# Vacuum polarization for compactified $\mathrm{QED}_{4+1}$ in a magnetic flux background

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We evaluate one-loop effects for $\mathrm{QED}_{4+1}$ compactified to $\mathbf{R}^4 \times S^1$ in a nontrivial vacuum for the gauge field such that a nonvanishing magnetic flux is encircled along the extra dimension. We obtain the vacuum polarization tensor and evaluate the exact parity-breaking term, presenting the results from the point of view of the effective $(3+1)$-dimensional theory.

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## I. INTRODUCTION

Quantum field theory models with compactified dimensions have been used to describe different physical situations, ranging from finite size effects in critical phenomena [1–3], to the unification of fundamental interactions [4–6]. These ideas have recently attracted renewed interest; for instance, results about the electroweak phase transition has been presented in Ref. [7] in the context of a $(4+1)$-dimensional theory with a compactified dimension.

The presence of extra compactified dimensions may give rise to effects at different scales, not just those in high-energy physics realm; in particular on low-energy phenomena, like atomic physics. Following this idea, we evaluate, within a specific domain of physics, namely quantum electrodynamics (QED), the effects that would follow if our world were five dimensional. In particular, we investigate at one-loop order some effects due to the assumption of a nontrivial vacuum with a nonvanishing magnetic flux along the compactified dimension.

Effects on the anomalous magnetic moment of the muon associated with extra-dimensional excitations of the photon and of the $W$ and $Z$ bosons have been studied in Ref. [8] in a space whose extra dimensions have large compactification radii. Those authors have shown that when the extra-dimensional corrections to the Fermi constant are included, their effects on $(g_\mu - 2)$ become too small to be observable. They discuss a model which avoids the extra-dimensional corrections to the muon decay $\mu \to e\bar{\nu}_e\nu_\mu$ without suppression of their effects on $(g_\mu - 2)$. Eventual extra-dimensional effects on $(g_\mu - 2)$ would be very interesting. We know, since the $g-2$ experiment at Brookhaven National Laboratory (U.S.) in 2004, and subsequent experiments, that the expected value from standard theoretical calculations, that predict $g = 2$, could not be confirmed, since both theoretical prediction and experimental results have a large amount of uncertainty. Although a conclusive response is not available, a value of $g \neq 2$ is not excluded by experimentalists [9]. Indeed, in the experimental framework of QED a recent experiment for the electron magnetic moment, gives a much more precise value for $g_e$ (the claimed uncertainty is nearly 6 times lower than in the past). These authors still find a deviation from the value $g = 2$ [10]. In atomic physics, very accurate measurements of the asymptotic quantum effects on Rydberg excitations have also been carried out [11].

Another interesting consequence of the possible existence of extra dimensions is explored in Ref. [12]. This study shows that they would imply that electric charge might not be exactly conserved, which has been a subject of discussion for a long time [13–15]. As mentioned in Ref. [12], in four-dimensional theories, a tiny deviation from electric charge conservation would lead to contradictions with low-energy tests of QED. These could, in turn, be avoided by the introduction of hyphotetical millicharged particles [15]. However, as argued in Ref. [12], if our world were considered as a submanifold of a higher-dimensional space, this artifact would not be necessary. Indeed in this case, particles initially confined to our four-dimensional subspace could, under some circumstances migrate to the extra dimensions. The idea presented in Ref. [12] is that if they are electrically charged, their migration from our world into extra dimensions would appear for us as nonconservation of electric charge. Charge nonconservation and other possible effects of extra dimensions could perhaps be investigated in experiments similar to those in Refs. [10,11].

In Refs. [16] a U(1) gauge field theory with fermion or scalar fields defined on a space with extra compactified dimensions has been considered. These authors compute the fermion-induced quantum energy in the presence of a constant magnetic field directed toward the $z$ axis. They study the effect of extra dimensions on the asymptotic behavior of the quantum energy in the strong field limit and find that the weak logarithmic growth of the quantum energy in four dimensions is modified by a rapid power growth in a space-time with extra dimensions.

Because of the reasons described above, we believe that the study of effects due to extra dimensions in electromagnetic phenomena is a subject of actual interest. We present here new results about that topic; they correspond to quantum effects in QED with an extra dimension, in a magnetic flux background. In particular, we consider the modifications that the extra dimension produces on the vacuum polarization phenomenon.

