Theory of quenching quantum fluctuations of a laser system with a ladder-type configuration

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The theory of a laser system with a ladder-type configuration is studied in detail based on the quantum Langevin approach. By using an external field to link the lower lasing level with another atomic level, whose decay rate is much larger, laser intensity significantly increases and the quantum-limited linewidth can be quenched. We also discuss the spectrum of fluctuations of the output field, and the result shows that the fluctuations at low frequencies can be much suppressed too. On the other hand, this quenching approach can realize a laser output between two atomic levels, whose decay rates do not satisfy the usual lasing condition that the decay rate of the lower lasing level should be larger than that of the upper lasing level. It will be very useful to realize a laser output with the wavelength we want. This quenching approach has been widely used in the absorption spectrum of the ytterbium optical lattice clock and in the laser cooling approach for calcium atoms. Here we apply it in the stimulated emission of lasers.

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I. INTRODUCTION

The quantum-limited linewidth of a single-mode laser with homogeneously broadened medium was originally derived by Schawlow and Townes [1] as

$$\Delta\nu_{\mathrm{ST}} = \frac{\kappa}{2I_o}, \tag{1}$$

where κ is the cold-cavity loss rate and I_o is the intracavity intensity of laser light in units of photon number. However, the result of Schawlow and Townes is only valid for a good-cavity laser, for which the decay rate of the atomic polarization is much larger than the cavity loss rate. Several authors [2–4] have generalized the theory of Schawlow and Townes for both good- and bad-cavity regimes, and the quantum-limited linewidth of a bad-cavity laser has been experimentally researched in Ref. [5]. In Ref. [6], the authors have investigated the influence of the finite atom-field interaction time on the laser linewidth with particular emphasis on the linewidth of the active optical clock with atomic beam configuration [7].

How to exceed the quantum limit described by Eq. (1) has played an important role in quantum optics. In every laser the spontaneous emission, which inevitably occurs in the gain medium, acts as a fundamental noise source due to the random-phase diffusion process arising from the addition of spontaneously emitted photons with random phases to the laser field. References [8,9] show that the quantum noise can be suppressed below the standard Schawlow-Townes limit by preparing the atomic systems in coherent superposition of states as in the Hanle effect and quantum beat experiments. These lasers operating via such a phase coherence of atomic ensemble are known as correlated spontaneous emission lasers (CEL). An interesting aspect of CEL is that it is possible to eliminate the spontaneous emission quantum noise in the relative linewidths by correlating the two spontaneous emission noise events. In a two-photon CEL [10,11], a cascade transition involving three-level atoms is coupled to only one mode of the radiation field. A well-defined coherence between the upper and lower levels leads to a correlation of the light field.

There are many other schemes for quantum noise reduction. In Ref. [12], the authors have pointed out that light transmitted through a resonant atomic system with electromagnetically induced transparency displays reduced phase-noise fluctuations. In Refs. [13–15], a scheme based on the phase-matching effect of the nonadiabatic interaction of two quasimonochromatic fields with Λ-type atoms has been investigated, and the corresponding experimental verification has been demonstrated in Ref. [16], in which the initial beat linewidth of 1 MHz between two lasers can be reduced to 5 kHz.

In this article we theoretically investigate a method of quenching laser linewidth in the three-level system with a ladder-type configuration. As shown in Refs. [17–19], using an additional laser field to link the target level with another level can adjust the decay rate of the target level. If we apply this method to the three-level laser system (as shown in Fig. 1) by linking the lower lasing level $|b\rangle$ with another atomic level $|c\rangle$, whose decay rate is much larger, the quantum-limited laser linewidth of the corresponding two-level laser system ($|a\rangle$ and $|b\rangle$) can be exceeded. We also discuss the spectrum of fluctuations of the output field and show that fluctuations at low frequencies can be strongly reduced too. This quenching approach has been applied in the ytterbium lattice-based optical clock [20,21], in which the optical transition line ${}^1S_0$-${}^3P_0$ is quenched by applying magnetic field to link levels ${}^3P_0$ with ${}^3P_1$, and in the laser cooling approach for calcium atoms [17–19]. On the other hand, this quenching method can also realize a laser output between two levels whose decay rates do not satisfy the usual lasing condition, which requires that the decay rate of the lower lasing level should be larger than that of the upper lasing level (e.g., $\gamma_b + \gamma_b' > \gamma_a$ shown in Fig. 1). It will be very useful for us to obtain a laser output with the wavelength we want.

Our investigation is completely based on the standard quantum Langevin approach [22–25]. In Sec. II, we establish

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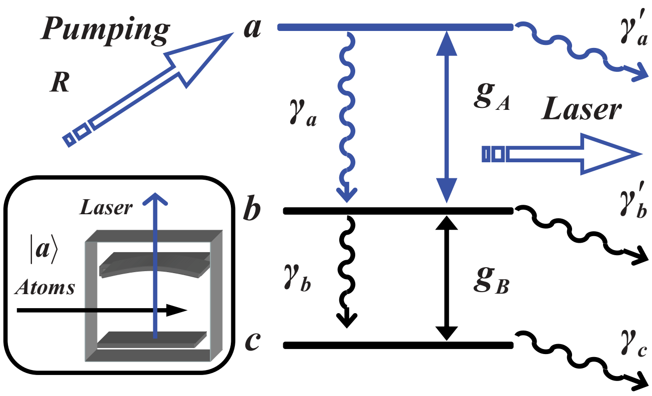

FIG. 1. (Color online) Relevant atomic levels. The inserted picture shows the scheme of a general laser system. Atoms with the ladder-type configuration fly into the single-mode cavity. The two lasing levels are $|a\rangle$ and $|b\rangle$. The pumping rate of the upper lasing level is R, and the coupling strength of the lower lasing level with another level $|c\rangle$ is g_B.

the basic Heisenberg-Langevin equations for the single-atom and macroscopic atomic variables and convert the quantum Langevin equations into c-number stochastic differential equations. In Sec. III, we discuss the steady-state solution. In Sec. IV, we discuss the non-Markovian behavior of phase fluctuations and the laser linewidth, and we use the helium-neon (He-Ne) laser as an example to specifically discuss this quenching method. We also calculate the spectrum of fluctuations of field outside the cavity, and we investigate the influence of fluctuations of the quenching field on the laser linewidth. Finally, our conclusion is summarized in Sec. V. All the diffusion coefficients for the single- and macroscopic atomic Langevin noise operators and for the c-number Langevin noise variables are listed in the Appendix.

II. QUANTUM LANGEVIN EQUATIONS

A. Physical model

In the two-level laser system, the quantum-limited laser linewidth is given by [4]

$$D = \left(\frac{\gamma_{ab}}{\gamma_{ab}+\kappa}\right)^2 \frac{g^2\mathcal{N}_{ao}}{I_o\gamma_{ab}}, \qquad (2)$$

where $\mathcal{N}_{ao}$ is the steady-state value of the upper lasing level, γ_{ab} is the damping rate of atomic polarization, κ is the loss rate of the cavity, g is the atom-cavity coupling constant, and I_o is the photon number inside the cavity. Here we have used the same symbols as in Ref. [4]. From Eq. (2) one can see that for a given system, the quantum-limited laser linewidth is determined by the efficiency of atoms on the upper lasing level to produce the coherent photons, $I_o/\mathcal{N}_{ao}$. If this efficiency is improved under the same pumping rate, the quantum-limited laser linewidth can be quenched. This can be realized by applying the similar approaches of Refs. [17–21], in which the decay rate of a certain level can be adjusted by linking it with another level.

Here we consider the laser system shown in Fig. 1. Atoms with a ladder-type structure fly into a single-mode cavity. The cross section of the laser beam is so wide (or the speed of the atom is so slow) that the atom-field interaction time is determined by the lifetime of atomic levels, not the atomic transit time. In this case, one need not consider the influence of finite atom-field interaction time on laser linewidth [13]. Additionally, since the direction of atomic movement is perpendicular to the transmission direction of the laser field, it is unnecessary to consider the Doppler effect. Thus, what we consider here is a homogeneously broadened laser system. The laser transition is $|a\rangle \leftrightarrow |b\rangle$, which is coupled with the cavity mode with detuning $\Delta = \omega_L - \omega_{ab}$ (ω_L is the frequency of cavity mode and ω_{ab} is the frequency of atomic transition $|a\rangle \leftrightarrow |b\rangle$). The atom-cavity coupling constant g_A is given by

$$g_A = \sqrt{\frac{1}{2\hbar\epsilon_o\omega_L V}}\,\omega_{ab}|\mathrm{d}_{ab}|,$$

where $|\mathrm{d}_{ab}|$ is the magnitude of the atomic dipole moment corresponding to transition $|a\rangle \leftrightarrow |b\rangle$. The lower lasing level $|b\rangle$ is coupled with another level $|c\rangle$ (which is not the atomic ground state) by an external field (laser field or magnetic field) with the coupling strength g_B. Before entering the cavity, all atoms are pumped onto the upper lasing state $|a\rangle$. γ_a', γ_b', and γ_c are the decay rates of atoms on levels $|a,b,c\rangle$ to other levels; γ_a and γ_b are the spontaneous decay rates of the transitions $|a\rangle \leftrightarrow |b\rangle$ and $|b\rangle \leftrightarrow |c\rangle$; and Γ_A, Γ_B, and Γ_O are the damping rates of atomic polarizations, which obey the inequalities $2\Gamma_A \geqslant \gamma_a+\gamma_a'+\gamma_b+\gamma_b'$, $2\Gamma_B \geqslant \gamma_b+\gamma_b'+\gamma_c$, and $2\Gamma_O \geqslant \gamma_a+\gamma_a'+\gamma_c$. Here we do not need to consider the decay rate of the transition $|a\rangle \leftrightarrow |c\rangle$ since for usual laser systems it is forbidden to transition between $|a\rangle$ and $|c\rangle$.

