Squeezing components in linear quantum feedback networks

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The aim of this article is to extend linear quantum dynamical network theory to include static Bogoliubov components (such as squeezers). Within this integrated quantum network theory, we provide general methods for cascade or series connections, as well as feedback interconnections using linear fractional transformations. In addition, we define input-output maps and transfer functions for representing components and describing convergence. We also discuss the underlying group structure in this theory arising from series interconnection. Several examples illustrate the theory.

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I. INTRODUCTION

The aim of this article is to develop systematic methods for describing and manipulating a class of quantum feedback networks (QFNs). Quantum networks are important because of the fundamental role they play in quantum technology, in particular quantum information, computing, and control. Knowledge of how to interconnect quantum (and classical) components in a network is a prerequisite for feedback control.

A QFN [1,2] is a physical network whose nodes (components) are open quantum systems and whose branches (wires) are quantum fields. Quantum dynamical components have linear relations between their input and output fields, but in general their internal physical variables may evolve nonlinearly. Furthermore, delays may be present due to the nonzero physical length of the branches and the finite speed of light. The network theory developed in [1,2] is based on quantum stochastic models of open quantum systems (Hudson-Parthasarathy [3], Gardiner-Collett [4]) and provides methods and tools for QFN modeling including series (or cascade) connections and feedback loops. The series connection defines a group operation in the class of open quantum dynamical systems. In this article, we consider the subclass of networks consisting of dynamical components whose internal variables evolve linearly. In [5], the theory was presented for components based on unitary transformations such as beam splitters and phase-shift modulators, but it does not include static components that require an external source of quanta for their operation, such as amplifiers and squeezers.

In quantum optics, one encounters the class of quantum linear networks with components implementing static linear transformations, called *Bogoliubov transformations*; see [6]. Examples of Bogoliubov components include devices capable of creating squeezed states of field out of a vacuum (squeezers). A series connection of static Bogoluibov components is given by matrix multiplication of representations of the Bogoliubov transformations and defines a group product for this class of systems.

These two classes of quantum networks, which are characterized by the nature of their components, are distinct but have elements in common. The beam splitter and phase-shift modulator are components in both classes. However, the squeezer does not belong to the class of open dynamical quantum systems (in the framework of Hudson-Parthasarathy [3] and Gardiner-Collett [4]), though it may be approximated by systems that are in this class.

The purpose of this article is merge together these two classes of linear networks in a unified, *multivariable* algebraic framework. By "multivariable," we mean that the framework allows for systems composed of multiple oscillator modes and multiple field channels; accordingly, a vector-matrix notation is used. The new class of linear QFNs (LQFNs) we consider in this article are therefore assembled from components of the following two types: (i) dynamical components, with linear evolution of physical variables, and (ii) static components characterized by Bogoliubov transformations. An example of such a network is shown in Fig. 1 [1,7].

To this end, we consider linear open quantum components that in general are a series connection of a linear dynamical part and a static Bogoliubov part. We define series connections of these components and extend linear fractional transformation (LFT) methods for describing feedback loops. The series connection defines a group structure for this new class of systems, which includes the linear dynamic and static Bogoliubov classes as subgroups. This group structure is interesting from physical as well as systems and control theoretic points of view.

Open quantum systems have a natural input-output structure. We define and make use of input-output maps for our class of linear open quantum systems and discuss convergence of systems in these terms. This input-output notion of convergence is important for applications and is weaker than stronger notions of convergence involving all system variables.

We begin in Sec. II by describing Bogoliubov transformations, which is followed in Sec. III by a discussion of open linear dynamical models of the Hudson-Parthasarathy type. In Sec. V, we discuss quantum components involving

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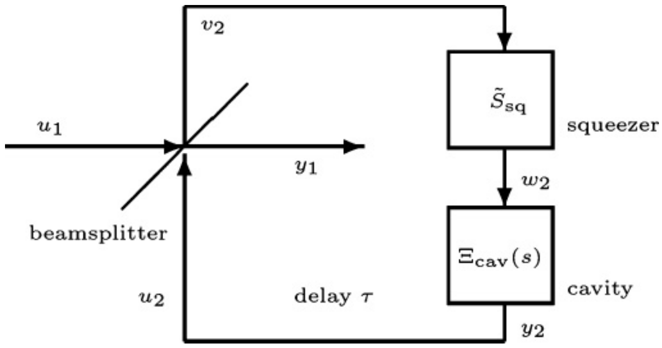

FIG. 1. LQFN consisting of a beam splitter, squeezer, and cavity. The time delay around the optical loop is τ.