This article is organized as follows: in Sec. II we introduce the model and study its more important features, mostly related to the realization of gauge invariance within the context of a theory with a compactified dimension, having an extra dimensional-like mode expansion in mind. Section III deals with the effective action, which is derived including both parity-conserving and parity-violating parts. In Sec. IV, we apply the general results of Sec. III to the exact calculation of the parity-conserving part of the vacuum polarization tensor. In Sec. V, we consider parity-breaking effects. Section VI contains our conclusions.

## II. THEORETICAL FRAMEWORK

From a general point of view, one can consider a simply or non simply connected $D$-dimensional manifold with a topology $\mathbf{R}^D_d = \mathbf{R}^{D-d} \times S_{l_1} \times \mathcal{S}_{l_2} \times, \ldots, \times \mathcal{S}_{l_d}$, with $l_1$ corresponding to the inverse temperature and $l_2, \ldots l_d$ to the compactification of $d-1$ spatial dimensions (this case has been considered, within the context of spontaneous symmetry breaking, in Ref. [17]). An interesting yet simple example of this, corresponds to the compactification of one dimension in an $\mathbf{R}^D$ Euclidean space-time, such that the topology of the resulting manifold $\mathcal{M}$ is that of $\mathbf{R}^{D-1} \times \mathcal{S}_1$, i.e., "circular compactification." Although the compelling features that emerge in this situation have been studied using several different techniques in the literature, one can take advantage of a (formal) common feature; indeed, they share many properties with the imaginary-time formulation of quantum field theory at finite temperature [18,19]. This allows one, for example, to take advantage of the many well-known methods and results developed in this context, such as Feynman diagrams and renormalization techniques, to import them to the case under consideration.

For just one compactified dimension (imaginary time or a spatial dimension) the Feynman rules are modified, the most characteristic new feature is the Matsubara prescription for momentum integrals,

$$\int \frac{dk_s}{2\pi} \to \frac{1}{\xi} \sum_{n=-\infty}^{+\infty}; \quad k_s \to \frac{2n\pi}{\xi}, \tag{1}$$

where $k_s$ amounts to the momentum component corresponding to the compactified dimension, while $\xi$ equals $\beta$ or $L$, for the finite temperature and compactified spatial dimension cases, respectively.

Within the previous general framework, we here investigate one-loop effects for $\mathrm{QED}_{3+1}$ with an extra compactified dimension, in a nontrivial vacuum for the gauge field, defined by a nonvanishing component along the extra dimension.

The system we shall deal with may be conveniently defined in terms of an Euclidean action, $S$, which has the structure:

$$S(\mathcal{A}; \bar{\Psi}, \Psi) = S_g(\mathcal{A}) + S_f(\mathcal{A}; \bar{\Psi}, \Psi), \tag{2}$$

where $S_g$ and $S_f$ denote the U(1) gauge field and fermionic actions, respectively. The former is assumed to have a standard Maxwell form, namely:

$$S_g(\mathcal{A}) = \frac{1}{4} \int d^5x \, \mathcal{F}_{\alpha\beta} \mathcal{F}_{\alpha\beta}, \tag{3}$$

with $\mathcal{F}_{\alpha\beta} \equiv \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha$, where we adopted the convention that indices from the beginning of the Greek alphabet ($\alpha$, $\beta, \ldots$) label all the coordinates of the spacetime manifold and therefore run from 0 to 4. Since we will be specially interested in the model as it is seen from a $(3+1)$-dimensional point of view, we shall also use another convention: indices from the middle of the Greek alphabet ($\mu$, $\nu, \ldots$) are reserved for the $(3+1)$-dimensional space-time coordinates while, when this notation is used, the extra-dimension coordinate shall be denoted by $s$. Then:

$$\begin{aligned} \alpha = 0,\ 1,\ 2,\ 3,\ 4, \quad \mu &= 0,\ 1,\ 2,\ 3, \\ d^5x \equiv d^{3+1}x \, dx_4 &= d^{3+1}x \, ds, \end{aligned} \tag{4}$$

and $x$ will be assumed to denote the $(3+1)$-dimensional coordinates $x_\mu$, unless explicit indication on the contrary. The extra dimension is assumed to be compactified with a radius $R$, so $s \sim s + L$, $L = 2\pi R$.

On the other hand, the Dirac action, $S_f$, is given by

$$S_f(\bar{\Psi}, \Psi; \mathcal{A}) = \int d^{3+1}x \, ds \, \bar{\Psi}(x,s)(\not{\mathcal{D}} + m)\Psi(x,s), \tag{5}$$

where $\not{\mathcal{D}}$ is the $(4+1)$-dimensional Dirac operator, $\not{\mathcal{D}} = \gamma_\alpha D_\alpha$. The covariant derivative $D_\alpha \equiv \partial_\alpha + ig\mathcal{A}_\alpha$ includes a coupling constant $g$ with the dimensions of $(\text{mass})^{-\frac{1}{2}}$. For Dirac's $\gamma$ matrices, we assume that $\gamma_s \equiv \gamma_5$, where the latter is the $\gamma_5$ matrix for the $3+1$ world.