As one knows, for a certain number of atoms on the upper lasing level, a larger decay rate of the lower lasing level can improve the efficiency of atoms to produce coherent photons. In this case, the link between levels $|b\rangle$ and $|c\rangle$ is very important here. Besides the spontaneous emission, atoms on state $|b\rangle$ can also transit to state $|c\rangle$ via the stimulated emission transition, for which atoms can be quickly reduced from state $|b\rangle$, and consequently the photon number will increase under the same pumping rate, especially if the damping rate of level $|c\rangle$ is much larger than that of $|b\rangle$. Therefore, the external field used to link $|b\rangle$ and $|c\rangle$ plays a major role in two terms: to introduce the stimulated emission transition from $|b\rangle$ to $|c\rangle$, and to increase the effective decay rate of level $|b\rangle$ (as shown in Ref. [26]).

Here we should note that laser systems with a ladder-type configuration like that shown in Fig. 1 can be widely found in nature, for example, He-Ne lasers, CO_2 lasers, and argon lasers. In this case, the physical model considered here is quite a universal system.

B. Quantum Langevin equations for fields interacting with single atom

In the interaction picture with the rotating wave approximation, the Hamiltonian of the laser system shown in Fig. 1 is given by

$$H = \hbar g_A \sum_j \theta(t-t_j)\big(a^\dagger\sigma_A^j e^{i\Delta t} + \sigma_A^{j\dagger} a e^{-i\Delta t}\big) + \hbar\sum_j \theta(t-t_j)\big(g_B^*\sigma_B^j + g_B\sigma_B^{j\dagger}\big). \qquad (3)$$

Here $a^\dagger$ and a are the creation and annihilation operators for the electromagnetic field. σ_A^j and σ_B^j are the atomic polarization operators $(|b\rangle\langle a|)_j$ and $(|c\rangle\langle b|)_j$ for the jth atom. $\theta(t)$ is the unit step function [$\theta(t)=1$ for $t>0$, $\theta(t)=1/2$ for $t=0$, and $\theta(t)=0$ for $t<0$]. The cavity loss and atomic decay are modeled in the standard way by coupling the radiation field and each atom to heat reservoirs. From this interaction Hamiltonian, one can find the following quantum Langevin equations of field and atomic operators,

$$\dot{a}(t) = -(\kappa/2)a(t) - ig_A\sum_j \theta(t-t_j)\sigma_A^j(t)e^{i\Delta t} + F_\gamma(t), \quad (4)$$

$$\begin{aligned}\dot{\sigma}_{aa}^j(t) &= -(\gamma_a+\gamma_a')\sigma_{aa}^j(t) - i\theta(t-t_j)g_A \\ &\quad\times\left[\sigma_A^{j\dagger}(t)a(t)e^{-i\Delta t} - a^\dagger(t)\sigma_A^j(t)e^{i\Delta t}\right] + f_{aa}^j(t), \quad (5)\\ \dot{\sigma}_{bb}^j(t) &= -(\gamma_b+\gamma_b')\sigma_{bb}^j(t) + \gamma_a\sigma_{aa}^j(t) + i\theta(t-t_j)g_A \\ &\quad\times\left[\sigma_A^{j\dagger}(t)a(t)e^{-i\Delta t} - a^\dagger(t)\sigma_A^j(t)e^{i\Delta t}\right] \\ &\quad - i\theta(t-t_j)\left[g_B\sigma_B^{j\dagger}(t) - g_B^*\sigma_B^j(t)\right] + f_{bb}^j(t), \quad (6)\\ \dot{\sigma}_{cc}^j(t) &= -\gamma_c\sigma_{cc}^j(t) + \gamma_b\sigma_{bb}^j(t) + i\theta(t-t_j) \\ &\quad\times\left[g_B\sigma_B^{j\dagger}(t) - g_B^*\sigma_B^j(t)\right] + f_{cc}^j(t), \quad (7)\\ \dot{\sigma}_A^j(t) &= -\Gamma_A\sigma_A^j(t) + i\theta(t-t_j)g_A\left[\sigma_{aa}^j(t) - \sigma_{bb}^j(t)\right] \\ &\quad\times a(t)e^{-i\Delta t} + i\theta(t-t_j)g_B^*\sigma_O^j(t) + f_A^j(t), \quad (8)\\ \dot{\sigma}_B^j(t) &= -\Gamma_B\sigma_B^j(t) + i\theta(t-t_j)g_B\left[\sigma_{bb}^j(t) - \sigma_{cc}^j(t)\right] \\ &\quad - i\theta(t-t_j)g_A a^\dagger(t)\sigma_O^j(t)e^{i\Delta t} + f_B^j(t), \quad (9)\\ \dot{\sigma}_O^j(t) &= -\Gamma_O\sigma_O^j(t) + i\theta(t-t_j)g_B\sigma_A^j(t) \\ &\quad - i\theta(t-t_j)g_A\sigma_B^j(t)a(t)e^{-i\Delta t} + f_O^j(t), \quad (10)\end{aligned}$$

where the single-atom operators for the jth atom are defined as $\sigma_{aa}^j = (|a\rangle\langle a|)^j$, $\sigma_{bb}^j = (|b\rangle\langle b|)^j$, $\sigma_{cc}^j = (|c\rangle\langle c|)^j$, $\sigma_O^j = (|c\rangle\langle a|)^j$, $\sigma_A^{j\dagger} = (|a\rangle\langle b|)^j$, $\sigma_B^{j\dagger} = (|b\rangle\langle c|)^j$, and $\sigma_O^{j\dagger} = (|a\rangle\langle c|)^j$. Above Heisenberg-Langevin equations have the same structure,

$$\dot{x}(t) = A_x(t) + f_x(t), \quad (11)$$

where $A_x(t)$ is the deterministic part of the equation and $f_x(t)$ is the quantum noise operator. The noise operators for the single-atom variables are δ-correlated in time,

$$\left\langle f_x^i(t)f_y^j(t')\right\rangle = \mathrm{d}(x,y)\delta_{ij}\delta(t-t'), \quad (12)$$

where $\mathrm{d}(x,y)$ is the diffusion coefficient. δ_{ij} makes only correlations between noise operators corresponding to the same atom be nonzero. Using the generalized dissipation-fluctuation theorem,

$$\mathrm{d}(x,y) = -\langle xA_y\rangle - \langle A_x y\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\langle xy\rangle, \quad (13)$$

one can calculate the diffusion coefficients. The nonvanishing terms are listed in the Appendix. Here we should denote that it is unnecessary to consider the influence of noise fluctuations of the external field, which is used to link state $|b\rangle$ with state $|c\rangle$. Since the noise of the external field can be controlled much lower than that of the atomic polarization σ_B, one can ignore its influence. More discussion can be found in Sec. IV C.

C. Quantum Langevin equations for fields interacting with macroscopic atoms

The macroscopic atomic operators can be defined by adding up all the individual atomic operators and taking into account the corresponding injection times into the cavity. Then, we have

$$N_{aa}(t) = \sum_j \theta(t-t_j)\sigma_{aa}^j(t), \quad (14)$$

$$N_{bb}(t) = \sum_j \theta(t-t_j)\sigma_{bb}^j(t), \quad (15)$$

$$N_{cc}(t) = \sum_j \theta(t-t_j)\sigma_{cc}^j(t), \quad (16)$$

$$M_A(t) = -i\sum_j \theta(t-t_j)\sigma_A^j(t), \quad (17)$$

$$M_B(t) = -i\sum_j \theta(t-t_j)\sigma_B^j(t), \quad (18)$$

$$M_O(t) = \sum_j \theta(t-t_j)\sigma_O^j(t). \quad (19)$$

The additional factor $(-i)$ is introduced for mathematical convenience. Operators $M_A(t)$, $M_B(t)$, and $M_O(t)$ represent the macroscopic atomic polarizations, and $N_{aa}(t)$, $N_{bb}(t)$, and $N_{cc}(t)$ represent the macroscopic populations of levels $|a,b,c\rangle$, respectively. With the aforementioned definitions and Eqs. (4)–(10), quantum Langevin equations for the electromagnetic field and macroscopic atomic operators can be expressed as

$$\begin{aligned}\dot{a}(t) &= -(\kappa/2)a(t) + g_A M_A(t)e^{i\Delta t} + F_\gamma(t), \quad (20)\\ \dot{N}_{aa}(t) &= R - (\gamma_a+\gamma_a')N_{aa}(t) - g_A[M_A^\dagger(t)a(t)e^{-i\Delta t} \\ &\quad + a^\dagger(t)M_A(t)e^{i\Delta t}] + F_{aa}(t), \quad (21)\\ \dot{N}_{bb}(t) &= -(\gamma_b+\gamma_b')N_{bb}(t) + \gamma_a N_{aa}(t) \\ &\quad + g_A[M_A^\dagger(t)a(t)e^{-i\Delta t} + a^\dagger(t)M_A(t)e^{i\Delta t}] \\ &\quad - [g_B M_B^\dagger(t) + g_B^* M_B(t)] + F_{bb}(t), \quad (22)\\ \dot{N}_{cc}(t) &= -\gamma_c N_{cc}(t) + \gamma_b N_{bb}(t) \\ &\quad + [g_B M_B^\dagger(t) + g^* M_B(t)] + F_{cc}(t), \quad (23)\\ \dot{M}_A(t) &= -\Gamma_A M_A(t) + g_A[N_{aa}(t) - N_{bb}(t)]a(t)e^{-i\Delta t} \\ &\quad + g_B^* M_O(t) + F_A(t), \quad (24)\\ \dot{M}_B(t) &= -\Gamma_B M_B(t) + g_B[N_{bb}(t) - N_{cc}(t)] \\ &\quad - g_A a^\dagger(t)M_O(t)e^{i\Delta t} + F_B(t), \quad (25)\\ \dot{M}_O(t) &= -\Gamma_O M_O(t) + g_A M_B(t)a(t)e^{-i\Delta t} \\ &\quad - g_B M_A(t) + F_O(t), \quad (26)\end{aligned}$$

where the total noise operators for the macroscopic atomic operators are given by

$$F_{aa}(t) = \sum_j \delta(t-t_j)\sigma_{aa}^j(t_j) + \sum_j \theta(t-t_j)f_{aa}^j(t) - R, \quad (27)$$

$$F_{bb}(t) = \sum_j \delta(t-t_j)\sigma_{bb}^j(t_j) + \sum_j \theta(t-t_j)f_{bb}^j(t), \quad (28)$$

$$F_{cc}(t) = \sum_j \delta(t - t_j)\sigma_{cc}^j(t_j) + \sum_j \theta(t - t_j) f_{cc}^j(t), \quad (29)$$

$$F_A(t) = -i \sum_j \delta(t - t_j)\sigma_A^j(t_j) - i \sum_j \theta(t - t_j) f_A^j(t), \quad (30)$$

$$F_B(t) = -i \sum_j \delta(t - t_j)\sigma_B^j(t_j) - i \sum_j \theta(t - t_j) f_B^j(t), \quad (31)$$

$$F_O(t) = \sum_j \delta(t - t_j)\sigma_O^j(t_j) + \sum_j \theta(t - t_j) f_O^j(t), \quad (32)$$

and the average of each macroscopic noise operator is zero. In deriving these macroscopic noise operators, we have used

$$\langle \delta(t - t_j)\rangle_S = R \int_{-\infty}^{+\infty} \mathrm{d}t_j \delta(t - t_j) = R, \quad (33)$$

where R is the mean pumping rate and $\langle \cdots \rangle_S$ denotes the classical average over the injection times and the fact that each atom was on the upper lasing state $|a\rangle$ at its injection time. The correlation function between two quantum Langevin forces $F_\alpha(t)$ and $F_\beta(t)$ can be expressed as

$$\langle F_\alpha(t) F_\beta(t)\rangle = \mathrm{D}(\alpha, \beta)\delta(t - t'), \quad (34)$$

where the diffusion coefficients $\mathrm{D}(\alpha, \beta)$ are listed in Appendix.