Bogoliubov transformations, both static and dynamic. LQFNs are described in Sec. VI. Several examples are discussed in Secs. IV and VII.

II. BOGOLIUBOV TRANSFORMATIONS

In this section, we present models for the quantum components considered in this article. Before this can be done, some notation is needed.

A. Notation

Let $X=(X_{jk})$, $j,k=1,\ldots,n$, denote a matrix whose entries X_{jk} are operators on a Hilbert space $\mathfrak{H}$ or are complex numbers. We define the matrices

$$X^{\#}=(X_{jk}^{*}),\quad X^{\top}=(X_{kj}),\quad X^{\dagger}=(X_{kj}^{*}).$$

Here, the asterisk, $*$, indicates Hilbert space adjoint or complex conjugation.

For a column vector x of operators of length k, we introduce the *doubled-up column vector*

$$\breve{x}\triangleq\begin{bmatrix}x\\x^{\#}\end{bmatrix}\tag{1}$$

of length $2k$, so that $\breve{x}^{\dagger}=(x^{\dagger},x^{\top})$.

Given a linear transformation of the form

$$y=E_{-}x+E_{+}x^{\#},$$

where x and y are vectors of operators of lengths k and r, respectively, and $E_{\pm}\in\mathbb{C}^{r\times k}$, we define the transformation $y^{\#}=E_{-}^{\#}x^{\#}+E_{+}^{\#}x$, and in doubled-up notation we have

$$\breve{y}=\Delta(E_{-},E_{+})\breve{x},$$

in which we introduce the $(2r\times 2k)$ *doubled-up matrix*

$$\Delta(E_{-},E_{+})\triangleq\begin{bmatrix}E_{-}&E_{+}\\E_{+}^{\#}&E_{-}^{\#}\end{bmatrix}.\tag{2}$$

We note that $\Delta(E_{-},E_{+})^{\dagger}=\Delta(E_{-}^{\dagger},E_{+}^{\top})$, and when the dimensions are compatible, $\Delta(E_{-},E_{+})\Delta(F_{-},F_{+})=\Delta(E_{-}F_{-}+E_{+}F_{+}^{\#},E_{-}F_{+}+E_{+}F_{-}^{\#})$. In the examples we consider, the linear transformations are between vectors of equal dimensions, and so the matrices $E_{\pm}$, etc., are square.

For a $(2n\times 2m)$ matrix X, we define an involution $\flat$ by

$$X^{\flat}\triangleq J_{m}X^{\dagger}J_{n},\tag{3}$$

where

$$J_{n}\triangleq\begin{bmatrix}I_{n}&0\\0&-I_{n}\end{bmatrix},\tag{4}$$

with I_n as the $(n\times n)$ identity matrix. When understood, we often drop the dimension index and just write I and J. For doubled-up matrices, we then have

$$\Delta(E_{-},E_{+})^{\flat}=\Delta(E_{-}^{\dagger},-E_{+}^{\top}).\tag{5}$$

The doubled-up Hilbert space $\mathfrak{H}\oplus\mathfrak{H}$ endowed with the indefinite inner product $\langle\cdot|J\cdot\rangle$ is an example of a Krein space, often called a J space or Pontryagin space [8].

B. Canonical commutation relations

We consider a collection of m harmonic oscillators, whose behavior is characterized by independent annihilation a_j and creation a_j^{*} operators ($j=1,\ldots,m$) satisfying the canonical commutation relations $[a_j,a_k^{*}]=\delta_{jk}$, with $[a_j,a_k]=0=[a_j^{*},a_k^{*}]$. The commutation relations may be written compactly as

$$[\breve{a}_j,\breve{a}_k^{\#}]=J_{jk},$$

where $(J_{jk})=J_m$ is the matrix defined in (4).

Systems consisting of m oscillator modes are equivalent, for fixed m, and it is convenient to consider just the category $\mathcal{S}(m)$ of such systems with representative described by column vector $a=(a_1,\ldots,a_m)^{\top}$.