To proceed, we discuss now the mode expansion and its relation to gauge invariance. To that end, we follow [2], where this issue is discussed at length, albeit in the finite temperature theory context, in the Matsubara formulation of thermal field theory. Due to the formal analogy with this situation, a quite straightforward procedure allows us to adapt the results derived there to our case. The necessary changes that follow from the fact that our compactified dimension is spatial rather than temporal are taken into account by using (1). In that analogy, the length $L$ plays the same role of the inverse temperature in Ref. [2]: $L \sim \beta$, $\beta = T^{-1}$.

What follows is a brief review of some of those properties (the ones which are relevant to our study), adapted to our case and conventions. To begin with, the gauge field configuration $\mathcal{A}_\alpha(x,s)$ may be decomposed into its zero ($A_\alpha$) and nonzero ($Q_\alpha$) mode components:

$$\mathcal{A}_\alpha(x,s) = L^{-\frac{1}{2}} A_\alpha(x) + Q_\alpha(x,s), \tag{6}$$

where the two terms in this decomposition may be defined by:

$$A_\alpha(x) = L^{-\frac{1}{2}} \int_0^L ds \, \mathcal{A}_\alpha(x,s), \tag{7}$$

and

$$Q_\alpha(x,s) = \mathcal{A}_\alpha(x,s) - L^{-\frac{1}{2}} A_\alpha(x), \tag{8}$$

so $\int_0^L ds \, Q_\alpha(x,s) = 0$. An $L^{-\frac{1}{2}}$ factor has been included in the zero mode term in order to make this field have the usual mass dimensions in $(3+1)$ space-time dimensions; this property will become useful after dimensional reduction.

The decomposition above finds a natural interpretation when one considers the Fourier expansion of the gauge field along the extra dimension:

$$\mathcal{A}_\alpha(x,s) = L^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{i\omega_n s} \widetilde{\mathcal{A}}_\alpha(x,n), \tag{9}$$

with $\omega_n \equiv \frac{2\pi n}{L}$, where one identifies:

$$A_\alpha(x) = \widetilde{\mathcal{A}}_\alpha(x,0), \quad Q_\alpha(x,s) = L^{-\frac{1}{2}} \sum_{n\neq 0} e^{i\omega_n s} \widetilde{\mathcal{A}}_\alpha(x,n). \tag{10}$$

Then we dimensionally reduce the theory, what, for the gauge field action, amounts to keeping just the zero mode component of the gauge field. Thus:

$$S_g(\mathcal{A}) \;\to\; S_g(A) \;=\; S_g(A_\mu,\, A_s), \tag{11}$$

where:

$$S_g(A_\mu, A_s) = \int d^{3+1}x \left[\frac{1}{2}\partial_\mu A_s \partial_\mu A_s + \frac{1}{4}F_{\mu\nu}(A)F_{\mu\nu}(A)\right], \tag{12}$$

with $F_{\mu\nu}(A) \equiv \partial_\nu A_\nu - \partial_\nu A_\mu$.

Regarding the fermionic action $S_f$, the reduction amounts to:

$$S_f(\mathcal{A};\bar\Psi,\Psi) \;\to\; S_f(A_\mu, A_s;\bar\Psi,\Psi). \tag{13}$$

The fermionic field is not dimensionally reduced for the simple reason that, in the calculation of the effective gauge field action, its only contribution comes from the fermion loop. That loop may be represented as a series of $(3+1)$-dimensional loops, each one with a different mass. Although the contributions of heavier modes may be relatively suppressed, the very fact that there is an infinite number of them forbids us to truncate that series (even if there were a zero mode).

Thus, the following explicit expression for the fermionic action shall be used after dimensional reduction:

$$S_f \;=\; \int d^{3+1}x \int_0^L ds\; \bar\Psi(x,s)\big(\not{D} + \gamma_s D_s + m\big)\Psi(x,s), \tag{14}$$

where

$$\not{D} \;=\; \gamma_\mu(\partial_\mu + ieA_\mu) \;\; D_s \;=\; \partial_s + ieA_s. \tag{15}$$

We have introduced a new, dimensionless coupling constant $e \equiv gL^{-\frac{1}{2}}$, which shall play the role of the electric charge in $(3+1)$ dimensions.