D. Equivalent c-number stochastic Langevin equations for a normally ordered product of operators

Now we derive the stochastic c-number Langevin equations, which are equivalent to the quantum Langevin equations. For this we should choose some particular ordering for products of atomic and field operators, because the c-number variables commute with each other while the operators do not. Here we choose the normal ordering of atomic and field operators, that is, $a^\dagger(t)$, $M_A^\dagger(t)$, $M_B^\dagger(t)$, $M_O^\dagger(t)$, $N_{aa}(t)$, $N_{bb}(t)$, $N_{cc}(t)$, $M_O(t)$, $M_B(t)$, $M_A(t)$, $a(t)$. The stochastic c-number variables corresponding to the operators $a(t)$, $M_A(t)$, $M_B(t)$, $M_O(t)$, $N_{aa}(t)$, $N_{bb}(t)$, and $N_{cc}(t)$ are denoted by $\mathcal{A}(t)$, $\mathcal{M}_A(t)$, $\mathcal{M}_B(t)$, $\mathcal{M}_O(t)$, $\mathcal{N}_{aa}(t)$, $\mathcal{N}_{bb}(t)$, and $\mathcal{N}_{cc}(t)$, respectively. Equations (20)–(26) are already written in normal order and one can directly obtain the equations for the corresponding c-number variables. Here we do not list them. The stochastic c-number Langevin forces of the corresponding quantum noise operators are denoted by $\mathcal{F}_\mu(t)$ with $\mu = \gamma$, aa, bb, cc, A, B, and O, and we have the properties $\langle \mathcal{F}_\mu(t)\rangle = 0$ and $\langle \mathcal{F}_\mu(t)\mathcal{F}_\nu(t')\rangle = \mathcal{D}_{\mu\nu}\delta(t - t')$. The c-number diffusion coefficients $\mathcal{D}_{\mu\nu}$ can be obtained from the quantum diffusion coefficients by transforming the expressions in the fluctuation-dissipation theorem of Eq. (13) into the normally ordered operator products. If the operator product $\hat{x}\hat{y}$ is normally ordered, its expectation value is equal to the expectation value of the corresponding c-number product. Hence we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{x}\hat{y}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle xy\rangle. \quad (35)$$

Using again the generalized dissipation-fluctuation theorem, we find that

$$\mathcal{D}_{xy} = \mathrm{D}_{xy} + \langle \hat{x}\hat{A}_y\rangle + \langle \hat{A}_x\hat{y}\rangle - \langle xA_y\rangle - \langle A_x y\rangle. \quad (36)$$

Thus one could derive the c-number diffusion coefficients, and all the nonvanishing ones are listed in the Appendix.

III. STEADY-STATE SOLUTION

The steady-state solution for the mean values of the field and atomic variables can be obtained from the c-number dynamic equations. These solutions are denoted by the subscript "o". Here, for the sake of simplicity, we only consider the resonant case $\Delta = 0$. In this case, the steady-state value of the field amplitude can be expressed as $\mathcal{A}_o = \frac{2g_A}{\kappa}\mathcal{M}_{Ao}$, which denotes that the field is completely determined by the atomic polarization $\mathcal{M}_{Ao}$. As we know, the optical phase is randomly distributed between 0 and 2π in the stationary state. Therefore, we can choose the arbitrary mean value of the optical phase to be zero, which is quite convenient since then both field $\mathcal{A}_o$ and polarization $\mathcal{M}_{Ao}$ become real, and further g_B, $\mathcal{M}_{Bo}$, and $\mathcal{M}_{Oo}$ are also real. The photon number inside the cavity is given by $I_o = \mathcal{A}_o^2$.

A. Special case $g_B = 0$

In the special case, $g_B = 0$, we have populations of levels $|a, b, c\rangle$,

$$\mathcal{N}_{aao} = \frac{1}{\gamma_a + \gamma_b + \gamma_b'}\left[R + (\gamma_b + \gamma_b')\frac{\Gamma_A\kappa}{2g_A^2}\right], \quad (37)$$

$$\mathcal{N}_{bbo} = \frac{1}{\gamma_a + \gamma_b + \gamma_b'}\left(R - \gamma_a'\frac{\Gamma_A\kappa}{2g_A^2}\right), \quad (38)$$

$$\mathcal{N}_{cco} = \frac{\gamma_b}{\gamma_c}\frac{1}{\gamma_a + \gamma_b + \gamma_b'}\left(R - \gamma_a'\frac{\Gamma_A\kappa}{2g_A^2}\right), \quad (39)$$

and the photon number I_o is given by

$$I_o = I_S(R/R_{\mathrm{TH}} - 1), \quad (40)$$

where I_S is the saturation intensity,

$$I_S = \frac{\gamma_a + \gamma_a'}{\gamma_a' + \gamma_b + \gamma_b'}\frac{\Gamma_A(\gamma_b + \gamma_b')}{2g_A^2}, \quad (41)$$

and R_{TH} is the threshold pumping rate,

$$R_{\mathrm{TH}} = \frac{\gamma_a + \gamma_a'}{\gamma_b + \gamma_b' - \gamma_a}\frac{\Gamma_A\kappa(\gamma_b + \gamma_b')}{2g_A^2}. \quad (42)$$

From (38) and (42) one can derive the necessary conditions for laser oscillation: $\gamma_b + \gamma_b' > \gamma_a$ and $R > \gamma_a'\frac{\Gamma_A\kappa}{2g_A^2}$. From (37)–(39) we have the relationship

$$\gamma_a'\mathcal{N}_{aao} + \gamma_b'\mathcal{N}_{bbo} + \gamma_c\mathcal{N}_{cco} = R, \quad (43)$$

which denotes that the loss rate of the atomic population is equal to the pumping rate. When $R \gg R_{\mathrm{TH}}$ and $I_o \gg I_S$, the populations of upper and lower lasing levels approach each other and the inversion goes to zero, corresponding to saturation.

B. General case

For the general case, $g_B \neq 0$, the steady-state values of atomic variables can be expressed in terms of the yet unknown photon number I_o,

$$\mathcal{N}_{aao} = \frac{R - \kappa I_o}{\gamma_a + \gamma_a'}, \tag{44}$$

$$\mathcal{N}_{bbo} = \frac{\eta_1(\gamma_c + 2\Gamma_O C)}{(\gamma_b + \gamma_b')\gamma_c + (\gamma_b' + \gamma_c)2\Gamma_O C} R + \frac{(\eta_2 - C)\gamma_c + \eta_2 2\Gamma_O C}{(\gamma_b + \gamma_b')\gamma_c + (\gamma_b' + \gamma_c)2\Gamma_O C} \kappa I_o, \tag{45}$$

$$\mathcal{N}_{cco} = \frac{\eta_1(\gamma_b + 2\Gamma_O C)}{(\gamma_b + \gamma_b')\gamma_c + (\gamma_b' + \gamma_c)2\Gamma_O C} R + \frac{\gamma_b' C + \eta_2(\gamma_b + 2\Gamma_O C)}{(\gamma_b + \gamma_b')\gamma_c + (\gamma_b' + \gamma_c)2\Gamma_O C} \kappa I_o, \tag{46}$$

and the photon number inside the cavity I_o can be obtained from the following equation

$$\frac{C}{2} + \frac{\Gamma_O}{\kappa I_o}[\mathcal{N}_{aao} - (1 - C)\mathcal{N}_{bbo} - C\mathcal{N}_{cco}] = \frac{\Gamma_O \Gamma_A + g_B^2}{2g_A^2 I_o}, \tag{47}$$

where the dimensionless parameter,

$$C \equiv \frac{g_B^2}{\Gamma_B \Gamma_O + g_A^2 I_o}, \tag{48}$$

is also related to I_o, and

$$\eta_1 \equiv \frac{\gamma_a}{\gamma_a + \gamma_a'}, \quad \eta_2 \equiv \frac{\gamma_a'}{\gamma_a + \gamma_a'}.$$

From Eqs. (44)–(47), one can numerically calculate $\mathcal{N}_{aao}$, $\mathcal{N}_{bbo}$, $\mathcal{N}_{cco}$, and I_o (or $\mathcal{A}_o$), and further obtain $\mathcal{M}_{Ao}$, $\mathcal{M}_{Bo}$, and $\mathcal{M}_{Oo}$. In the linear approximation $\Gamma_B \Gamma_O \gg g_A^2 I_o$, Eq. (47) can be expressed in the same form as Eq. (40), of which the photon number I_o is proportional to the pumping rate R.

