

Vortices and ring dark solitons in nonlinear amplifying waveguides

Jie-Fang Zhang,^{1,*} Lei Wu (吴雷),² Lu Li,³ Dumitru Mihalache,⁴ and Boris A. Malomed⁵

¹*Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua, Zhejiang 321004, People's Republic of China*

²*Tianmu College, Zhejiang Forestry University, Lin'an 311300, People's Republic of China*

³*Institute of Theoretical Physics, Shanxi University, Taiyuan, Shanxi 030006, People's Republic of China*

⁴*Horia Hulubei National Institute for Physics and Nuclear Engineering, Magurele, Bucharest R-077125, Romania*

⁵*Department of Physical Electronics, Faculty of Engineering, Tel Aviv University, Tel Aviv IL-69978, Israel*

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We consider the generation and propagation of $(2 + 1)$ -dimensional beams in a nonlinear waveguide with the linear gain. Simple self-similar evolution of the beams is achieved at the asymptotic stage if the input beams represent the fundamental mode. On the contrary, if they carry vorticity or amplitude nodes (or phase slips), vortex tori and ring dark solitons (RDSs) are generated, featuring another type of the self-similar evolution, with an exponentially shrinking vortex core or notch of the RDS. Numerical and analytical considerations reveal that these self-similar structures are robust entities in amplifying waveguides, being *stable* against azimuthal perturbations.

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I. INTRODUCTION

The topic of self-similarity has been the focus of intense research interest in versatile areas of physics [1]. In nonlinear optics, the intrinsic self-similarity of the nonlinear Schrödinger (NLS) equation has led to the development of the symmetry-reduction method, which has been widely applied to search for exact and asymptotic self-similar solutions [2–9]. Such optical waves (*similaritons*) possess many attractive features that make them potentially useful for various applications in fiber-optic telecommunications and photonics, as they can maintain the overall shape while allowing their amplitudes and widths to vary, following the modulation of a system's parameters—dispersion, nonlinearity, gain, inhomogeneity, and others. Recent studies reveal that similaritons also exist in other fields of physics, including Bose-Einstein condensates and plasmas [10–12].

Among various types of self-similar modes, there has been an increasing interest in the study of asymptotically exact parabolic similaritons since their first experimental realization in normally dispersive fiber amplifiers [2]. One important advantage of these waves is that they are free of collapse and filamentation at high power levels. Another remarkable property of the similaritons is that the corresponding output profile is completely determined by the input power; that is, all initial profiles with the same power evolve toward the same parabolic similariton [2,3,13], which is very attractive for applications such as pulse (or beam) shaping or optical regeneration [14,15]. This conclusion, however, is correct only when the initial beams are of the fundamental type; that is, they do not have phase slips or phase singularities, and their amplitude profiles do not have nodes. If the input beam carries a phase slip, a dark soliton in the self-similar parabolic background is generated [16]. On the other hand, a spatiotemporal vortex torus will emerge if the input beam has an embedded vorticity [17–19].

In this work we study the generation and propagation of optical beams inside the two-dimensional nonlinear waveguide amplifier, focusing on effects of initial phase singularities, phase slips, and amplitude nodes.

II. MODEL AND ASYMPTOTIC EXACT SELF-SIMILAR BEAMS

The nonlinearity of the medium is assumed to be *self-defocusing*, with refractive index $n = n_0 - n_2 I$, where I is the beam's intensity and n_2 is positive (the cubic self-defocusing is possible in semiconductor waveguides). The respective governing equation for the paraxial optical beam in such an amplifying waveguide is the $(2 + 1)$ -dimensional NLS equation,

$$i \frac{\partial u}{\partial z} + \frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u - |u|^2 u = i \frac{g}{2} u, \quad (1)$$

where the beam's envelope u , propagation distance z , spatial coordinates x and y , and gain g are normalized by $(k_0 n_2 L_D)^{-1/2}$, L_D , w_0 , and L_D^{-1} , respectively, with $k_0 = 2\pi n_0/\lambda$, $L_D = k_0 w_0^2$, and w_0 being, respectively, the wave number at the input wavelength λ , the diffraction length, and a characteristic transverse width.

