_1 plus the one between a_2 and b_2. It is a very important relation since it is connected to the ability to extract arbitrary entangled states from a standard one and the ability to communicate classical information using a quantum channel. Moreover, if a measure E is additive for pure states and extends for mixed states through its convex roof, the SS implies the additivity of the measure for mixed states as well. Although the additivity of EOF was proved previously in some very particular cases [4], recently, Hastings demonstrated in a remarkable work [5] that once all of these conjectures were equivalent they were also in general false due to the existence of counterexamples for the minimal output entropy for sufficiently large dimensions of the Hilbert spaces involved. Whether there is a violation of SS of EOF for lower dimensions is unknown. Perhaps finding counterexamples for lower dimensions requires new insights from the Theory of Information.

In this article we derive an entanglement measure based on the α-quantum Rényi entropy. For $\alpha > 1$ we numerically find counterexamples violating the SS (1) for four qubits systems, the smallest possible situation for which SS can be written. This suggests that counterexamples for SS of EOF ($\alpha = 1$) may exist for smaller dimensions. Moreover, this measure also provides an important relation between SS and the so-called *monogamy of entanglement* [6]. The last is related to the way in that quantum correlation (entanglement) can be distributed between many parties. A measure of entanglement E satisfying the monogamy relation with Alice's subsystem a and Bob's subsystems b_1 and b_2 must follow

$$E_{a|b_1b_2} \geqslant E_{a|b_1} + E_{a|b_2}. \tag{2}$$

Important measures of entanglement, and particularly the EOF, fail to satisfy monogamy [6]. In some sense it seems that these two properties, in principle unrelated, SS and monogamy of entanglement, may actually be related and this could be important for quantum information tasks since it would be also equivalent to the other existent conjectures. We start by discussing entanglement monogamy relations and we show how a second-order monogamy relation implies the SS inequality independently of the measure of entanglement. Then we show that the Rényi measure, for $\alpha = 2$, satisfies the standard monogamy inequality for qubits. Numerically, we investigate the interrelation between these two inequalities using that measure. Interestingly, we find that violation of these two inequalities happens quite rarely but always simultaneously. Thus, we conjecture that SS violation of the Rényi measure for $\alpha = 2$ is necessary and sufficient for the second-order monogamy violation. After that we show how numerical methods can be used to find violations of SS of the Rényi measure for α very close to one.

II. SECOND-ORDER MONOGAMY AND STRONG SUPERADDITIVITY

Monogamy of entanglement shows how quantum correlation is special and different from the classical one. While classical correlation can be arbitrarily shared with as many individuals as desired, quantum correlation cannot. This impossibility of sharing quantum entanglement was first quantified by Coffman, Kundu, and Wooters (CKW) [6],

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through the squared concurrence C^2, as follows:

$$C^2_{a|b_1b_2}(\rho_{ab_1b_2}) \geqslant C^2_{ab_1}(\rho_{ab_1b_2}) + C^2_{ab_1b_2}(\rho_{ab_1b_2}) \quad (3)$$

for any pure or mixed state $\rho_{ab_1b_2}$ of a tripartite system built of qubits a, b_1, and b_2.

Surprisingly, not all measures of entanglement satisfy monogamy relations, for increased Hilbert space dimension and/or number of systems, with the exception of the squashed entanglement [7]. Moreover, there exists a constraint in the CKW monogamy relation: it is true only when a, b_1, and b_2 are qubits. In Ref. [8], the authors extended its validity when b_2 is an n-level system, allowing them to prove the CKW monogamy for N-qubits, $C^2_{1|23\ldots N} \geqslant C^2_{12} + C^2_{13} + \cdots + C^2_{1N}$, as conjectured in Ref. [6]. However, the inequality (3) is not satisfied by increasing the dimension of a [9]. In fact, a measure of entanglement which is monogamous when the subsystem a has higher dimensions implies directly the SS as we now show. Let us consider the case of subsystem a being broken into two subsystems, a_1 and a_2, and apply the monogamy relation (2) again to obtain

$$E_{a_1a_2|b_1b_2} \geqslant E_{a_1|b_1} + E_{a_2|b_1} + E_{a_1|b_2} + E_{a_2|b_2}. \quad (4)$$

We call this relation second-order monogamy, whose meaning is similar to that of (2): The amount of bipartite entanglement shared between $a_1 \otimes a_2$ and $b_1 \otimes b_2$ gives us an upper bound to the sum of entanglement shared by a_1 and b_1, a_2 and b_1, a_1 and b_2, and a_2 and b_2. This idea can be generalized and we can obtain higher-order monogamy relations by successive applications of (2). We are, however, more interested in the fact that a measure E satisfying this second-order relation (4) also satisfies the SS inequality (1). Note, however, that by this reasoning it is not possible to show whether SS implies the second-order monogamy (4) directly. Instead, the SS is a necessary condition for satisfying monogamy for *any* measure of entanglement. Then we question if it is sufficient as well. To investigate this point we choose the family of Rényi entropies which are known to be additive [10,11].