For simplicity, we introduce the dimensionless parameters

$$a \equiv \gamma_a/\kappa, \quad a' \equiv \gamma_a'/\kappa, \quad b \equiv \gamma_b/\kappa, \quad b' \equiv \gamma_b/\kappa, \quad c \equiv \gamma_c/\kappa,$$

$$A \equiv \Gamma_A/\kappa, \quad B \equiv \Gamma_B/\kappa, \quad O \equiv \Gamma_O/\kappa, \quad g \equiv g_A/\kappa.$$

In Fig. 2, we show the dependence of the photon number I_o on the pumping rate R and the coupling strength g_B calculated from Eqs. (44)–(47). The boundary of the contour lines in Fig. 2(a) gives the laser threshold, and photon number I_o as a function of g_B under different pumping rates R is shown in Fig. 2(b). One can see that for a certain pumping rate R photon number I_o increases for suitable g_B and decreases for larger g_B. This is because under suitable g_B the atomic population of level $|b\rangle$ can be reduced via the stimulated emission transition to level $|c\rangle$, and since the decay rate of level $|c\rangle$ is larger than that of $|b\rangle$ ($c > b + b'$), atoms on state $|b\rangle$ can be further decreased. However, for larger g_B, we are left with $\mathcal{N}_{bbo} < \mathcal{N}_{cco}$ and the stimulated absorption transition from $|c\rangle$ to level $|b\rangle$ plays a major role, which can suppress the stimulated emission transition from $|a\rangle$ to level $|b\rangle$. All these can be found in Fig. 2(c), which shows the atomic populations of levels $|a, b, c\rangle$ as functions of g_B under a certain pumping rate R. One can see that $\mathcal{N}_{bbo}$ is larger than $\mathcal{N}_{cco}$ for small g_B, and as a result atoms on level $|b\rangle$ can transit to level $|c\rangle$ via the stimulated emission with transition rate g_B and the photon number I_o increases. The population of level $|a\rangle$ also decreases for small g_B. However, for much larger g_B, the populations of levels $|b\rangle$ and $|c\rangle$ approach each other and atoms accumulate on the upper lasing level $|a\rangle$. In this case, laser output is quenched. Equation (44) gives the opposite changing trend between $\mathcal{N}_{aao}$ and I_o.

There are two ways to reduce the atoms on state $|b\rangle$: (i) go back to the ground state via state $|c\rangle$ for small g_B; (ii) stimulate absorption of laser photons and transition back to state $|a\rangle$ for large g_B. The former process will enlarge the laser field, while the latter reduces the laser photons inside the cavity. From the steady-state equation of $\mathcal{M}_{Oo}$

$$\mathcal{M}_{Oo} = \frac{1}{\Gamma_O}(g_A \mathcal{A}_o \mathcal{M}_{Bo} - g_B \mathcal{M}_{Ao}), \tag{49}$$

one can see that $\mathcal{M}_{Oo}$ is negative for larger g_B, while it is positive for small g_B. $\mathcal{M}_{Oo}$ can strongly influence the polarizations of $\mathcal{M}_{Ao}$ and $\mathcal{M}_{Bo}$. Positive $\mathcal{M}_{Oo}$ will enhance $\mathcal{M}_{Ao}$, increase the coherence between two lasing levels, and enlarge the photon number I_o, while negative $\mathcal{M}_{Oo}$ can reduce the photon number.

C. Laser with $\gamma_b + \gamma_b' < \gamma_a$

As concluded in Sec. III A, the lasing condition for the case of $g_B = 0$, is given by $\gamma_b + \gamma_b' > \gamma_a$. However, it is unnecessary for the case of $g_B \neq 0$. As we have pointed out

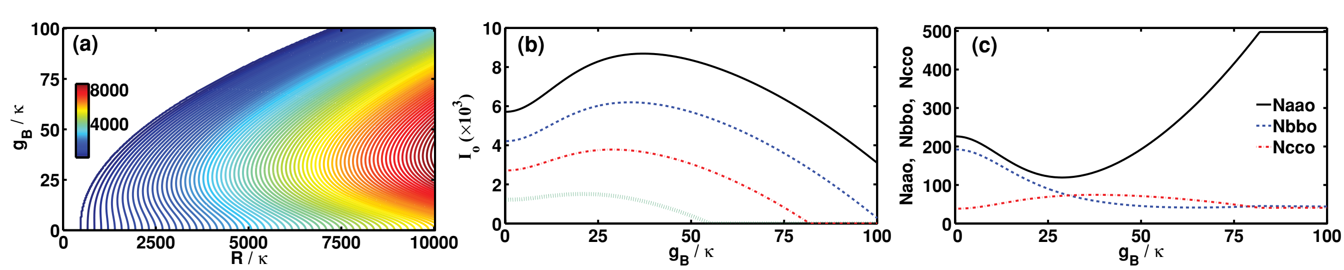

FIG. 2. (Color online) Dependence of photon number I_o on pumping rate R and coupling strength g_B. (a) The contour lines of photon number I_o. (b) I_o as a function of g_B under different pumping rates R [$R/\kappa = 2.5 \times 10^3$ for the green line (bottom line); 5×10^3 for the red line (second line from the bottom); 7.5×10^3 for the blue line (third line from the bottom); 10^4 for the black line (top line)]. (c) Atomic populations on difference levels with pumping rate $R/\kappa = 5000$. $\mathcal{N}_{aao}$, top line; $\mathcal{N}_{bbo}$, second line from the top; $\mathcal{N}_{cco}$, bottom line. For all curves $a = 5$, $a' = 5$, $b = 10$, $b' = 10$, $g = 0.5$, $c = 50$, $A = 17$, $B = 40$, and $O = 30$.

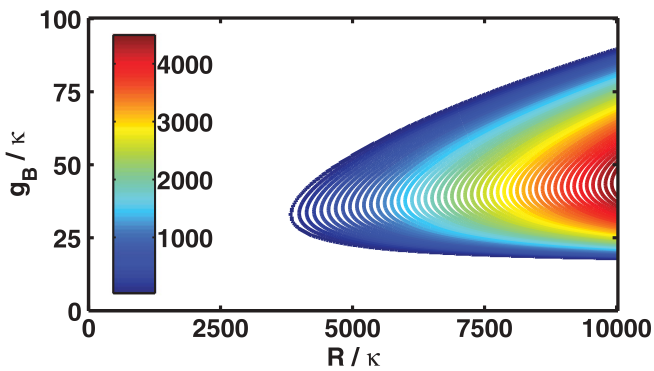

FIG. 3. (Color online) Dependence of photon number I_o on pumping rate R and coupling strength g_B for the case of $\gamma_b + \gamma_b' < \gamma_a$ with $a = 20$, $a' = 0$, $b = 5$, $b' = 5$, $g = 0.5$, $c = 50$, $A = 30$, $B = 40$, and $O = 50$.

previously, the coupling strength g_B can change the decay rate of lower lasing state $|b\rangle$. The laser field can be produced if we use an external field to link states $|b\rangle$ and $|c\rangle$ with a suitable strength g_B, despite $\gamma_b + \gamma_b' < \gamma_a$. In this case, this quenching method is an effective way to realize a laser generation with the wavelength we want, since for a certain atomic transition line the common lasing condition $\gamma_b + \gamma_b' > \gamma_a$ may not be satisfied.

Figure 3 displays the dependence of photon number I_o on pumping rate R and coupling strength g_B, and the boundary of the contour lines gives the laser threshold. One can see that, due to the coupling between $|b\rangle$ and $|c\rangle$, laser output can be realized for nonzero g_B, although the common lasing condition ($\gamma_b + \gamma_b' > \gamma_a$) is not satisfied.

Here we have discussed the steady-state solutions of the laser system. Next, we investigate the evolution of the quantum fluctuations around the steady-state solutions.

IV. QUANTUM FLUCTUATIONS OF THE LASER FIELD AROUND STEADY STATE

To investigate the small fluctuations of the laser field and atomic variables around steady states we consider all the variables, as usual, as the sum of the steady-state solution and a small fluctuating term. For example, for $\mathcal{N}_{aa}(t)$ we set $\mathcal{N}_{aa}(t) = \mathcal{N}_{aao} + \delta\mathcal{N}_{aa}(t)$ and in the same way for the other variables. Based on Eqs. (20)–(26), one can obtain a set of linear equations. Here, we have expressed the dynamic variables as a sum of the steady-state values and small fluctuations. It should be assumed that the laser is operating sufficiently above threshold so that the fluctuations of dynamic variables are much smaller than their steady-state values. Next, we take the Fourier transform of all variables and convert the differential equations into algebraic equations, for example,

$$\delta\mathcal{N}_{aa}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt\, \delta\mathcal{N}_{aa}(t) e^{i\omega t}.$$

In order not to overcharge the notation, we adopt the same symbol for both members of a Fourier-transform pair, which will therefore get distinguished through the time or frequency argument. We may set $\mathcal{F}_\gamma(t) = 0$ since the mean value and the correlation functions of this force with all the other variables, as well as the autocorrelation function, are zero. Here, for simplicity, we do not list the linear equations for the Fourier amplitudes. The Fourier-transformed fluctuation forces satisfy the equation

$$\langle \mathcal{F}_\alpha(\omega)\mathcal{F}_\beta(\omega')\rangle = 2\mathcal{D}_{\alpha\beta}\delta(\omega + \omega'). \tag{50}$$

The solution of this linear system is straightforward. The field phase quadrature component of field fluctuations inside the cavity, which is defined as $\delta Y(\omega) \equiv \frac{1}{2i}[\delta\mathcal{A}(\omega) - \delta\mathcal{A}^*(-\omega)]$, can be expressed as

$$\delta Y(\omega) = \frac{g_A}{2i} \frac{g_A g_B \mathcal{A}_o[\mathcal{F}_B(\omega) - \mathcal{F}_B^*(-\omega)] + g_B \tilde{\Gamma}_B(\omega)[\mathcal{F}_O(\omega) - \mathcal{F}_O^*(-\omega)] + \Theta(\omega)[\mathcal{F}_A(\omega) - \mathcal{F}_A^*(-\omega)]}{(\kappa/2 - i\omega)\left[\tilde{\Gamma}_A(\omega)\Theta(\omega) + g_B^2\tilde{\Gamma}_B(\omega)\right] - g_A^2\Delta(\omega)}, \tag{51}$$

with the following shorthand,

$$\Delta(\omega) = \Theta(\omega)(\mathcal{N}_{aao} - \mathcal{N}_{bbo}) + g_A g_B \mathcal{A}_o \mathcal{M}_{Oo} + g_B \mathcal{M}_{Bo}\tilde{\Gamma}_B(\omega), \tag{52}$$

$$\Theta(\omega) = \tilde{\Gamma}_O(\omega)\tilde{\Gamma}_B(\omega) + g_A^2\mathcal{A}_o^2, \tag{53}$$

and $\tilde{\Gamma}_{A,B,O}(\omega) = (\Gamma_{A,B,O} - i\omega)$. From Eq. (51) one can see that the field phase fluctuation comes from all the noises of atomic polarizations $\mathcal{F}_A(t)$, $\mathcal{F}_B(t)$, and $\mathcal{F}_O(t)$. In the special case $g_B = 0$, we obtain

$$\delta Y(\omega) = \frac{g_A}{2i} \frac{[\mathcal{F}_A(\omega) - \mathcal{F}_A^*(-\omega)]}{(\kappa/2 - i\omega)\tilde{\Gamma}_A(\omega) - g_A^2(\mathcal{N}_{aao} - \mathcal{N}_{bbo})}, \tag{54}$$

which is the same as the result of the two-level laser system [4]. The field phase fluctuation completely comes from the noise of atomic polarizations $\mathcal{F}_A(t)$.