As explained in Ref. [2], when considering the form of the gauge transformations in terms of the decomposition into zero and nonzero modes, one finds that it $A_\mu$ transforms as a standard gauge field [in $(3+1)$ dimensions]:

$$\delta A_\mu(x) \;=\; \partial_\mu\alpha(x) \tag{16}$$

while its extra-dimensional component $A_s$, a scalar from the $(3+1)$-dimensional point of view, is shifted by a constant:

$$\delta A_s(x) \;=\; \Omega. \tag{17}$$

The constant $\Omega$ has to be of the form $\Omega = \frac{2\pi k}{Le}$, where $k$ is an integer, since the gauge field is coupled to a (charged) fermionic field, whose transformation law under simultaneous action of the previous gauge transformations is:

$$\begin{aligned}\Psi(x,s) &\to e^{-ie[\alpha(x)+\Omega s]}\,\Psi(x,s)\\ \bar\Psi(x,s) &\to e^{ie[\alpha(x)+\Omega s]}\,\bar\Psi(x,s).\end{aligned} \tag{18}$$

## III. EFFECTIVE ACTION

We now define the part of the effective action that only depends on the (dimensionally reduced) gauge field, $\Gamma(A)$,

$$\Gamma(A) \;\equiv\; \Gamma(A;\bar\Psi,\Psi)|_{\bar\Psi=\Psi=0}, \tag{19}$$

where $\Gamma(A;\bar\Psi,\Psi)$ is the full effective action. The functional $\Gamma(A)$ allows one to derive one-particle irreducible (1PI) functions containing only $A_\mu$, $A_s$ external lines. The former have an immediate $(3+1)$-dimensional interpretation, while the latter shall be assumed to have a constant (but otherwise arbitrary) value, which is determined by a condition which is external to the model.

On the other hand, at the one-loop order, the only nontrivial term comes from the fermionic loop:

$$\Gamma(A) \;=\; \Gamma^{(0)}(A) \;+\; \Gamma^{(1)}(A) \;+\; \cdots, \tag{20}$$

where $\Gamma^{(0)}(A) = S_g(A)$ and

$$e^{-\Gamma^{(1)}(A)} \;=\; \int \mathcal{D}\Psi\mathcal{D}\bar\Psi e^{-S_f(A;\bar\Psi,\Psi)}. \tag{21}$$

We shall focus on the effective action for the gauge field components $A_\mu$ that have a direct physical interpretation from a $(3+1)$-dimensional perspective. Regarding the scalar component, $A_s$, as we have said above, it will be assumed to yield a nonvanishing flux:

$$e\int_0^L ds A_s \;=\; \theta, \tag{22}$$

where $\theta$ is a constant. This condition may be conveniently solved by means of a constant $A_s$:

$$A_s \;=\; \frac{\theta}{eL}, \tag{23}$$

which is the gauge fixing that we shall assume. Note that, since the gauge transformations shift $A_s$ by an integer multiple of $\frac{2\pi}{eL}$, we may fix the value of $\theta$ to the fundamental region:

$$0 \leqslant \theta < 2\pi, \tag{24}$$

which we shall assume in what follows.

It is worth noting that this kind of gauge field configuration may be interpreted as "topological," in the sense that it corresponds locally (althougth not globally) to a "pure gauge" field configuration. Indeed, it cannot be gauged away, since the corresponding gauge transformation would be multivalued (when the extra dimension is encircled). Charged fields feel this kind of configuration when they encircle the extra coordinate, in a fashion that resembles the Aharonov-Bohm effect. The field configuration may be realized in a similar way to this effect: a singular field strength pointing in a direction orthogonal to the plane of the circle. Besides, as in the Aharonov-Bohm effect, the region of space where the field strength is nonvanishing cannot be reached by the charged fields. The situation can be easily visualized in a lower dimensional example, namely the case of a $(2+1)$-dimensional theory, if one assumes $x_2$ to be the extra, compactified dimension. Here, space is a cylinder, and the gauge field configuration corresponding to the vacuum field would be a singular flux string along the cylinder axis. This means that it is outside of the assumed cylindrical space, since it needs a third coordinate to be realized. In a similar way, the kind of configuration we consider could be realized by singular, monopolelike field strengths in a higher [more than $(4+1)$] dimensional manifold.