III. RÉNYI MEASURE OF ENTANGLEMENT

The quantum Rényi entropy of order α [12] is defined as

$$R_\alpha = \frac{1}{1-\alpha}\log \mathrm{Tr}\rho^\alpha, \quad (5)$$

where $\alpha \geqslant 0$ and the logarithmic function will always be assumed to be base 2 in this article. In this way, for any pure bipartite system, the Rényi α entropy of one of the subsystems is a good and additive measure of entanglement. The natural way to define the Rényi measure of entanglement, $\mathcal{R}_\alpha$, for a bipartite mixed-state ρ_{ab} is to use the convex roof reasoning of Ref. [1]. We consider the set $\mathcal{E}$ of all ensembles of pure states $|\varphi_i\rangle$ with weight p_i realizing the state ρ_{ab}, $\rho_{ab} = \sum_i p_i|\varphi_i\rangle\langle\varphi_i|$. For each ensemble, we can define an average value of $\mathcal{R}_\alpha$. Then we define $\mathcal{R}_\alpha(\rho_{ab})$ as the minimal value of this average over all the possible ensembles,[1]

$$\mathcal{R}_\alpha(\rho_{ab}) = \min_{\mathcal{E}}\left\{\sum_i p_i\mathcal{R}_\alpha(|\varphi_i\rangle)\right\}. \quad (6)$$

In the case of two qubits, we can show an analytical expression for $\mathcal{R}_\alpha$ for all $\alpha > 1$. For pure states,

$$\mathcal{R}_\alpha(\rho_{ab}) = \frac{1}{1-\alpha}\log[x^\alpha + (1-x)^\alpha, \quad (7)$$

where $x = (1+\sqrt{1-C^2})/2$ and C is the concurrence [1]. When $\alpha \to 1$, this formula goes to the usual one for the EOF [16]. To see that this relation is also true for mixed states, we must notice that $\mathcal{R}_\alpha$ is a convex function of C for $\alpha \geqslant 1$ and the ensemble realizing the convex roof of concurrence is an ensemble composed of states with the same value of C [16]. By this construction, $\mathcal{R}_\alpha$ would be an additive measure if the SS was true.

Now we show that $\mathcal{R}_2$ satisfies the CKW monogamy for systems of N qubits. First we consider the case of an N-partite pure state. Noticing that Eq. (7) simplifies for $\alpha = 2$, we can write the $\mathcal{R}_2$ between subsystem 1 and the other $(N-1)$ subsystems as $\mathcal{R}_2^{1|23\ldots N} = -\log\frac{(2-C^2_{1|234\ldots N})}{2} \geqslant -\log\frac{(2-\sum_i C^2_{1i})}{2}$, where the second inequality comes from the CKW monogamy [6,8]. The entanglement between the two subsystems is given by Eq. (7). Then, if we can show

$$-\log\frac{(2-\sum_i C^2_{1i})}{2} \geqslant -\sum_i \log\frac{2-C^2_{1i}}{2}, \quad (8)$$

we obtain the CKW monogamy for $\mathcal{R}_2$. However, the inequality (8) is equivalent to

$$0 \leqslant \frac{1}{2^2}\sum_{i\neq j}\frac{1}{2!}C^2_{1i}C^2_{1j} - \frac{1}{2^3}\sum_{i\neq j\neq k}\frac{1}{3!}C^2_{1i}C^2_{1j}C^2_{1k} + \frac{1}{2^4}\sum_{i\neq j\neq k\neq m}\frac{1}{4!}C^2_{1i}C^2_{1j}C^2_{1k}C^2_{1m} - \frac{1}{2^5}\sum_{i\neq j\neq k\neq m\neq n}\frac{1}{5!}C^2_{1i}C^2_{1j}C^2_{1k}C^2_{1m}C^2_{1n} + \cdots. \quad (9)$$

It is easily seen that this inequality is always true since each negative term is always smaller than its preceding positive one. Since it implies the CKW monogamy, we have proved our claim. The result generalizes for mixed states by straightforward use of the definition of $\mathcal{R}_2$ as a convex roof and the fact the monogamy is true for pure states.