A. Laser linewidth

The quantum-limited laser linewidth is also called the intrinsic or natural linewidth, which originates from the quantum fluctuations of field around its mean value. The usual treatment rests on the approximation that linewidth arises from fluctuations of the field phase described by phase diffusion. The autocorrelation function of phase quadratures is δ function correlated,

$$\langle \delta Y(\omega)\delta Y(\omega')\rangle = (\delta^2 Y)_\omega \delta(\omega + \omega'). \tag{55}$$

For a small fluctuation of the field phase, the spectrum of phase fluctuations is simply related to the spectrum of the phase quadrature component of field fluctuations, namely,

$$(\delta\phi^2)_\omega = \frac{1}{I_o}(\delta Y^2)_\omega. \tag{56}$$

Based on Eq. (56) one can directly calculate the correlation function of the time derivative of field phase fluctuation:

$$\langle \delta\dot{\varphi}(t)\delta\dot{\varphi}(t')\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-t')}\omega^2(\delta\phi^2)_\omega. \quad (57)$$

When $|t-t'|$ is much shorter than all the other characteristic times of the laser system, this expression becomes

$$\langle \delta\dot{\varphi}(t)\delta\dot{\varphi}(t')\rangle = D\delta(t-t'), \quad (58)$$

which corresponds to a Markovian time evolution for the field phase, and D gives the laser linewidth [27]. From Eq. (51), we obtain

$$D = \frac{g_A^2}{I_o} \times \frac{g_A^2 g_B^2 I_o \Gamma_B \mathcal{N}_{bbo} + \left[g_B^2\Gamma_B^2\Gamma_O + \Theta(\omega=0)^2\Gamma_A\right]\mathcal{N}_{aao} + g_B^2 g_A \mathcal{A}_o \Gamma_B \Gamma_A \mathcal{M}_{Ao} + g_A \mathcal{A}_o g_B \Theta(\omega=0)\Gamma_A \mathcal{M}_{Oo}}{\left\{\frac{\kappa}{2}(\Gamma_B\Gamma_O + \Gamma_A\Gamma_B + \Gamma_A\Gamma_O) + \Gamma_A\Gamma_B\Gamma_O + \left(\frac{\kappa}{2}+\Gamma_A\right)g_A^2 I_o + \left(\frac{\kappa}{2}+\Gamma_B\right)g_B^2 - g_A^2[(\Gamma_O+\Gamma_B)(\mathcal{N}_{aao}-\mathcal{N}_{bbo}) + g_B\mathcal{M}_{Bo}]\right\}^2}, \quad (59)$$

which is unrelated to the pumping statistics.

1. Special case: $g_B = 0$

In this special case, laser linewidth can be simplified as

$$D = \left(\frac{\Gamma_A}{\Gamma_A + \kappa/2}\right)^2 D_{\rm ST}, \quad (60)$$

where $D_{\rm ST}$ is the usual Schawlow-Townes diffusion coefficient [22]

$$D_{\rm ST} = \frac{g_A^2 \mathcal{N}_{aao}}{I_o\Gamma_A}. \quad (61)$$

This expression coincides with the one given in Refs. [2,3]. If the atomic polarization decay rate is much faster than the cavity loss rate, that is, $\Gamma_A \gg \kappa/2$, the additional factor $[\Gamma_A/(\Gamma_A+\kappa/2)]^2$ is close to unity and we are left with the usual Schawlow-Townes diffusion coefficient. But in the opposite case, $\Gamma_A \ll \kappa/2$, we have $D \ll D_{\rm ST}$. Figure 4 shows the dependence of laser linewidth D on the pumping rate R. One can see that laser linewidth decreases with increasing R (actually, increasing the photon number I_o).

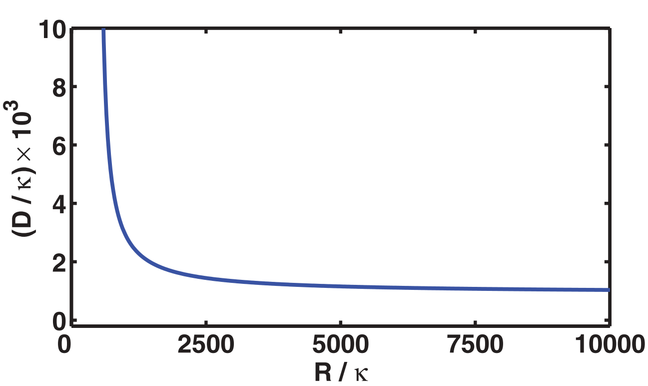

FIG. 4. (Color online) Dependence of laser linewidth D on pumping rate R in the special case $g_B = 0$ with the same parameters as in Fig. 2.

2. General case: $g_B \neq 0$

In Fig. 5, we show the dependence of laser linewidth D on the pumping rate R and coupling strength g_B. As shown in Fig. 5(a), laser linewidth close to threshold is much larger than that far away from threshold. Thus, increasing the pumping rate R (in fact increasing the photon number I_o) can reduce the laser linewidth, but what we are mostly concerned about is the influence of the coupling strength g_B on the laser linewidth D under a certain pumping rate R. As shown in Fig. 5(b), laser linewidth D decreases for suitable g_B and increases for larger g_B.

Generally, laser linewidth D is inversely proportional to the efficiency of atoms on the upper lasing level to produce the coherent photons, $I_o/\mathcal{N}_{aao}$. As shown in Fig. 2(c), for lower coupling strength g_B, we have $\mathcal{N}_{bbo} > \mathcal{N}_{cco}$, and the stimulated emission transition from $|b\rangle$ to $|c\rangle$ plays a major role in the atom-field interaction. In this case, besides the spontaneous emission the atomic population of level $|b\rangle$ can be reduced via the stimulated emission transition from $|b\rangle$ to $|c\rangle$. On the other hand, for the nonzero g_B, atoms will be on a superposition state of $|b\rangle$ and $|c\rangle$, whose damping rate can be adjusted by turning the coupling strength g_B. If $\gamma_c > \gamma_b + \gamma_b'$, the decay rate of atoms on the lower lasing level can be enhanced. Thus, the atomic population of level $|a\rangle$ can be further decreased via the stimulated emission transition from $|a\rangle$ to $|b\rangle$ and the photon number can greatly increase. For both reasons, the efficiency of atoms on the upper lasing level in producing photons $I_o/\mathcal{N}_{aao}$ can increase under the same pumping rate R, and the laser linewidth can be quenched, as shown in Fig. 5(b). However, with a further increase in the coupling strength g_B, we have $\mathcal{N}_{bbo} < \mathcal{N}_{cco}$, and the stimulated absorption transition from $|c\rangle$ to $|b\rangle$ plays a major role in the atom-field interaction, which can suppress the stimulated emission from $|a\rangle$ to $|b\rangle$. Atoms will accumulate on level $|a\rangle$, and the intensity of the laser field will decrease to zero, which reduces the efficiency of atoms to produce coherent photons. Therefore, the laser linewidth reaches a minimum and then increases again.

As we have said before, the positive $\mathcal{M}_{Oo}$ denotes that atoms on level $|b\rangle$ can be reduced via the stimulated emission transition from $|b\rangle$ to $|c\rangle$, while the negative $\mathcal{M}_{Oo}$ denotes that atoms on level $|b\rangle$ can be reduced by the stimulated absorption

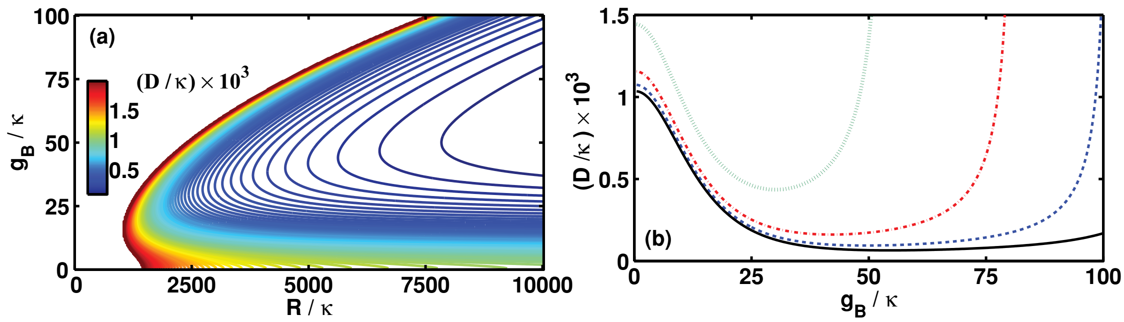

FIG. 5. (Color online) Dependence of laser linewidth D on pumping rate R and coupling strength g_B. (a) The contour lines of laser linewidth D. (b) D as a function of g_B under different pumping rates R [$R/\kappa = 2.5 \times 10^3$ for the green line (top line); 5×10^3 for the red line (second line from the top); 7.5×10^3 for the blue line (third line from the top); 10^4 for the black solid line (bottom line)]. All parameters are the same as in Fig. 2.

transition (absorb the laser photons inside the cavity) from $|b\rangle$ to $|a\rangle$. From Eq. (49) one can see that for the small photon number I_o (corresponding small pumping rate R), $\mathcal{M}_{Oo}$ can easily change from the positive value to the negative one by the lower coupling strength g_B. Thus, the laser linewidth cannot be much quenched, such as the green line (top line) in Fig. 5(b). However, for the larger photon number I_o (corresponding large pumping rate R), $\mathcal{M}_{Oo}$ is still positive under the larger coupling strength g_B, and the laser linewidth can be much quenched. Therefore, increasing the pumping rate R is a good way to quench the laser linewidth.