We then proceed to Fourier expand the fermionic fields along the $s$ coordinate:

$$\Psi(x,s) = L^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{i\omega_n s}\psi_n(x)$$
$$\bar{\Psi}(x,s) = L^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-i\omega_n s}\bar{\psi}_n(x) \tag{25}$$

and insert this into the functional expression for $\Gamma^{(1)}(A)$, to obtain:

$$S_f = \sum_{n=-\infty}^{n=+\infty} \int d^{3+1}x\, \bar{\psi}_n(x) \left[ \not{D} + i\gamma_s \left(\omega_n + \frac{\theta}{L}\right) + m \right] \psi_n(x). \tag{26}$$

Under the same expansion, the fermionic measure factorizes:

$$\mathcal{D}\Psi\mathcal{D}\bar{\Psi} = \prod_{n=-\infty}^{n=+\infty} \mathcal{D}\psi_n(x)\mathcal{D}\bar{\psi}_n(x), \tag{27}$$

and, finally, the Euclidean action corresponding to each mode $n$ may be equivalently written as follows:

$$\int d^{3+1}x\, \bar{\psi}_n(x) \left[ \not{D} + i\gamma_s \left(\omega_n + \frac{\theta}{L}\right) + m \right] \psi_n(x)$$
$$= \int d^{3+1}x\; \bar{\psi}_n(x)(\not{D} + M_n\, e^{-i\varphi_n\gamma_5})\psi_n(x) \tag{28}$$

with

$$M_n \equiv \sqrt{m^2 + (\omega_n + \theta/L)^2}, \quad \varphi_n = \arctan\left(\frac{\omega_n + \theta/L}{m}\right). \tag{29}$$

The existence of a $\gamma_5$ term means that parity symmetry will generally be broken; to study that phenomenon more clearly, we perform a change in the fermionic variables that gets rid of the dependence in $\gamma_5$,

$$\psi_n(x) \to e^{-i\gamma_5\varphi_n/2}\psi_n(x), \quad \bar{\psi}_n(x) \to \bar{\psi}_n(x)e^{-i\gamma_5\varphi_n/2}, \tag{30}$$

after which the mode labeled by $n$ has the action:

$$\int d^{3+1}x\; \bar{\psi}_n(x)(\not{D} + M_n\, e^{-i\varphi_n\gamma_5})\psi_n(x). \tag{31}$$

This chiral rotation in the $(3+1)$ Euclidean fermionic variables induces, however, an anomalous Jacobian $\mathcal{J}_n$ for each mode. Then, $\Gamma^{(1)}$ may be written as follows:

$$e^{-\Gamma^{(1)}(A)} = \prod_{n=-\infty}^{+\infty} \left[\mathcal{J}_n\, e^{-\Gamma^{(1)}_{3+1}(A,M_n)}\right], \tag{32}$$

where

$$\mathcal{J}_n = \exp\left(\frac{ie^2}{16\pi^2}\,\varphi_n \int d^{3+1}x\, \tilde{F}_{\mu\nu}F_{\mu\nu}\right), \tag{33}$$

with $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F_{\rho\lambda}$, and $\Gamma^{(1)}_{3+1}(A, M_n)$ is the one-loop fermionic contribution to the effective action, for a fermion whose mass is $M_n$, in $(3+1)$ dimensions. Of course, it may be expressed as a fermionic determinant:

$$e^{-\Gamma^{(1)}_{3+1}(A,M_n)} = \det(\not{D} + M_n). \tag{34}$$

Then, we arrive to a general expression for the one loop effective action,

$$\Gamma^{(1)}(A) = \Gamma^{(1)}_e(A) + \Gamma^{(1)}_o(A) \tag{35}$$

where the $e$ and $o$ subscripts stand for the even an odd components (regarding parity transformations) and are given by

$$\Gamma^{(1)}_e(A) = \sum_{n=-\infty}^{\infty} \Gamma^{(1)}_{3+1}(A, M_n) \tag{36}$$

and

$$\Gamma^{(1)}_o(A) = -\sum_{n=-\infty}^{\infty} \ln \mathcal{J}_n, \tag{37}$$

respectively.