C. The Bogoliubov matrix Lie group, $\mathrm{Sp}(\mathbb{C}^m)$

Definition. A $(2m\times 2m)$ complex matrix $\tilde{S}$ is said to be $\flat$ *unitary* if it is invertible and

$$\tilde{S}^{\flat}\tilde{S}=\tilde{S}\tilde{S}^{\flat}=I_{2m}.$$

The group of *Bogoliubov matrices* $\mathrm{Sp}(\mathbb{C}^m)$ is the subgroup of $\flat$-unitary matrices that are of doubled-up form, that is, $\tilde{S}=\Delta(S_{-},S_{+})$, for suitable $S_{-},S_{+}\in\mathbb{C}^{m\times m}$. This is also known as the *symplectic group* [9,10].

The transformation $a'=S_{-}a+S_{+}a^{\#}$ is called a *Bogoliubov transformation* for $a\in\mathcal{S}(m)$. In doubled-up notation, this takes the simpler form

$$\breve{a}'=\tilde{S}\breve{a}.\tag{6}$$

Note that $a'\in\mathcal{S}(m)$ and, in particular, that the transformation preserves the canonical commutation relations.

A Bogoliubov matrix $\tilde{S}\in\mathrm{Sp}(\mathbb{C}^m)$ admits a *Shale decomposition* [6],

$$\tilde{S}=\Delta(S_{\rm out}^{\dagger},0)\Delta(\cosh R,\sinh R)\Delta(S_{\rm in},0),\tag{7}$$

where $S_{\rm in}$, $S_{\rm out}$ are $(m\times m)$ unitary matrices and R is a real diagonal $(m\times m)$ matrix. Note that $\Delta(\cosh R,\sinh R)=\exp\Delta(0,R)$. The middle term in (7) corresponds to *squeezing*, an important characteristic widely exploited in applications of quantum optics. To see what this means, suppose $\tilde{S}=\Delta(\cosh R,\sinh R)$. Define the quadratures $a^{x}=\frac{1}{2}(a+a^{\#})$ and $a^{y}=\frac{1}{2i}(a-a^{\#})$ and similarly for a'. Then

$$(a')^{x}=e^{R}a^{x},\quad(a')^{y}=e^{-R}a^{y},\tag{8}$$

which shows that if the y quadrature is scaled by less than unity, the x quadrature must correspondingly be expanded by an amount greater than unity. Also, note that the unitary group $U(m)$ of unitary $(m \times m)$ matrices can be viewed as a subgroup of $\mathrm{Sp}(\mathbb{C}^m)$ via the correspondence $\Delta(S,0) \in \mathrm{Sp}(\mathbb{C}^m)$ whenever $S \in U(m)$.

The Bogoliubov transformation (6) defined by a fixed $\tilde{S} \in \mathrm{Sp}(\mathbb{C}^m)$ corresponds to the action of a physical device acting on a vector $a \in \mathcal{S}(m)$. By Shale's theorem [6], the Bogoliubov transformation may be unitarily implemented; that is, there exists a unitary operator U such that

$$\tilde{S}\breve{a} = U^* \breve{a} U. \tag{9}$$

D. The Bogoliubov Lie algebra, $\mathfrak{sp}(\mathbb{C}^m)$

We remark that the Lie algebra $\mathfrak{sp}(\mathbb{C}^m)$ consists of matrices $-i\tilde{\Omega} \in \mathbb{C}^{2m\times 2m}$ that are of doubled-up form (in order to generate doubled-up matrices $\tilde{S} = e^{-i\tilde{\Omega}}$) and satisfy $\tilde{\Omega}^\flat = \tilde{\Omega}$. The second condition can be written as $J\tilde{\Omega} - \tilde{\Omega}^\dagger J = 0$. We therefore deduce that the infinitesimal generators take the form

$$-i\tilde{\Omega} = -\Delta(i\Omega_-, i\Omega_+), \tag{10}$$

with complex matrices Ω_- and Ω_+ having the symmetries $\Omega_-^\dagger = \Omega_-$ and $\Omega_+^\top = \Omega_+$. We remark that we may construct a Hermitean operator H on the oscillator Hilbert space as

$$H = \sum_{\alpha,\beta=1}^{m} \left(a_\alpha^* \omega_{\alpha\beta}^- a_\beta + \frac{1}{2} a_\alpha^* a_\beta^* \omega_{\alpha\beta}^+ + \frac{1}{2} a_\alpha a_\beta \omega_{\alpha\beta}^{+*} \right), \tag{11}$$

where the coefficients are the entries of the matrices $\Omega_\pm = (\omega_{\alpha\beta}^\pm) \in \mathbb{C}^{m\times m}$. From the familiar quantum mechanical point of view, H is the Hamiltonian generating the canonical transformation (6); that is, in (9), we have $U = e^{-iH}$.