Our first aim is to investigate the self-similar evolution of the optical beam in the asymptotic limit of $z \rightarrow \infty$ by considering the intrinsic self-similarity of Eq. (1). For the evolution to be exactly self-similar, the functional form of the beam's intensity profile must remain unchanged at different propagation distances. Accordingly, one can express the beam's intensity as $|u(z, x, y)|^2 = \exp[\int_0^z g(z') dz'] |U(X, Y)|^2 / \ell^2$, where $(X, Y) \equiv (x, y)/\ell$, with $\ell(z)$ being a positive function of the propagation distance that characterizes the evolution of the beam's width, and $|U(X, Y)|^2$ is a functional form satisfying the energy conservation condition, obtained after the subtraction of the gain effect,

$$\iint |U(X, Y)|^2 dX dY = U_0, \quad (2)$$

*Corresponding author: jf_zhang@zjnu.cn

with U_0 being the input power of the optical beam. With this intensity shape, the expansion velocity of the beam's intensity profile is $\mathbf{v} = \ell^{-1} (d\ell/dz) \mathbf{r}$, where $\mathbf{r} \equiv \{x, y\}$. This velocity is equal to the gradient of the beam's phase, $\mathbf{v} = \nabla\phi$ [20]. Therefore, the phase of the beam undergoing the self-similar evolution is $\phi = (2\ell)^{-1} (d\ell/dz) r^2 - \phi_0(z)$, where $\phi_0(z)$ is a phase offset. Thus, by introducing the following self-similar transformation,

$$u(x, y, z) = \frac{U(X, Y)}{\ell} \exp\left[\frac{1}{2} \int_0^z g(z') dz' + i\phi\right], \quad (3)$$

we cast Eq. (1) into the form of

$$\begin{aligned} & |U|^2 U - \frac{1}{2} \exp\left[-\int_0^z g(z') dz'\right] \nabla^2 U \\ & = \exp\left[-\int_0^z g(z') dz'\right] \left[\frac{d\phi_0}{dz} \ell^2 - \frac{1}{2} \frac{d^2\ell}{dz^2} \ell^3 R^2 \right] U, \end{aligned} \quad (4)$$

where $\nabla^2 = \partial^2/\partial X^2 + \partial^2/\partial Y^2$ and $R^2 = X^2 + Y^2$.

Without the loss of generality, we let $(d^2\ell/dz^2)\ell^3 = \exp[\int_0^z g(z') dz']$ and $(d\phi_0/dz)\ell^2 = \mu \exp[\int_0^z g(z') dz']$, where μ is a positive constant to be determined. Note that the coefficient in front of the diffraction term in Eq. (4), $\exp[-\int_0^z g(z') dz']/2$, vanishes at $z \rightarrow \infty$. Therefore, if $U(X, Y)$ is smooth with a finite value of $\nabla^2 U$, the diffraction term may be neglected in the asymptotic regime. The aforementioned consideration shows that the exact self-similar evolution of the optical beam can be achieved in the asymptotic limit, under which Eq. (4) yields

$$|U|^2 = \mu - \frac{R^2}{2}, \quad (5)$$

for $R^2 \leq 2\mu$, and $U = 0$ otherwise, where the positive constant $\mu = \sqrt{U_0/\pi}$ is determined by Eq. (2). Note that this profile is essentially the same as that produced by the Thomas-Fermi approximation for the ground-state solution of the two-dimensional Bose-Einstein condensates described by the Gross-Pitaevskii equation with the isotropic parabolic potential and repulsive interatomic interaction [21]. Thus, the exact self-similar beams are robust entities.

Hereafter we assume g to be a constant; hence the variable characterizing the change of the beam width is

$$\ell(z) = \sqrt{\frac{4}{g}} \exp\left(\frac{gz}{4}\right), \quad (6)$$

and the phase offset is

$$\phi_0(z) = \frac{\mu}{2} \left[\exp\left(\frac{gz}{2}\right) - 1 \right]. \quad (7)$$

From Eqs. (3) and (6), it follows that the amplitude and the half width of the exact self-similar beam increase exponentially as $\sqrt{\mu g} \exp(gz/4)/2$ and $\sqrt{8\mu/g} \exp(gz/4)$, respectively. As the optical beam expands, a phase-front curvature (*chirp*) develops. Using the stationary-phase method [22], one can find that the spatial spectrum of the optical beam is parabolic too. Note that the asymptotically exact self-similar evolution of the optical beam is predicated upon the assumption that U is smooth, so that the relative strength of the diffraction becomes negligible when compared to

that of the nonlinearity [6]. This condition does not hold if the input beam carries vorticity, nodes in its amplitude profile, or phase slips. As stated previously, the generation of vortex tori and ring dark solitons (RDSs) is expected in that case.