## IV. PARITY-CONSERVING TERM

The parity-conserving part of the effective action may be obtained by performing the sum of the required $QED_{3+1}$ object, with an $n$-dependent mass, $M_n$. We shall focus on that part of $\Gamma^{(1)}_e$ that contributes to the vacuum polarization tensor for the $A_\mu$ gauge field components. Since we are not interested in response functions which involve the $s$ component of the currents, it is useful to define:

$$\Gamma^{(1)}_e(A_\mu) \equiv \Gamma^{(1)}_e(A_\mu, A_s) - \Gamma^{(1)}_e(0, A_s). \tag{38}$$

Note that $\Gamma^{(1)}_e(0, A_s) \equiv \Gamma_s(A_s)$ does not contribute to response functions involving $A_\mu$, although it can be used to study the fermion-loop corrections to an $A_s$ effective potential. The explicit form of this function is [2]:

$$\Gamma_s(A_s) = -2L \int d^{3+1}x \int \frac{d^4k}{(2\pi)^4} \ln[\cosh(Lk) + \cos\theta]. \tag{39}$$

The vacuum polarization tensor $\Pi_{\mu\nu}$ is obtained from the quadratic term in a functional expansion in the gauge field:

$$\Gamma^{(1)}_e(A_\mu) = \frac{1}{2}\int d^{3+1}x \int d^{3+1}y A_\mu(x)\Pi_{\mu\nu}(x,y)A_\nu(y) + \cdots. \tag{40}$$

It is then sufficient to resort to the analogous expansion for the $(3+1)$-dimensional effective action,

$$\Gamma^{(1)}_{3+1}(A, M_n) = \frac{1}{2}\int d^{3+1}x \int d^{3+1}y$$
$$\times\left[A_\mu(x)\Pi^{(n)}_{\mu\nu}(x,y)A_\nu(y)\right] + \cdots \tag{41}$$

(which is even) so the vacuum polarization receives contributions from all the modes:

$$\Pi^e_{\mu\nu} = \sum_n \Pi^{(n)}_{\mu\nu}, \tag{42}$$

where $\Pi^{(n)}_{\mu\nu} = \Pi^{(n)}(k^2)\delta^T_{\mu\nu}(k)$, with:

$$\Pi^{(n)}(k^2) = \frac{2\,e^2}{\pi}\int_0^1 d\beta\ \beta(1-\beta)\ \ln\left[1+\beta(1-\beta)\frac{k^2}{M_n^2}\right], \tag{43}$$

which is formally identical to the renormalized scalar part of the vacuum polarization tensor for a $(3+1)$-dimensional theory, and the transverse projector is defined by $\delta^T_{\mu\nu}(k) \equiv \delta_{\mu\nu} - k_\mu k_\nu/k^2$. Note that the renormalization performed for $\Pi^{(n)}(k^2)$ should in fact be interpreted as a subtraction for the $(4+1)$-dimensional theory, which (see below) yields a logarithmically divergent vacuum polarization, as in $(3+1)$ dimensions, once all the symmetries have been taken into account. The subtraction already performed in $(3+1)$ dimensions does not yet fulfill the renormalization conditions for the $(4+1)$-dimensional theory: the zero of $\Pi^{(n)}$ is at $k^2=0$ for each term, but the limit $k^2 \to 0$ does not necessarily commute with the (infinite) sum over modes. Indeed, that commutativity is not guaranteed, since the series in (42) does not converge uniformly.

To do have the proper pole in the propagator, we shall need to perform also a finite renormalization. Indeed, the sum in (42) may be explicitly evaluated using $\zeta$ function regularization techniques [20]; we can write $\Pi(k^2) = \sum_n \Pi^{(n)}(k^2)$, with

$$\Pi(k^2) = \frac{2\,e^2}{\pi}\int_0^1 d\beta\ \beta(1-\beta)\Pi(k^2,\beta), \tag{44}$$

and

$$\Pi(k^2,\beta) = \sum_{n=-\infty}^{+\infty} \ln\left[\frac{\left(bn+\frac{\theta}{L}\right)^2 + m^2 + \beta(1-\beta)k^2}{\left(bn+\frac{\theta}{L}\right)^2+m^2}\right], \tag{45}$$

where $b = 2\pi/L$. Then it can be readily seen that,

$$\Pi(k^2,\beta) = \lim_{s\to 0_-}\left\{\frac{d}{ds}\left[Z_1^{m^2}\left(s,b^2,\frac{\theta}{L}\right)\right]\right\} - \lim_{s\to 0_-}\left\{\frac{d}{ds}\left[Z_1^{m^2+\beta(1-\beta)k^2}\left(s,b^2,\frac{\theta}{L}\right)\right]\right\}, \tag{46}$$

where $Z_1(s,\ldots)$ are generalized inhomogeneous $\zeta$ functions,

$$Z_1^{M^2}\left(s,b^2,\frac{\theta}{L}\right) = \sum_{n=-\infty}^{\infty}\left[\left(bn+\frac{\theta}{L}\right)^2+m^2\right]^{-s}, \tag{47}$$

with $M^2 = m^2$ or $M^2 = m^2+\beta(1-\beta)k^2$.