IV. NUMERICAL RESULTS

The Rényi measure of entanglement does not satisfy the SS only in some very particular cases. Numerically, we

[1] Due to the Schur concavity of Rényi entropy, $\mathcal{R}_\alpha$ does not increase under deterministic local operations and classical communication (LOCC) [13]. However, the Rényi entropy for $\alpha \leqslant 2$ is known to be concave only if the dimension of the space is 2 [14]. There is a counterexample to the concavity for a dimension larger than 8 and $\alpha = 2$ [14]. This implies that $\mathcal{R}_\alpha$ might increase on average under probabilistic LOCC for higher-dimensional systems [15].

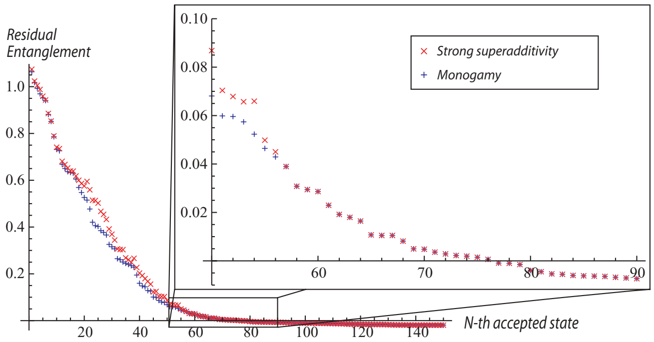

FIG. 1. (Color online) Evolution of the minimization. Each point is a new state found with a smaller residual entanglement than the previous one. The total number of states generated in this example was about 10 000. One can see that the residues of monogamy and strong additivity start to coincide when they are close to zero.

were able to find counterexamples for α very close to one with systems of only four qubits (Fig. 1). The violation is smaller as $\alpha \rightarrow 1$ and vanishes for the case of EOF (Fig. 2). These counterexamples suggest that counterexamples to the additivity of EOF and Holevo capacity may exist for smaller dimensions. Furthermore, in the particular case of $\alpha = 2$, we could not find any violation of monogamy inequality not corresponding to the violation of the SS as well. In fact, all states numerically found where this violation occurs are such that two of the bipartite entanglements appearing on the right side of Eq. (4) vanish, being thus equivalent to the SS inequality (1). Thus, we conjecture that SS is necessary and sufficient for monogamy.

Violations of inequalities (1) and (4) are not easy to find. For the case of $\alpha = 2$, we were not able to find a violation by choosing 50 million pure states randomly (according to the Haar measure), which takes about a week of computing time on a standard PC. To find one, we had to employ a simple Monte Carlo minimization algorithm. The function to be minimized is the difference between the first and the second members of (1) and (4), also called residual entanglement [6]. So there are two residual entanglements, one for SS (1) and one for monogamy (4). The algorithm works as follows. First, we choose randomly a state as a seed and we fix a distance δ. Then we look randomly for a state with smaller residual entanglement within a distance (trace distance) δ from the seed. We also use a counter to count

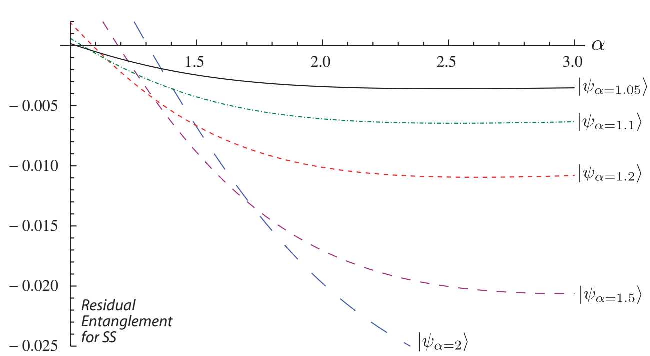

FIG. 2. (Color online) Violation of SS for some states numerically found. These states are found successively minimizing the residual entanglement for SS for $\alpha = 2$, 1.5, 1.2, 1.1, and 1.05.

the number of random states generated until we find a state with smaller residual entanglement. When we find one we always reset the counter and start the search from this state as a new seed. When the counter gets some large value (1000 is usually large enough), we divide by 2 the distance δ from the seed and reset the counter. When the distance gets smaller than 10^{-4}, we stop (this is sufficient to get a precision of order 10^{-8}). A standard PC can run this in some minutes for four-qubit systems and the results are very reasonable. Figure 1 shows the progress of the algorithm for one particular case.