It is instructive to look at the $m = 1$ case. Here, the Hamiltonian is $H = \omega_- a^* a + \frac{1}{2}\omega_+ a^{*2} + \frac{1}{2}\omega_+^* a^2$ with ω_- real and $\omega_+ = |\omega_+| e^{i\theta}$ complex. The corresponding element $-i\tilde{\Omega} \in \mathfrak{sp}(\mathbb{C})$ can be written in terms of Pauli matrices as as

$$\begin{aligned} -i\tilde{\Omega} &= -\Delta(i\omega_-, i\omega_+) \\ &= \omega_{+y}\sigma_x + \omega_{+x}\sigma_y - i\omega_-\sigma_z, \end{aligned}$$

where $\omega_+ = \omega_{+x} + i\omega_{+y}$, or

$$\tilde{\Omega} = \begin{bmatrix} \omega_- & \omega_{+x} + i\omega_{+y} \\ \omega_{+x} - i\omega_{+y} & -\omega_- \end{bmatrix}.$$

The eigenvalues of $-i\tilde{\Omega}$ are $\pm\sqrt{\zeta}$ where

$$\zeta = |\omega_+|^2 - \omega_-^2. \tag{12}$$

We note that Heisenberg dynamical equations is trigonometric for $\zeta < 0$ and hyperbolic for $\zeta > 0$. Let us try and diagonalize the Hamiltonian by introducing the Bogoliubov transformation $e^{i\theta} a = \cosh r\, a' - \sinh r\, a'^*$ (this is the original purpose of Bogoliubov transformations). For $\zeta < 0$, we may choose $\tanh 2r = |\omega_+|/\omega_-$ to get $H \equiv \omega_- \sqrt{1 - |\frac{\omega_+}{\omega_-}|^2} a'^* a'$. For $\zeta < 0$, we may choose $\tanh 2r = \omega_-/|\omega_+|$ to get $H \equiv \frac{1}{2}|\omega_+| \sqrt{1 - |\frac{\omega_-}{\omega_+}|^2} (a'^{*2} + a'^2)$. That is, for $\zeta < 0$, we may diagonalize Ω as $\tilde{S}^\flat \Omega \tilde{S} = \Delta(\omega_-', 0)$ using a Bogoliubov matrix $\tilde{S}$, but for $\zeta > 0$, the best we can do is to put it into the form $\Delta(0, \omega_+')$.

We say that H is *passive* if $\tilde{\Omega}$ has only real eigenvalues. In this case, we may find an $\tilde{S}$ in $\mathrm{Sp}(\mathbb{C})$ such that $\tilde{S}^\flat \tilde{\Omega} \tilde{S} = \Delta(\Omega_-', 0)$ for some Ω_-'. The term "passive" means that such Hamiltonians do not describe energy flow into the system from an external pumping source and that the dynamical equations are always of trigonometric type.

[For $m = 1$, H is passive if and only if the parameter $\zeta \leqslant 0$ in (12), as the eigenvalues are $\pm\sqrt{-\zeta}$.]

The group $\mathrm{Sp}(\mathbb{C}^m)$ is a noncompact group and, in fact, is not covered by the exponential mapping on its Lie algebra $\mathfrak{sp}(\mathbb{C}^m)$. We see this in the case $m = 1$, where the form of $e^{-i\tilde{\Omega}}$ depends on the sign of ζ. We have $e^{-i\tilde{\Omega}}$ given respectively by $(\zeta > 0)$

$$\begin{bmatrix} \cosh\sqrt{\zeta} - i\omega_- \frac{\sinh\sqrt{\zeta}}{\sqrt{\zeta}}, & \omega_+ \frac{\sinh\sqrt{\zeta}}{\sqrt{\zeta}} \\ \omega_+^* \frac{\sinh\sqrt{\zeta}}{\sqrt{\zeta}}, & \cosh\sqrt{\zeta} + i\omega_- \frac{\sinh\sqrt{\zeta}}{\sqrt{\zeta}} \end{bmatrix}$$

and $(\zeta < 0)$

$$\begin{bmatrix} \cos\sqrt{-\zeta} - i\omega_- \frac{\sin\sqrt{-\zeta}}{\sqrt{-\zeta}}, & \omega_+ \frac{\sin\sqrt{-\zeta}}{\sqrt{-\zeta}} \\ \omega_+^* \frac{\sin\sqrt{-\zeta}}{\sqrt{-\zeta}}, & \cos\sqrt{-\zeta} + i\omega_- \frac{\sin\sqrt{-\zeta}}{\sqrt{-\zeta}} \end{bmatrix}.$$