III. VORTICES AND RDSs EMBEDDED IN THE PARABOLIC BACKGROUND

Now we proceed to the investigation of the dynamics of a vortex torus of topological charge S in the asymptotic limit. As the local beam's intensity vanishes at the center of the vortex, the relative strength of the diffraction in the vortical core is much larger than that of the nonlinearity. The size of the core range is determined by the coherence length, $L = S\sqrt{1/2|u|^2}$. In the presence of the gain, the beam's amplitude increases exponentially; therefore L decreases at the same rate. Therefore, at the asymptotic stage, the vortex becomes very narrow when compared to the whole beam, which makes it reasonable to seek a solution to Eq. (4) in the asymptotic limit by setting

$$U = \Psi(R) \exp(iS\theta), \quad (8)$$

where θ is the azimuthal angle, and neglecting terms containing gradients of Ψ . Then, we obtain an asymptotic expression for the local intensity of the vortex torus,

$$|\Psi|^2 = \mu_{\text{vort}} - \frac{R^2}{2} - \frac{p}{R^2}, \quad (9)$$

where $p \equiv (S^2/2) \exp(-gz)$ and μ in Eq. (5) is replaced by μ_{vort} such that the phase offset keeps the form of Eq. (7) with μ replaced by μ_{vort} .

From Eq. (9), it follows that, in this approximation, the power in the core of the vortex vanishes at $R^2 < \mu_{\text{vort}} - \sqrt{\mu_{\text{vort}}^2 - 2p} \approx p/\mu_{\text{vort}}$ (in the exact solution, the beam's intensity vanishes as $R^{|S|}$ at $R \rightarrow 0$). On the other hand, the last term in Eq. (9) may be neglected at large R ; hence the beam's intensity profile keeps the parabolic form outside of the core. Note that the local beam's intensity of the vortex torus at large distances is slightly greater than that of the fundamental soliton [Eq. (5)] for the same input power, since the optical field is removed in the narrow core of the vortex. Therefore, the difference of μ_{vort} , which is generated by the local nonlinearity, from μ is almost negligible. Thus it can be concluded, from the aforementioned expressions, that the radius of the vortex decreases exponentially, as

$$r = R\ell \simeq \sqrt{\frac{2}{g\mu}} S \exp\left(-\frac{gz}{4}\right), \quad (10)$$

whereas the width and amplitude of the whole beam increase exponentially, like those of the fundamental spatial soliton. This analysis predicts peculiarities of the self-similar evolution of the vortex torus, in comparison with the vortex-free beam.

It should be emphasized that now U is not only a function of X and Y , but also a function of z [Eqs. (8) and (9)]. The latter contradicts the definition of the function $U = U(X, Y)$ in the transformation [Eq. (3)]. This contradiction, however, is negligible: if U is also a function of z , Eq. (4) will contain an additional term $i\ell^2 \exp[-\int_0^z g(z') dz'] U_z$. From