Using explicit formulas for $K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$ and $\sum_{n=1}^{\infty}\frac{e^{-a}}{n} = -\ln\left(1-e^{-a}\right)$, we get after some manipulations, remembering $b = 2\pi/L$,

$$\Pi(k^2,\beta) - \Pi(0,\beta) = \ln\left\{\frac{\cosh\left[mL\sqrt{1+\frac{\beta(1-\beta)k^2}{m^2}}\right]-\cos\theta}{\cosh(mL)-\cos\theta}\right\}. \tag{48}$$

This leads directly to the result

$$\Pi^e_R(k^2) = \Pi_e(k^2) - \Pi_e(0) = \frac{2\,e^2}{\pi}\int_0^1 d\beta\ \beta(1-\beta)\ \ln[1+F(k^2)], \tag{49}$$

with

$$F(k^2) = \frac{\cosh\left[mL\sqrt{1+\beta(1-\beta)\frac{k^2}{m^2}}\right]-\cosh(mL)}{\cosh(mL)-\cos(\theta)}. \tag{50}$$

It is interesting to note that, even though the theory is five dimensional, the vacuum polarization tensor requires, to be renormalized, just fixing the position and residue of one pole, as in four dimensions. Indeed, the superficial degree of divergence, $\delta(\gamma)$, for 1PI Feynman graph $\gamma$ in $QED_5$ is

$$\delta(\gamma) = 5 - \tfrac{3}{2}E_G - 2E_F + \tfrac{1}{2}V, \tag{51}$$

where $E_G$ and $E_F$ are the number of external gauge and fermion lines, respectively, and $V$ is the number of vertices. For the one-loop vacuum polarization tensor, we then have $\delta(\gamma) = 3$, which, taking into account gauge invariance is reduced to 1. Moreover, since the divergent terms can only be *even* polynomials in the momentum, we are left with a zero degree divergence: this is the logarithmic divergence already tamed in (44).

Let us now study some immediate properties and consequences that follow from expressions (49) and (50) above. The natural approach is perhaps to look at its predictions for different momentum regimes. Let us thus begin by considering the low-momentum regime, namely $k^2 \ll m^2$. The leading term, $k^2/m^2 \to 0$, has already been considered to impose the renormalization condition $\Pi^e_R \to 0$, which is not actually a prediction but rather is used to demonstrate the fact that the model contains Coulomb's law at long distances.

The next-to-leading term already contains a nontrivial effect. Indeed, a simple effect that will be sensible to the presence of the flux can be seen by expanding the renormalized tensor to $(\frac{k}{m})^2$ order in a momentum expansion:

$$\Pi_R(k^2) \sim -\frac{e^2}{30\pi}\left[\frac{mL\ \sinh(mL)}{\cos(\theta)-\cosh(mL)}\right]\frac{k^2}{m^2}, \quad k^2\sim 0. \tag{52}$$

The corresponding modification in the photon's effective action produces, for example, a correction in the electrostatic potential due to a point charge. For the hydrogen atom, the corrected potential energy becomes:

$$V_{\rm eff}(r) = -\frac{e^2}{4\pi r} - \frac{e^4}{120\pi^2 m^2}\left[\frac{mL\ \sinh(mL)}{\cosh(mL)-\cos\theta}\right]\delta^{(3)}(\mathbf{r}). \tag{53}$$

The usual correction is obtained when $\theta\to 0$ and $mL\to 0$:

$$V_{\rm eff}(r) \to -\frac{e^2}{4\pi r} - \frac{e^4}{60\pi^2 m^2}\delta^{(3)}(\mathbf{r}). \tag{54}$$

It is interesting to study the shape of the ratio between the corrected and usual strengths of the respective terms:

$$\xi(mL,\theta) \equiv \frac{\frac{mL}{2}\,\sinh(mL)}{\cosh(mL)-\cos\theta}. \tag{55}$$

The case of a vanishing flux yields simply

$$\xi(mL,0) \;=\; \frac{\frac{mL}{2}}{\tanh\left(\frac{mL}{2}\right)},$$

which for small values of $mL$ approaches 1 and grows linearly with $mL$ when $mL \gg 1$.

The opposite regime, when the effect of the flux is maximum, corresponds to $\theta = \pi/2$:

$$\xi\left(mL,\frac{\pi}{2}\right) = \frac{mL}{2}\,\tanh(mL). \tag{56}$$

The behavior in this case is quite different; it tends to zero quadratically for small $mL$ and also grows linearly in the opposite case, albeit with a different slope.