Also, $e^{-i\tilde{\Omega}} = 1 - i\tilde{\Omega}$ if $\zeta = 0$ (note that $\tilde{\Omega}^2 = 0$ in this case). As observed in [11], we must have $\mathrm{tr}\, e^{-i\tilde{\Omega}} \geqslant -2$ so that there exist matrices in $\tilde{S} \in \mathrm{Sp}(\mathbb{C})$ that do not possess a logarithm in $\mathfrak{sp}(\mathbb{C})$, for example, $\tilde{S} = -\Delta(\cosh u, \sinh u) \equiv \exp[-\Delta(i\pi, 0)] \exp[-\Delta(0, -u)]$. In particular, such Bogoliubov transformations are not generated by a single Hamiltonian H. The best that can be done is to write the unitary U in (9) as $U = U_1 \ldots U_k$ where each U_i has a logarithm in $\mathfrak{sp}(\mathbb{C}^m)$; see [11] for higher order cases.

E. Gaussian states

A state on $\mathcal{S}(m)$ is said to be Gaussian if we have

$$\langle \exp i(\breve{u}^\dagger \breve{a}) \rangle = \exp\left(-\tfrac{1}{2}\breve{u}^\dagger F \breve{u} + i\breve{u}^\dagger \breve{\alpha}\right),$$

where $F \geqslant 0$. For simplicity, we consider mean zero states $(\alpha = 0)$. In particular, $F = \langle \breve{a}\breve{a}^\dagger \rangle$ takes the specific form

$$F = \begin{bmatrix} \langle a a^\dagger \rangle & \langle a a^\top \rangle \\ \langle a^\# a^\dagger \rangle & \langle a^\# a^\top \rangle \end{bmatrix} = \begin{bmatrix} I + N^\top & M \\ M^\dagger & N \end{bmatrix} \tag{13}$$

with

$$N_{jk} = \langle a_j^* a_k \rangle, \quad M_{jk} = \langle a_j a_k \rangle, \tag{14}$$

and we note that $N = N^\dagger$ and $M = M^\top$. In particular, positivity of F implies that $N \geqslant 0$. The vacuum state is the special state determined by the choice $N = 0$, $M = 0$, for

$$F_{\text{vac}} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{15}$$

For fixed $N \geqslant 0$, the choice of M is constrained by the requirement that F be positive. For the $n = 1$ case, N and M are scalars, and the positivity condition is easily seen to be $N \geqslant 0$ with $|M|^2 \leqslant N(N+1)$. More generally, we should have a diagonalization $V^\dagger N V = \mathrm{diag}(N_1, \ldots, N_n)$ for unitary V, in which case we could consider new fields $a' = Va$. Here N_j can be interpreted as the average number of quanta in the mode a'_j. In general, we cannot expect to simultaneously diagonalize N and M.

1. Generalized Araki-Woods representation

Given a Gaussian state determined by F in Eq. (13), we now show that we can construct modes having that state through canonical transformations of vacuum modes. That is, given a state for which $a \in \mathcal{S}(m)$ has covariance F given by (13), there exists a $(2m \times 4m)$ matrix $\tilde{S}_0$ such that

$$\breve{a} = \tilde{S}_0 \breve{a}_0, \tag{16}$$

where

$$a_0 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathcal{S}(m+m)$$

has vacuum statistics, and

$$\tilde{S}_0 \tilde{S}_0^\flat = I. \tag{17}$$

Indeed, we construct $\tilde{S}_0 = \Delta(E_-^0, E_+^0)$ for some $(m \times 2m)$ matrices E_-^0, E_+^0. This generalizes a construction originally due to Araki and Woods [12] for nonsqueezed thermal states; see [13,14].