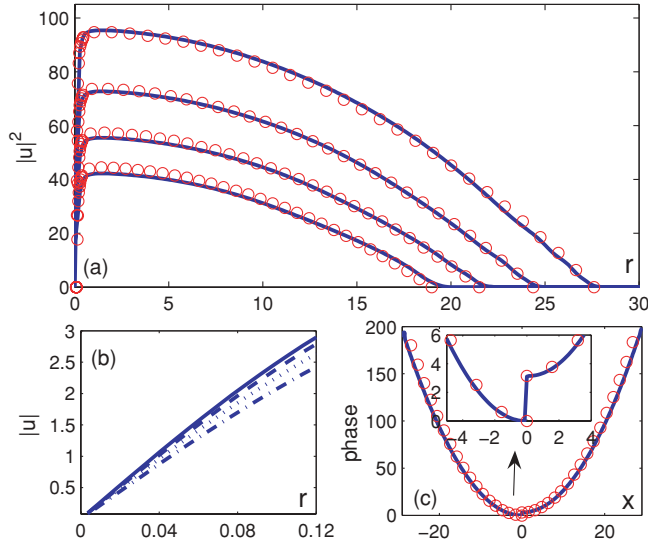


FIG. 1. (Color online) (a) The self-similar evolution of the vortex torus of topological charge $S = 1$ in the waveguide amplifier, starting with input beam $u(0, r) = r \exp(-r^2/2) \exp(i\theta)$, for $g = 2$. From the top to the bottom, the propagation distance is 5.25, 5, 4.75, and 4.5, respectively. (b) The beam's amplitude in the core of the vortex, which is a linear function of r . (c) The phase [phase offset $\phi_0(z)$ is ignored] of the vortex torus at propagation distance $z = 5.25$, with the inset showing the phase slip π in the x direction, to confirm that the topological charge is 1. Here and in the following figures, the solid lines and circles represent results of numerical simulations of Eq. (1) and the analytical predictions, respectively.

Eq. (9) it follows that this term can be approximated by $2i \exp(-gz/2) \mu_{\text{vort}}^{-1/2} p R^{-2}$, which is about $\exp(gz/2)$ times less than the last term in Eq. (9), and hence it is negligible at the asymptotic limit. The same argument also holds for the dark solitons embedded in the parabolic background shown in Fig. 2.

Figure 1 displays the evolution of an initial Laguerre-Gaussian beam of topological charge $S = 1$ toward the vortex torus, whose intensity and phase profiles at the asymptotic stage of the evolution are found to be in good agreement with the predictions based on Eqs. (3), (6), (8), and (9). Further numerical simulations show that initial Laguerre-Gaussian beams with topological charges up to $S = 10$ evolve into the corresponding vortex tori, indicating that the latter are robust entities in the waveguide amplifiers.

Now, we proceed to the generation and propagation of dark solitons in waveguide amplifiers. Since quasi-one-dimensional (stripe-shaped) dark solitons eventually decay into vortices, we focus on the RDSs [23]. Numerical simulations reveal that dark solitons of this type can be generated from input beams with nodes in the amplitude profile [Fig. 2(a)] or from Gaussian beams with ring phase slips [Fig. 2(b)]. In the asymptotic limit, the approximate RDS solution of Eq. (4) can be written as

$$|U|^2 = \left(\mu - \frac{r^2}{2\ell^2} \right) \left\{ 1 - A^2 \text{sech}^2 \left[\sqrt{\mu} A \exp \left(\frac{gz}{2} \right) \frac{r - r_c}{\ell} \right] \right\}, \quad (11)$$

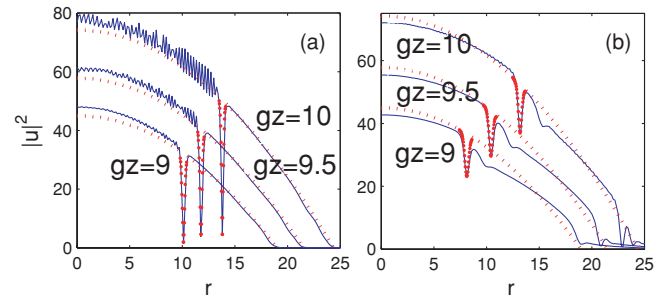


FIG. 2. (Color online) The self-similar evolution of radial dark solitons in the amplifying waveguide, obtained from input beams of total power π with initial profiles (a) $u(0, r) = W^{-3} \exp(-r^2/2W^2)(r^2 - W^2)$ and (b) $u(0, r) = W^{-1} \exp(-r^2/2W^2) \exp(i\pi \tanh[2(r - W)]/2)$. The gain parameter is $g = 2$ and the initial width is $W = 1.5$.

where A and r_c characterize the depth and location of the center of the RDS, respectively. From Eq. (11) it follows that the effective width of the black RDS decreases exponentially, as $(2/\sqrt{g\mu}) \exp(-gz/4)$ [cf. Eq. (10)]. The character of self-similar evolution of the RDS, which is qualitatively similar to that of the vortex, is confirmed by simulations displayed in Fig. 2, where one could also find that (i) the RDSs generated from the initial phase slip are much more shallow than those generated from the initial amplitude node and (ii) the RDSs move outward due to their self-repulsive nature and the expanding parabolic background. Finally, Fig. 3 shows the output beam intensity at $z = 5$ in the amplifying waveguide with $g = 2$, as obtained from the input beam of power π , where one could see that the vortex and RDSs may coexist, being embedded in the self-similar parabolic background.