It is noteworthy that, from Eq. (55), one can obtain a crude estimate for the length $L$. To that end, we need some assumptions. First, we consider that we are within the vanishing flux approximation. Second, as the typical contribution of the vacuum polarization term for the energy shift in muonic atoms is of the order of 0.5 % [21], we may then take $\xi(mL,\,0) \lesssim 1.0001$; such a choice implies that a correction due to an extra dimension does not significantly change the values from the present data. Having this in mind, we obtain through this simple reasoning that $L \lesssim 0.03[m]^{-1}$ which in natural units can be translated to $L \lesssim 10^{-14}$ m. In order to get a more stringent bound, one should take into account other effects, which may show other dependencies on the physics of the extra dimensions. That investigation is, however, outside the scope of the present article.

Let us now consider the would-be large-momentum region for the vacuum polarization. This regime will be defined by the condition that $k^2 \gg m^2$, although $k$ (and $m$) will be assumed to be much smaller than $L^{-1}$. The latter is enforced in order to say that the mass of the Kaluza-Klein modes is much larger than the photon momentum. Under this assumption, one gets the expression:

$$\Pi_R^e(k^2) \sim \frac{2\,e^2}{\pi}\int_0^1 d\beta\;\beta(1-\beta)\,\ln\left[1+\beta(1-\beta)\frac{k^2}{m_{\rm eff}^2}\right], \tag{57}$$

where

$$m_{\rm eff} \equiv \frac{2\left|\sin\frac{\theta}{2}\right|}{L}. \tag{58}$$

We conclude that, as a consequence of the existence of the nonvanishing flux, the large-momentum behavior differs from the one that one has in standard QED, by the emergence of an effective mass $m_{\rm eff}$. This mass should, in order not to spoil the known antiscreening effect at short distances, be very small. Since $L$ is assumed to be very small, that can only be achieved with an extremely small $\theta$, namely $\theta \ll 1$. Hence,

$$m_{\rm eff} \equiv \frac{2\left|\sin\frac{\theta}{2}\right|}{L} \sim \frac{|\theta|}{L} \ll \frac{1}{L}. \tag{59}$$

In natural units, if $L^{-1} \equiv \Lambda$ is the large-momentum scale set by the Kaluza-Klein modes, and we want $m_{\rm eff}$ to be much smaller than the electron mass, since only in that situation we recover the expected behavior for the effective charge at small distances. Then we should have:

$$|\theta| \ll \frac{m}{\Lambda}. \tag{60}$$

## V. PARITY-BREAKING TERM

The parity-breaking term, $\Gamma_o$ is simply obtained by taking into account (37) and (33):

$$\Gamma_o \;=\; -\frac{ie^2}{16\pi^2}\,\Phi\int d^{3+1}x\,\tilde{F}_{\mu\nu}F_{\mu\nu}, \tag{61}$$

where we introduced the factor:

$$\Phi \;=\; \sum_{n=-\infty}^{\infty}\varphi_n; \tag{62}$$

the sum of this series is well-known [22], the result being:

$$\Phi \;=\; \arctan\left[\tanh\left(\frac{mL}{2}\right)\tan(\theta/2)\right]. \tag{63}$$

The possible effects due to this term are more difficult to elucidate, since they would require the existence of nontrivial Abelian gauge field background to manifest themselves. Within the present model, there is no room to accommodate them, except if singular configurations were included by hand.

## VI. CONCLUSIONS

To conclude, we summarize the main points we have explored this article: The vacuum polarization function exhibits physical effects due to the extra dimension and flux. Among those, the strongest one is due to the nonvanishing flux, parametrized by $\theta$, and manifests itself in the large-momentum behavior of the effective charge. Indeed, $\theta$ should be much smaller than the ratio between the electron mass and the (momentum) scale induced by the inverse of the compactification radius for this effect to be supressed. Besides, the effect of the nonvanishing flux is maximum when it reaches $\pi$. This is to be expected, since in that case there is no massless mode, and hence there is no natural way to dimensionally reduce the theory at the level of the fermionic field. That is, on the other hand, the case when $\theta = 0$, since it means that the $n = 0$ mode finds a natural $(3+1)$-dimensional interpretation and there is a smooth limit when $L \to 0$. We find that, finally, parity-breaking effects might be expected only if there were a compelling reason to know that the gauge field itself adopts a topologically nontrivial configuration; this cannot be done within the context of the present model.

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