2. Construction of Araki-Woods vacuum representation

Step 1: Diagonalize N. We may find a unitary matrix $V \in \mathbb{C}^{m\times m}$ such that $V^\dagger N V = \mathrm{diag}(N_1, \ldots, N_m)$. The eigenvalues are assumed to be ordered such that $N_1 \geqslant \cdots \geqslant N_m \geqslant 0$. Therefore we can restrict our attention to the case where N is diagonalized in this way.

Step 2: Ignore zero eigenvalues. Take the first m_+ eigenvalues to be strictly positive, with the remaining $m_0 = m - m_+$ eigenvalues to be zero. We respect to the eigen decomposition $\mathbb{C}^m = \mathbb{C}^{m_+} \oplus \mathbb{C}^{m_0}$, we decompose F as

$$F = \begin{bmatrix} I + N_{++} & 0 & M_{++} & M_{+0} \\ 0 & I & M_{0+} & M_{00} \\ M_{++}^\top & M_{+0}^\top & N_{++} & 0 \\ M_{0+}^\top & M_{00}^\top & 0 & 0 \end{bmatrix}.$$

However, we observe that if a positive matrix has a zero on a diagonal, then every entry on the corresponding row and column must vanish[1] so that actually

$$F \equiv \begin{bmatrix} I + N_{++} & 0 & M_{++} & 0 \\ 0 & I & 0 & 0 \\ M_{++}^\top & 0 & N_{++} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, we can restrict our attention to the case where N is diagonal, strictly positive, and in particular invertible.

Step 3: Explicit construction. We begin by noting the constraint $I + N \geqslant M(1/N)M^\dagger$, which follows from noting the positivity of

$$\begin{bmatrix} I & -M\frac{1}{N} \\ 0 & 0 \end{bmatrix} F \begin{bmatrix} I & -M\frac{1}{N} \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} I + N - M\frac{1}{N}M^\dagger & 0 \\ 0 & 0 \end{bmatrix}.$$

We then introduce the following matrices:

$$X = \sqrt{I + N - M(1/N)M^\dagger},$$
$$Y = \sqrt{N} = \mathrm{diag}(\sqrt{N_1}, \ldots, \sqrt{N_m}),$$
$$Z = MY^{-1}.$$

Note that $Y = Y^\top$, and from $Z = MY^{-1}$ we have that $YZ^\top = M^\top = M = ZY = ZY^\top$. These matrices satisfy the conditions

$$XX^\dagger - YY^\dagger + ZZ^\dagger = I \quad \text{and} \quad YZ^\top = ZY^\top. \tag{18}$$

Now take b_1 and b_2 to be independent (commuting) modes in $\mathcal{S}(m)$. We fix the state to be the joint vacuum state for both of these modes. Then we may represent a as

$$a = Xb_1 + Yb_2^\sharp + Zb_2; \tag{19}$$

indeed, it is straightforward to check that

$$\langle a^\# a^\top \rangle = Y^2 = N,$$
$$\langle a a^\top \rangle = ZY^\top = ZY = M.$$

Therefore, we have obtained the representation (16) with $\tilde{S}_0 = \Delta(E_-^0, E_+^0)$ and

$$E_-^0 = (X \quad 0), \quad E_+^0 = (Z \quad Y).$$

Property (17) follows from (18).

III. QUANTUM OPEN LINEAR DYNAMICS

In this section, we consider the general class of open linear dynamical models arising from a unitary model for the joint system and field. The system is a collection $\mathcal{S}(m)$ of m harmonic modes with representative $a = (a_1, \ldots, a_m)^\top$.

A. Boson fields: vacuum states

The open quantum systems described here are driven by n quantum noise fields (input processes) represented by annihilation $b_j(t)$ and creation $b_j^*(t)$ operators ($j = 1, \ldots, n$) satisfying canonical commutation relations $[b_j(t), b_k^*(t')] = \delta_{jk}\delta(t - t')$, with $[b_j(t), b_k(t')] = 0$. This may be written compactly as

$$[\breve{b}_j(t), \breve{b}_k^\#(t')] = J_{jk}\, \delta(t - t'). \tag{20}$$

We denote the class of m independent input processes by $\mathcal{F}(n)$ with representative described by column vector $b = (b_1, \ldots, b_n)^\top$.