3. Example: Helium-neon laser

Here we take a helium-neon (He-Ne) laser as an example to show the approach of quenching laser linewidth. Figure 6 displays the level configuration of a He-Ne laser expressed in Paschen notation. Although a He-Ne laser usually operators with inhomogeneous broadening, one can always use some methods (e.g., the external-cavity feedback induced by an external optical grating) to realize the homogeneous output. Thus, we only consider the homogeneously broadened case.

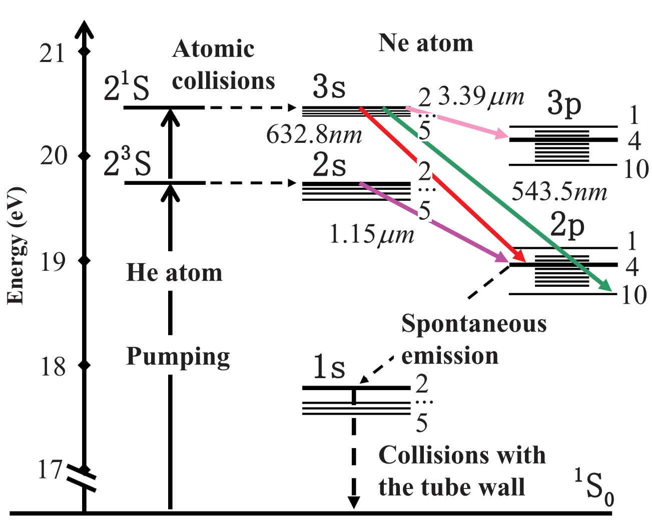

FIG. 6. (Color online) The level configuration of a He-Ne laser expressed in Paschen notation.

Atomic pumping can be realized by the energy transfer from He atoms to Ne atoms. We only consider the laser transition line $2s_2$-$2p_4$ (wavelength 1152.3 nm) of the Ne atom with the upper lasing level $2s_2$ ($\gamma_a = 2\pi \times 1.65$ MHz, $\gamma_a \gg \gamma_a'$) and the lower state $2p_4$ ($\gamma_b = 2\pi \times 13.27$ MHz, $\gamma_b \gg \gamma_b'$) [28]. Atoms on the lower lasing state can transit to state $1s_2$ via spontaneous emission transition. Level $1s_2$ is a metastable state and must be destroyed by the collisions with the discharge tube walls. Therefore, the damping rate of state $1s_2$ (γ_c) can be larger than γ_b (here we assume $\gamma_c \approx 10\gamma_b$) and atoms on state $1s_2$ can quickly go back to the ground state $^1S_0$. The typical cavity length (L) and laser diameter are about 20 cm and 4 mm, respectively. It is easy to obtain the atom-photon coupling strength $g_A = 2\pi \times 4.5$ kHz. The intensity reflectivity of the output cavity mirror is about 99%. In this case, the loss rate of photons inside the optical resonator is about $\kappa = 2\pi \times 24$ MHz. The relaxation rates for the polarization between two lasing states Γ_A is much larger than that for the population inversion [4]. Here we assume $\Gamma_A = 2\pi \times 75$ MHz and the other two relaxation rates, $\Gamma_B = 2\pi \times 90$ MHz and $\Gamma_O = 2\pi \times 80$ MHz. Additionally, we use an external laser field to couple the atomic transition $2p_4$-$1s_2$ with the coupling strength g_B, as in the same method discussed previously. In this case the quantum-limited laser linewidth can be quenched.

Figure 7 displays the dependence of intensity and linewidth of a He-Ne laser on the coupling strength g_B. All curves are calculated based on the laser transition line $2s_2$-$2p_4$. For suitable g_B, the laser intensity can be strongly enhanced (from 10.5 to 16 mW with $R = 10^{16}$ s^{-1}) with the laser linewidth decreasing (from 6.4 to 1.6 rad/s with $R = 10^{16}$ s^{-1}). If we further increase the pumping rate R, the laser linewidth can be strongly reduced (from 4 to 0.1 rad/s with a factor of 40 as shown in Fig. 7).

We have shown that the quantum-limited linewidth of a two-level laser system can be quenched by linking the lower lasing level $|b\rangle$ with another level $|c\rangle$, whose decay rate is much larger than that of $|b\rangle$. Next, we consider the spectrum of amplitude fluctuations of the output field.

B. Spectrum of the output field

We have discussed the laser field inside the cavity. Moreover, one is interested in the output field. The relation between

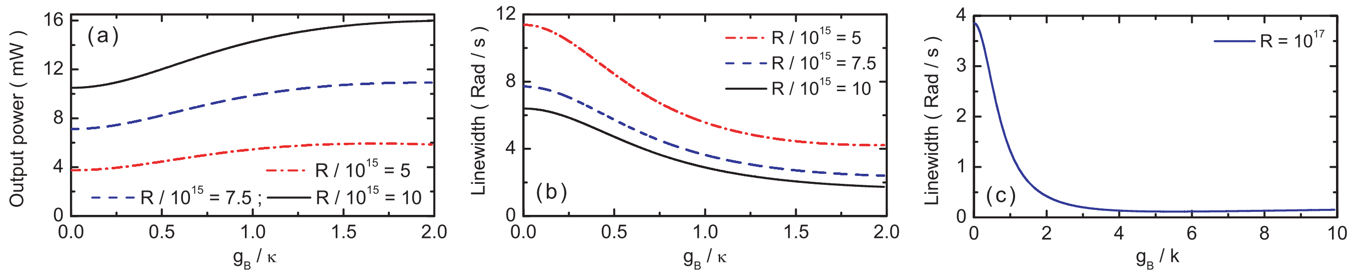

FIG. 7. (Color online) Dependence of intensity (a) and linewidth (b, c) of a He-Ne laser on the coupling strength g_B with different pumping rates R.

fields inside and outside the cavity has been established in Refs. [29–32]. Now we investigate the spectrum of fluctuations for the field transmitted through the cavity port. From Ref. [4], the spectrum of the output field can be expressed as

$$V_A(\omega) = 1 + 4\kappa(\delta X^2)_\omega, \tag{62}$$

where $(\delta X^2)_\omega$ is the spectrum of the amplitude quadrature component and can be derived from the autocorrelation function of the amplitude quadrature,

$$\langle \delta X(\omega)\delta X(\omega')\rangle = (\delta X^2)_\omega \delta(\omega + \omega'), \tag{63}$$

where the amplitude quadrature is defined as

$$\delta X(\omega) = \tfrac{1}{2}[\delta\mathcal{A}(\omega) + \delta\mathcal{A}^*(\omega)]. \tag{64}$$

The first term on the right-hand side of Eq. (62) corresponds to the shot-noise contribution. For a coherent state, we have $V_A = 1$. Therefore, $V_A < 1$ means squeezing in a quadrature component, and $V_A(\omega) = 0$ denotes the complete squeezing at some frequency ω [33]. Actually, this spectrum defined in this way corresponds to the normalized photocurrent obtained in a homodyne measurement of the field quadrature component. Since the expression of $V_A(\omega)$ is very complicated, we do not list it here and only show the consequences.

Figure 8 displays the spectrum of amplitude fluctuations for different statistical parameters p and coupling strength g_B, for operation far above threshold. As shown in Fig. 8(a), for $g_B = 0$ amplitude noise at low frequencies is reduced with increasing parameter p, and for a regular statistics $p = 1$ we obtain the limited noise reduction. The influence of coupling strength g_B on the spectrum of amplitude fluctuations for $p = 1$ has been shown in Fig. 8(b). One can see that the limited noise reduction for $p = 1$ and $g_B = 0$ is much exceeded with suitable $g_B \neq 0$ since the efficiency of atoms on the upper lasing level in producing photons increases for suitable g_B. However, for larger g_B, noise reduction at low frequencies increases again because of the saturation of $|b\rangle$ and $|c\rangle$.

C. Quantum fluctuations of the quenching field

In Sec. II, we ignore the influence of fluctuation of the quenching field on the laser field and assume the coupling strength g_B is constant in the case where the linewidth of the quenching field is smaller than the relaxation rate between levels $|b\rangle$ and $|c\rangle$. In this section, we discuss in detail the influence of fluctuation of the quenching field. Our present goal is to make sure that not too much effect is introduced into the laser linewidth when the two lower levels are linked by a fluctuation laser field.