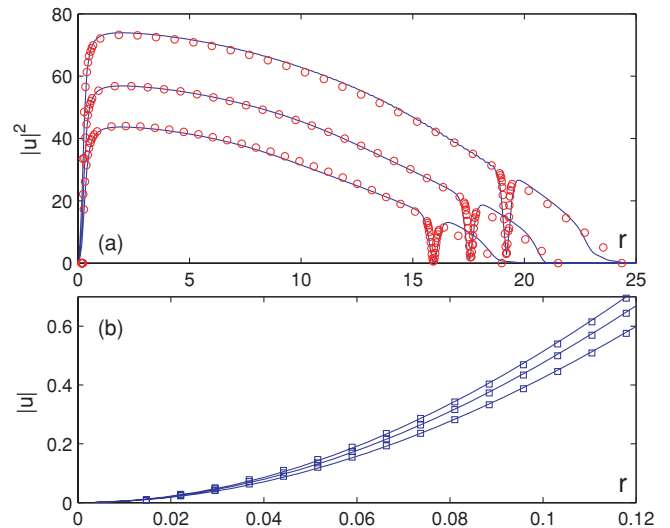


FIG. 3. (Color online) (a) The coexistence of a vortex with topological charge $S = 2$ and radial dark soliton. (b) The beam's amplitude in the core of the vortex (solid lines), where the squares represent the numerical fits of $|u|$ as quadratic functions of r . The initial condition is $u(0, r) = 0.309 \exp(-r^2/2)(r^2 - 3/2)r^2 \exp(2i\theta)$, and the gain parameter is $g = 2$. From the top to the bottom, the propagation distance is 5, 4.75, and 4.5, respectively.

Next we investigate the stability of RDSs. By introducing the transformation $U \rightarrow U \exp(i\phi_0)$, we rewrite Eq. (4) as $iU_Z + \nabla^2 U / [2 \exp(gz)] - |U|^2 U = R^2 U / 2$, where the new propagation distance $Z \equiv (1/2)[\exp(gz/2) - 1]$, as obtained from Eq. (7). If the coefficient in front of the diffraction term is 1 ($g = 0$), this equation is just the governing equation for the dynamics of RDS in Bose-Einstein condensates with the harmonic potential [23], and it was demonstrated that: (i) in the limit case of a quasi-plane soliton, the mass of the deep RDS is 2 and its oscillation frequency is $\omega = \sqrt{1/2}$, and (ii) the RDSs persists up to $Z \simeq 3T = 6\pi/\omega$ and then begins to decay due to the azimuthal modulation instability (AMI). When $g \neq 0$, the effective mass of the deep RDS is $2 \exp(gz)$ and its oscillation frequency is approximately $\sqrt{1/2} \exp(-gz/2)$. Thus, the RDS would reach the AMI threshold at $Z' \approx 6\sqrt{2}\pi \exp(gz/2)$. In fact, because Z is always smaller than Z' , we infer that the instability threshold will never be reached in the present model, which is indeed confirmed by direct numerical simulations. Thus, the RDSs are robust entities in the waveguide amplifiers, contrary to the common opinion that AMI is

inevitable for two-dimensional topological modes, such as RDS [17,24,25].

IV. CONCLUSIONS

In summary, within the framework of the $(2+1)$ -dimensional NLS equation including the linear gain, we have found three types of self-similar optical beams in nonlinear amplifying waveguides, namely, fundamental beams with parabolic profiles, robust vortex tori with different values of the topological charge, and radial dark solitons with a parabolic background. These results were obtained in the analytical form using a Thomas-Fermi type of asymptotic approximation and were confirmed by numerical simulations.

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