The vacuum state for the field is characterized by

$$\left\langle \exp i \int_0^\infty \{u(t)b^\dagger(t) + u^\dagger(t)b(t)\}dt \right\rangle_{\text{vac}}$$
$$= \exp -\frac{1}{2}\int_0^\infty u^\dagger(t)u(t)dt.$$

[1]To see this, let $E \geqslant 0$ be a complex $k \times k$ matrix with $E_{11} = 0$. Set $u(t) = (tx_1, x_2, \ldots, x_k)^\top$. Then we have $0 \leqslant u(t)^\dagger E u(t) = 2t\, \mathrm{Re}\sum_{j>1} x_1 E_{1j} x_j + \sum_{j,k>1} x_j^* E_{jk} x_k$. However, the only way to guarantee this inequality for all real t is to require that $\mathrm{Re}\sum_{j>1} x_1 E_{1j} x_j = 0$. Replacing t with it shows that $\mathrm{Im}\sum_{j>1} x_1 E_{1j} x_j$ likewise vanishes. As this must be true for arbitrary x_j, we conclude that $E_{1j} = E_{j1}^* = 0$ for all $j > 1$.

It is convenient to introduce the integrated fields

$$B_j(t) = \int_0^t b_j(r)dr,$$

and in the vacuum representation their future-pointing (Itō) increments satisfy the quantum Itō table

$$\begin{array}{c|cc} \times & dB_k^\dagger & dB_k \\ \hline dB_j & \delta_{jk}dt & 0 \\ dB_j^\dagger & 0 & 0 \end{array}.$$

We may write this more compactly as

$$\left\langle \exp i \int_0^\infty \breve{u}^\dagger(t)\breve{b}(t)dt \right\rangle_{\text{vac}} = \exp -\frac{1}{2}\int_0^\infty \breve{u}^\dagger(t)F_{\text{vac}}\breve{u}(t)dt.$$

The vacuum state is then the Gaussian state for which $\langle \breve{b}(t)\breve{b}^\dagger(t')\rangle_{\text{vac}} = F_{\text{vac}}\delta(t-t')$, and the Itō table may be summarized by

$$d\breve{B}d\breve{B}^\dagger = F_{\text{vac}}dt.$$

Here $B = (B_1, B_2, \ldots, B_n)^\top$. In the vacuum case, we may also define the counting process

$$\Lambda_{jk}(t) = \int_0^t b_j^*(r)b_k(r)dr,$$

which may be included in the Itō table [3]. The additional nontrivial products of differentials are

$$d\Lambda_{jk}dB_l^\dagger = \delta_{kl}dB_j^\dagger, \quad dB_j d\Lambda_{kl} = \delta_{jk}dB_l,$$
$$d\Lambda_{jk}d\Lambda_{li} = \delta_{kl}d\Lambda_{ji}.$$

B. Boson fields: Gaussian field states

We may generalize the situation in Sec. III A to the case where the input fields are in Gaussian states with zero mean but with the correlation functions

$$\langle b_j^*(t)b_k(t')\rangle = N_{jk}\ \delta(t-t'),$$
$$\langle b_j(t)b_k(t')\rangle = M_{jk}\ \delta(t-t'),$$

with N and M as in (14). That is,

$$\langle \breve{b}(t)\breve{b}^\dagger(t')\rangle \equiv F\delta(t-t'), \tag{21}$$

where F has the same form encountered in the case of a finite number of modes in Eq. (13). The extended Itō table is then

$$\begin{array}{c|cc} \times & dB_k^\dagger & dB_k \\ \hline B_j & (\delta_{jk}+N_{kj})dt & M_{jk}dt \\ dB_j^\dagger & M_{kj}^*dt & N_{jk}dt \end{array}.$$

Generalized Araki-Woods representations for arbitrary Gaussian field states may be obtained based on a straightforward lifting of the constructions in Sec. II E [13,14].