As an admitted model [34] we elevate the Rabi frequency (coupling strength) g_B to an operator,

$$\hat{g}_B = g_B + G(t), \tag{65}$$

where g_B is a stationary real classical amplitude as before and $G(t)$ is a noise operator with the properties

$$[G(t), G^\dagger(t')] = \langle G(t)G^\dagger(t')\rangle = \frac{\Gamma_G}{2}\delta(t-t'),$$
$$\langle G(t)\rangle = \langle G^\dagger(t)\rangle = \langle G^\dagger(t)G(t')\rangle = 0. \tag{66}$$

Here, the effective bandwith Γ_G denotes the linewidth of the quenching field. Thus, the quenching field is treated not as a dynamical variable but as an externally imposed quantity with prescribed quantum statistics. Inserting the operator $\hat{g}_B$ into Eqs. (22)–(26), we get the new terms in the evolution

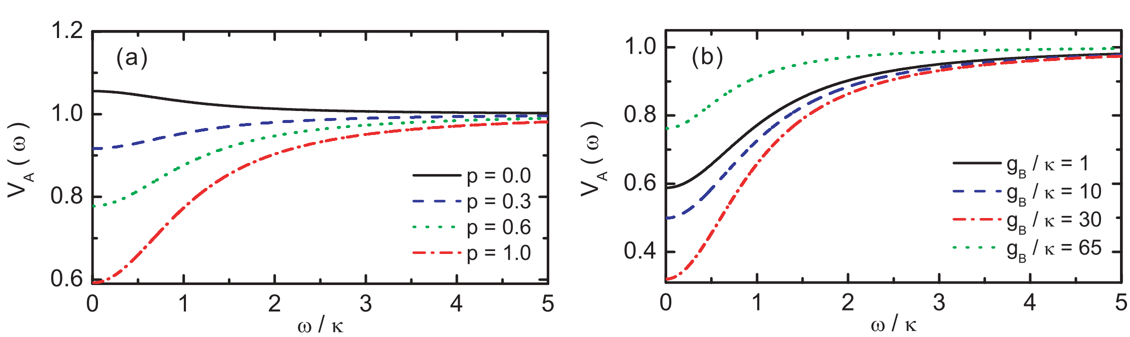

FIG. 8. (Color online) Spectrum of amplitude fluctuations with pumping rate $R/\kappa = 10^4$. (a) For different statistical parameters p with $g_B = 0$. (b) For different coupling strength g_B with $p = 1$. All the other parameters are the same as in Fig. 2.

equations of atomic variables:

$$[\dot{N}_{bb}(t)]_{\rm ad} = -[G(t)M_B^\dagger(t) + G^\dagger(t)M_B(t)], \quad (67)$$

$$[\dot{N}_{cc}(t)]_{\rm ad} = [G(t)M_B^\dagger(t) + G^\dagger(t)M_B(t)], \quad (68)$$

$$[\dot{M}_A(t)]_{\rm ad} = G^\dagger(t)M_O(t), \quad (69)$$

$$[\dot{M}_B(t)]_{\rm ad} = G(t)\,[N_{bb}(t) - N_{cc}(t)]\,, \quad (70)$$

$$[\dot{M}_O(t)]_{\rm ad} = -G(t)M_A(t). \quad (71)$$

Following the same calculating approach, one obtains an expression similar to Eq. (51) for the laser field phase quadrature $\delta Y(\omega)$, but an additional term,

$$\mathcal{W}(\omega) = \left[g_A g_B \mathcal{A}_o(\mathcal{N}_{bbo} - \mathcal{N}_{cco}) + g_B \frac{g_A^2 I_o}{\tilde{\Gamma}_O}\mathcal{M}_{Ao} - \Theta(\omega) \times \left(\mathcal{M}_{Oo} + \frac{g_B}{\tilde{\Gamma}_O}\mathcal{M}_{Ao}\right)\right]\left[\mathcal{G}(\omega) - \mathcal{G}^*(-\omega)\right], \quad (72)$$

is presented in the numerator. Here we have used the symbol $\mathcal{G}(\omega)$ to denote the c-number frequency variable corresponding to the noise operator $G(t)$.

Since the correlation functions of $\mathcal{W}(\omega)$ with all the other noise operators $\mathcal{F}_{A,B,O}(\omega')$ are zero, the additional term, which comes from the fluctuations of the quenching field and appears in the numerator of Eq. (59), is only $\langle\mathcal{W}(\omega)\mathcal{W}(\omega')\rangle_{\omega\to 0}$. By using Eqs. (49) and (53) we have

$$\langle\mathcal{W}(\omega)\mathcal{W}(\omega')\rangle_{\omega\to 0} = 0, \quad (73)$$

which is to say that the fluctuations of the quenching field do not influence the laser linewidth. It can be easily understood since the laser linewidth defined by Eq. (58) is the quantum-limited linewidth, which is also called the intrinsic linewidth and is certainly completely determined by the atomic system, not the fluctuations of external fields. This is also why the laser linewidth is unrelated to the pumping statistics, as discussed in Ref. [4]. However, that is not to say that the fluctuations of external fields do not influence the phase-noise spectrum of the laser field. One can see that the quantum-limited linewidth is defined around the lower frequency region of the phase-noise spectrum of the laser field. The fluctuations of external fields can influence the higher frequency region of a laser phase-noise spectrum, but not the lower frequency region. However, in the case of $\Gamma_G < \Gamma_B$, one can still ignore the influence of fluctuations of the quenching field.

From Eq. (51) of the field phase quadrature $\delta Y(\omega)$, one can obtain the noise contribution from the damping relaxation between levels $|b\rangle$ and $|c\rangle$ as

$$\mathcal{V}(\omega) = g_A g_B \mathcal{A}_o[\mathcal{F}_B(\omega) - \mathcal{F}_B^*(-\omega)], \quad (74)$$

and the contribution to the phase-noise spectrum of the laser field can be expressed as

$$\langle\mathcal{V}(\omega)\mathcal{V}(\omega')\rangle = D_{\mathcal{V}}(\omega)\delta(\omega + \omega'), \quad (75)$$

where $D_{\mathcal{V}}(\omega)$ is the corresponding diffusion coefficient. The contribution to laser phase-noise spectrum from the fluctuations of quenching field is given by

$$\langle\mathcal{W}(\omega)\mathcal{W}(\omega')\rangle = D_{\mathcal{W}}(\omega)\delta(\omega + \omega'), \quad (76)$$

where $D_{\mathcal{W}}$ is the corresponding diffusion coefficient. Figure 9 displays $D_{\mathcal{V}}$ and $D_{\mathcal{W}}$ as a function of noise frequency ω with

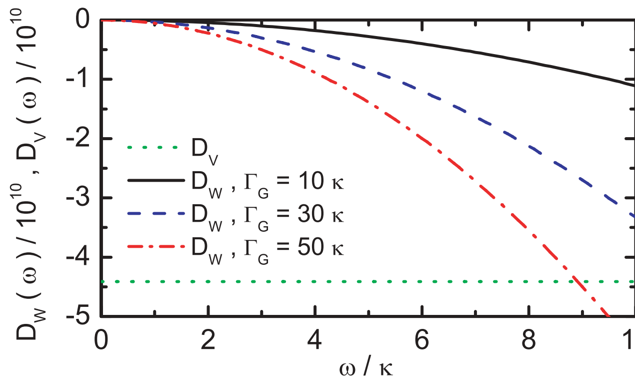

FIG. 9. (Color online) The diffusion coefficients of $D_{\mathcal{V}}(\omega)$ and $D_{\mathcal{W}}(\omega)$ as a function of noise frequency ω with $g_B/\kappa = 30$ and $R/\kappa = 10^4$. All the other parameters are the same as in Fig. 2.

$g_B/\kappa = 30$ and $R/\kappa = 10^4$. One can see that in the case of $\Gamma_G < \Gamma_B$, the influence of fluctuation of the quenching field is much smaller than that of the intrinsic damping relaxation of the atomic polarization between levels $|b\rangle$ and $|c\rangle$. Therefore, one can always ignore the influence of fluctuations of the quenching field if $\Gamma_G < \Gamma_B$, which is also suitable to the amplitude noise spectrum of the laser field, and we do not need any other conditions, such as $\Gamma_G < \Gamma_{A,O}$, $\Gamma_G < \gamma_{a,b}$, $\gamma'_{a,b}$, or even that Γ_G should be smaller than the laser linewidth D, since Γ_B is the largest intrinsic relaxation rate in a laser system.

V. CONCLUSION

Up to now, great theoretical and practical interest has been focused on the problem of quenching quantum noise in lasers, and many approaches have been proposed, including the regularization of pumping [25], CEL [8,9], and the reduction of spontaneous-emission noise for short measurement times due to the atomic memory effects [35].

Here we propose an approach to reduce the quantum noise in lasers by quenching the atomic population of the lower lasing level. The result shows that the standard quantum-limited linewidth for the two-level laser system can be exceeded and the amplitude fluctuations of the output field are also strongly reduced. On the other hand, this quenching approach can realize a laser output between two levels, whose decay rates do not satisfy the usual lasing condition $\gamma_b + \gamma'_b > \gamma_a$. It is very useful for us to obtain a laser generation with the wavelength we want.

Besides quenching the atomic population of the lower lasing level, this approach can also be used to quench the upper lasing level $|a\rangle$ by applying an external field to link $|a\rangle$ with another atomic level (e.g., $|d\rangle$), whose decay rate is *smaller* than $|a\rangle$. In this case, the damping rate of the atomic polarization between two lasing levels $|a\rangle$ and $|b\rangle$ could be smaller than the inherent value Γ_A. However, the pumping rate R will also be reduced since many atoms on the upper lasing level $|a\rangle$ are transferred to level $|d\rangle$, and the field intensity (photon number) will decrease. Therefore, we only apply this quenching method to the lower lasing level.

This quenching approach has been widely applied in experiments, such as laser cooling for calcium atoms [17,18] and the optical lattice ytterbium atomic clock [20,21]. Here we apply it in a laser system to reduce the laser noise. The laser system with a ladder-type configuration considered in this article is a universal physical model, which can be widely found in nature, for example, He-Ne gas lasers, CO_2 lasers, and argon lasers. This approach can also be used in other atomic configurations, for example, a V-type atomic system.