With this method, the algorithm finds a vanishing residual entanglement on 70% of the runnings and a negative residual entanglement of -0.0197 on the remaining 30%. This state, which we call $|\psi_{\text{vio}}\rangle$, has a reduced density matrix $\rho_{a_1a_2}$ with eigenvalues $\{0.66, 0.14, 0.14, 0.06\}$ and has a considerable entanglement, $\mathcal{R}_2 = 1.06$, between a and b. The density matrix $\rho_{a_1b_1}$ and $\rho_{a_2b_2}$ has eigenvalues $\{0.997, 0.003, 0, 0\}$; that is, they are almost pure. So $|\psi_{\text{vio}}\rangle$ is very close to a product state of the form $|\psi_{a_1b_1}\rangle \otimes |\psi_{a_2b_2}\rangle$. The entanglement between the subsystems a_1 and b_1 and a_2 and b_2 are all equal to 0.54. The entanglements between all the other qubits vanish. Therefore, these states can be characterized by showing entanglement only between the components relevant to the SS inequality and been close to product states of the subsystems 1 and 2.

We also conducted an extensive numerical test to check if all states violating monogamy have these properties. Using the search algorithm described, we obtain a sequence of states forming a path from an initially random state to one of maximum violation of (4). The states of this path start to violate monogamy at the same point that they violate SS and the value of violation is always the same (see Fig. 1), confirming that two bipartite entanglements of (4) vanish. Furthermore, during the process, thousands of random states are generated near this path and tested. With this method we tested more than 3×10^6 states in many different runnings of the algorithm. In order to check this more carefully, we made a modification in the algorithm for not staying always near this path. When we get inside the region of states having negative residual entanglement, we stop to decrease the distance and start a random walk in that region. With this modified method, we checked more than 4×10^6 states and all of them have the same residual entanglement for monogamy and SS inequalities. Therefore, in the case of the $\mathcal{R}_2$ for four-qubit systems, we conjecture that states violating the monogamy inequality (4) are the ones that also violate the SS (1) as well.

Finding a violation for the SS for α close to one is more difficult. The violation gets very small and the best strategy is to use a recurrence procedure. Instead of starting our search with a random state, we start it with the state that maximally violates SS for $\alpha = 2$ as a seed, but run the algorithm for minimizing the residual entanglement of SS with $\alpha = 1.5$ starting with a smaller distance of 10^{-2} and leave it decreasing until 10^{-8}. Then we go successively to $\alpha = 1.2$, 1.1, 1.05, ... and soon on. The progress of this process can be seen in Fig. 2 and illustrates how the violation of SS vanishes as $\alpha \rightarrow 1$. With this method, we found violation for $\alpha = 1.002$ of order of 10^{-6}. For large α, the violation saturates to a value depending on the state. These counterexamples strongly suggest that there are counterexamples to the SS of Rényi measure for all $\alpha > 1$.

We have made an extensive search for counterexamples to SS for $\alpha = 1$ using these methods. As the numerical methods were efficient for finding counterexamples for almost every α, we have a strong indication that there are no violations to SS of EOF for four-qubits systems. Despite that, the existence of counterexamples to SS for α close to one at these very small dimensions suggest that there can be counterexamples to SS of EOF, and for all the other equivalent additivity questions, for reasonably smaller dimensions than the ones necessary in the Hastings counterexamples. It is important to remember that his counterexamples were inspired by previous ones of Hayden and Winter [17] for the minimal Rényi entropy output of a quantum channel. So the counterexamples found here can be considered as a good indication of the existence of analogous ones for the EOF. The existence of such counterexamples, for smaller dimensions, would have great implications for quantum information. It would imply that the superadditivy of the Holevo capacity and the subadditivity of EOF can be used to improve the ability of communication over a quantum channel and of the ability of forming states from a standard resources like Einstein-Podolsky-Rosen pairs in more practical and simpler situations.

V. CONCLUSIONS

In this work, we connected the properties of monogamy and additivity of entanglement using the Rényi measure. We show that this measure satisfies the standard monogamy inequality for the particular case $\alpha = 2$. We also show that the second-order monogamy (4) implies the SS (1). Again in the case of $\alpha = 2$, we found numerically that the inequalities (1) and (4) are violated rarely, but always simultaneously and with the same magnitude. Further, we provided strong numerical support for conjecturing that the violation of monogamy inequality (4) is related to the SS (1) violation for the Rényi measure of order 2. This approach made it possible to find more counterexamples for SS as α gets closer to one. Also, there are counterexamples to the SS of the Rényi measure for every $\alpha > 1$. The violation of SS becomes very small as $\alpha \to 1$ and vanishes for $\alpha = 1$.

The results here can help in the understanding of why EOF turns out to be nonadditive. The counterexamples found can stimulate the research of new counterexamples to the additivity of EOF at small dimensions. Once the numerical methods employed are very simple, they can certainly be improved. This fact opens the possibility of numerical searching for such counterexamples for small dimensions larger than 4 by 4. Since additivity and monogamy seem to be connected through our findings, we expect that it may shed some light on the understanding of the way in which entanglement is distributed. The Rényi measure introduced here certainly plays an important role in this research, as well as the search for new counterexamples to the additivity of EOF.

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