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APPENDIX: DIFFUSION COEFFICIENTS

A. Diffusion coefficients of the single-atom noise operators

Here we list the nonvanishing diffusion coefficients of the single-atom noise operators calculated from Eq. (13):

$$\begin{aligned}
\mathrm{d}(\sigma_{aa}, \sigma_{aa}) &= (\gamma_a + \gamma_a')\langle\sigma_{aa}(t)\rangle,\\
\mathrm{d}(\sigma_{aa}, \sigma_{bb}) &= -\gamma_a\langle\sigma_{aa}(t)\rangle,\\
\mathrm{d}(\sigma_{bb}, \sigma_{bb}) &= (\gamma_b + \gamma_b')\langle\sigma_{bb}(t)\rangle + \gamma_a\langle\sigma_{aa}(t)\rangle,\\
\mathrm{d}(\sigma_{bb}, \sigma_{cc}) &= -\gamma_b\langle\sigma_{bb}(t)\rangle,\\
\mathrm{d}(\sigma_{bb}, \sigma_{A}) &= (\gamma_b + \gamma_b')\langle\sigma_{A}(t)\rangle,\\
\mathrm{d}(\sigma_{cc}, \sigma_{cc}) &= \gamma_c\langle\sigma_{cc}(t)\rangle + \gamma_b\langle\sigma_{bb}(t)\rangle,\\
\mathrm{d}(\sigma_{cc}, \sigma_{A}) &= -\gamma_b\langle\sigma_{A}(t)\rangle,\\
\mathrm{d}(\sigma_{cc}, \sigma_{B}) &= \gamma_c\langle\sigma_{B}(t)\rangle,\\
\mathrm{d}(\sigma_{cc}, \sigma_{O}) &= \gamma_c\langle\sigma_{O}(t)\rangle,\\
\mathrm{d}(\sigma_A^\dagger, \sigma_{A}) &= (2\Gamma_A - \gamma_a - \gamma_a')\langle\sigma_{aa}(t)\rangle,\\
\mathrm{d}(\sigma_A^\dagger, \sigma_B^\dagger) &= (\Gamma_A + \Gamma_B - \Gamma_O)\langle\sigma_O^\dagger(t)\rangle,\\
\mathrm{d}(\sigma_B^\dagger, \sigma_{B}) &= (2\Gamma_B - \gamma_b - \gamma_b')\langle\sigma_{bb}(t)\rangle + \gamma_a\langle\sigma_{aa}(t)\rangle,\\
\mathrm{d}(\sigma_B^\dagger, \sigma_{O}) &= (\Gamma_O + \Gamma_B - \Gamma_A)\langle\sigma_{A}(t)\rangle, a\\
\mathrm{d}(\sigma_O^\dagger, \sigma_{O}) &= (2\Gamma_O - \gamma_a - \gamma_a')\langle\sigma_{aa}(t)\rangle.
\end{aligned}$$

All the other diffusion coefficients are zero.

B. Diffusion coefficients of the macroscopic atomic noise operators

Using the definitions of macroscopic Langevin forces defined by Eqs. (27) to (32), one can derive the following nonvanishing diffusion coefficients:

$$\begin{aligned}
\mathrm{D}(N_{aa}, N_{aa}) &= (\gamma_a + \gamma_a')\langle N_{aa}(t)\rangle + R(1-p),\\
\mathrm{D}(N_{aa}, N_{bb}) &= -\gamma_a\langle N_{aa}(t)\rangle,\\
\mathrm{D}(N_{bb}, N_{bb}) &= (\gamma_b + \gamma_b')\langle N_{bb}(t)\rangle + \gamma_a\langle N_{aa}(t)\rangle,\\
\mathrm{D}(N_{bb}, N_{cc}) &= -\gamma_b\langle N_{bb}(t)\rangle,\\
\mathrm{D}(N_{bb}, M_{A}) &= (\gamma_b + \gamma_b')\langle M_{A}(t)\rangle,\\
\mathrm{D}(N_{cc}, N_{cc}) &= \gamma_c\langle N_{cc}(t)\rangle + \gamma_b\langle N_{bb}(t)\rangle,\\
\mathrm{D}(N_{cc}, M_{A}) &= -\gamma_b\langle M_{A}(t)\rangle,\\
\mathrm{D}(N_{cc}, M_{B}) &= \gamma_c\langle M_{B}(t)\rangle,\\
\mathrm{D}(N_{cc}, M_{O}) &= \gamma_c\langle M_{O}(t)\rangle,\\
\mathrm{D}(M_A^\dagger, M_{A}) &= (2\Gamma_A - \gamma_a - \gamma_a')\langle N_{aa}(t)\rangle + R,\\
\mathrm{D}(M_A^\dagger, M_B^\dagger) &= -(\Gamma_A + \Gamma_B - \Gamma_O)\langle M_O^\dagger(t)\rangle,\\
\mathrm{D}(M_B^\dagger, M_{B}) &= (2\Gamma_B - \gamma_b - \gamma_b')\langle N_{bb}(t)\rangle + \gamma_a\langle N_{aa}(t)\rangle,\\
\mathrm{D}(M_B^\dagger, M_{O}) &= -(\Gamma_O + \Gamma_B - \Gamma_A)\langle M_{A}(t)\rangle,\\
\mathrm{D}(M_O^\dagger, M_{O}) &= (2\Gamma_O - \gamma_a - \gamma_a')\langle N_{aa}(t)\rangle + R.
\end{aligned}$$

In deriving we have used

$$\left\langle \sum_{j\neq k} \delta(t-t_j)\delta(t'-t_k) \right\rangle_S = R^2 - pR\delta(t-t'),$$

where p is a parameter which characterizes the pumping statistics: Poissonian excitation statistics correspond to $p = 0$, and for regular statistics we have $p = 1$. The intermediate cases between these two extremes are described by values of p between 1 and 0.

C. Diffusion coefficients of the c-number macroscopic atomic noise variables

Following Eq. (36), one can find all the nonvanishing c-number diffusion coefficients as follows:

$$\begin{aligned}
\mathcal{D}(\mathcal{M}_A^*, \mathcal{M}_A^*) &= 2g_A\langle\mathcal{A}^*(t)\mathcal{M}_A^*(t)\rangle,\\
\mathcal{D}(\mathcal{M}_A^*, \mathcal{M}_B^*) &= -(\Gamma_A + \Gamma_B - \Gamma_O)\langle\mathcal{M}_O^*(t)\rangle\\
&\quad - g_A\langle\mathcal{A}^*(t)\mathcal{M}_B^*(t)\rangle,\\
\mathcal{D}(\mathcal{M}_A^*, \mathcal{M}_O^*) &= g_A\langle\mathcal{A}^*(t)\mathcal{M}_O^*(t)\rangle,\\
\mathcal{D}(\mathcal{M}_A^*, \mathcal{N}_{bb}) &= (\gamma_b + \gamma_b')\langle\mathcal{M}_A^*(t)\rangle,\\
\mathcal{D}(\mathcal{M}_A^*, \mathcal{N}_{cc}) &= -\gamma_b\langle\mathcal{M}_A^*(t)\rangle,\\
\mathcal{D}(\mathcal{M}_A^*, \mathcal{M}_A) &= (2\Gamma_A - \gamma_a - \gamma_a')\langle\mathcal{N}_{aa}(t)\rangle + R,\\
\mathcal{D}(\mathcal{M}_B^*, \mathcal{M}_B^*) &= 2g_B^*\langle\mathcal{M}_B^*(t)\rangle,\\
\mathcal{D}(\mathcal{M}_B^*, \mathcal{N}_{aa}) &= -g_A\langle\mathcal{M}_O^*(t)\mathcal{A}(t)\rangle,\\
\mathcal{D}(\mathcal{M}_B^*, \mathcal{N}_{bb}) &= g_A\langle\mathcal{M}_O^*(t)\mathcal{A}(t)\rangle,\\
\mathcal{D}(\mathcal{M}_B^*, \mathcal{N}_{cc}) &= \gamma_c\langle\mathcal{M}_B^*(t)\rangle,\\
\mathcal{D}(\mathcal{M}_B^*, \mathcal{M}_O) &= -(\Gamma_O + \Gamma_B - \Gamma_A)\langle\mathcal{M}_A(t)\rangle,\\
\mathcal{D}(\mathcal{M}_B^*, \mathcal{M}_B) &= (2\Gamma_B - \gamma_b - \gamma_b')\langle\mathcal{N}_{bb}(t)\rangle + \gamma_a\langle\mathcal{N}_{aa}(t)\rangle,\\
\mathcal{D}(\mathcal{M}_O^*, \mathcal{N}_{cc}) &= \gamma_c\langle\mathcal{M}_O^*(t)\rangle,\\
\mathcal{D}(\mathcal{M}_O^*, \mathcal{M}_O) &= (2\Gamma_O - \gamma_a - \gamma_a')\langle\mathcal{N}_{aa}(t)\rangle + R,\\
\mathcal{D}(\mathcal{N}_{aa}, \mathcal{N}_{aa}) &= (\gamma_a + \gamma_a')\langle\mathcal{N}_{aa}(t)\rangle + R(1-p)\\
&\quad - g_A\langle\mathcal{M}_A^*(t)\mathcal{A}(t)\rangle - g_A\langle\mathcal{A}^*(t)\mathcal{M}_A(t)\rangle,\\
\mathcal{D}(\mathcal{N}_{aa}, \mathcal{N}_{bb}) &= -\gamma_a\langle\mathcal{N}_{aa}(t)\rangle + g_A\langle\mathcal{M}_A^*(t)\mathcal{A}(t)\rangle\\
&\quad + g_A\langle\mathcal{A}^*(t)\mathcal{M}_A(t)\rangle,\\
\mathcal{D}(\mathcal{N}_{bb}, \mathcal{N}_{bb}) &= (\gamma_b + \gamma_b')\langle\mathcal{N}_{bb}(t)\rangle + \gamma_a\langle\mathcal{N}_{aa}(t)\rangle\\
&\quad - g_A\langle\mathcal{M}_A^*(t)\mathcal{A}(t)\rangle - g_A\langle\mathcal{A}^*(t)\mathcal{M}_A(t)\rangle\\
&\quad - g_B\langle\mathcal{M}_B^*(t)\rangle - g_B^*\langle\mathcal{M}_B(t)\rangle,\\
\mathcal{D}(\mathcal{N}_{bb}, \mathcal{N}_{cc}) &= -\gamma_b\langle\mathcal{N}_{bb}(t)\rangle + g_B\langle\mathcal{M}_B^*(t)\rangle + g_B^*\langle\mathcal{M}_B(t)\rangle,\\
\mathcal{D}(\mathcal{N}_{bb}, \mathcal{M}_A) &= (\gamma_b + \gamma_b')\langle\mathcal{M}_A(t)\rangle,\\
\mathcal{D}(\mathcal{N}_{bb}, \mathcal{M}_B) &= g_A\langle\mathcal{A}^*(t)\mathcal{M}_O(t)\rangle,\\
\mathcal{D}(\mathcal{N}_{cc}, \mathcal{N}_{cc}) &= \gamma_c\langle\mathcal{N}_{cc}(t)\rangle + \gamma_b\langle\mathcal{N}_{bb}(t)\rangle\\
&\quad - g_B\langle\mathcal{M}_B^*(t)\rangle - g_B^*\langle\mathcal{M}_B(t)\rangle.
\end{aligned}$$

All the c-number diffusion coefficients are calculated in the resonant case.

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