C. Quantum linear dynamical models

The dynamical behavior of a system composed of m oscillators interacting with n input fields (vacuum state) is given in terms of the Hudson-Parthasarathy-Schrödinger equation (in Itō form) [3]

$$dU(t) = \Bigg\{ \sum_{i,j=1}^n (S_{ij}-\delta_{ij})d\Lambda_{ij}(t) + \sum_{j=1}^n dB_j^*(t)L_j - \sum_{j,k=1}^n L_j^* S_{jk}dB_k(t) - \left(\frac{1}{2}\sum_{j=1}^n L_j^*L_j + iH\right)dt \Bigg\} U(t), \tag{22}$$

for a unitary operator $U(t)$, with $U(0) = I$. To obtain a unitary evolution leading to linear dynamics, we must take $S \in \mathbb{C}^{n\times n}$ to be unitary, H to be of the form encountered in (11), and the coupling of the system modes to the fields to be of the form

$$L_j = \sum_{\alpha=1}^m (C_{j\alpha}^- a_\alpha + C_{j\alpha}^+ a_\alpha^*), \tag{23}$$

where $C_\pm = (C_{j\alpha}^\pm) \in \mathbb{C}^{n\times m}$.

The oscillator variables evolve unitarily $a_j(t) = U^*(t)a_jU(t)$, and likewise the output field is $B_{\text{out}}(t) = U^*(t)B(t)U(t)$. The dynamical equations are

$$\begin{aligned} \dot{a}(t) &= C_+^\top S^\# b^\#(t) - C_-^\dagger S b(t) + A_- a(t) + A_+ a^\#(t), \\ b_{\text{out}}(t) &= Sb(t) + C_- a(t) + C_+ a^\#(t), \end{aligned} \tag{24}$$

where

$$A_\mp = -\tfrac{1}{2}(C_-^\dagger C_\mp - C_+^\top C_\pm^\#) - i\Omega_\mp. \tag{25}$$

Note that $-\frac{1}{2i}(A_- - A_-^\dagger) = \Omega_-$ and $-\frac{1}{2i}(A_+ + A_+^\top) = \Omega_+$, but in general $A_- \neq A_-^\dagger$ and $A_+ \neq A_+^\top$. Here differential equations are expressed in terms of quantum noise fields and may be interpreted in the Stratonovich or Itō senses; moreover, the evolution preserves the commutation relations of the oscillator variables.

The linear dynamical equations can be written in doubled-up form as

$$\begin{aligned} \dot{\breve{a}}(t) &= \Delta(A_-, A_+)\breve{a}(t) - \Delta(C_-, C_+)^\flat \Delta(S, 0)\breve{b}(t), \\ \breve{b}_{\text{out}}(t) &= \Delta(C_-, C_+)\breve{a}(t) + \Delta(S, 0)\breve{b}(t). \end{aligned} \tag{26}$$

Let us introduce the doubled-up matrices $\tilde{A} = \Delta(A_-, A_+)$, $\tilde{C} = \Delta(C_-, C_+)$, and $-i\tilde{\Omega} = -\Delta(i\Omega_-, i\Omega_+)$. Then we have the identities

$$2\text{Re}_\flat(A) = \tilde{A} + \tilde{A}^\flat = -\tilde{C}^\flat\tilde{C}, \quad \tilde{\Omega}^\flat = \tilde{\Omega}. \tag{27}$$

These are readily established by noting

$$\begin{aligned} \Delta(A_-, A_+) + \Delta(A_-, A_+)^\flat &= \Delta(A_- + A_-^\dagger, A_+ - A_+^\top) \\ &= -\Delta(C_-^\dagger C_- - C_+^\top C_+^\#, C_-^\dagger C_+ - C_+^\top C_-^\#) \\ &= -\Delta(C_-, C_+)^\flat \Delta(C_-, C_+), \end{aligned}$$

and

$$\Delta(i\Omega_-, i\Omega_+)^\flat = \Delta(-i\Omega_-^\dagger, -i\Omega_+^\top) = -\Delta(i\Omega_-, i\Omega_+).$$

The dynamical equations can then be recast as

$$\begin{aligned} \dot{\breve{a}}(t) &= \tilde{A}\breve{a}(t) + \tilde{B}\breve{b}(t), \\ \breve{b}_{\text{out}}(t) &= \tilde{C}\breve{a}(t) + \tilde{D}\breve{b}(t), \end{aligned} \tag{28}$$

where

$$\tilde{D} = \Delta(S, 0), \tag{29}$$

and

$$\tilde{B} = -\tilde{C}^{\flat}\tilde{D}, \quad \tilde{A} = -\tfrac{1}{2}\tilde{C}^{\flat}\tilde{C} - i\tilde{\Omega}. \tag{30}$$

We denote this class of linear Hudson-Parthasarathy systems with n input fields and m oscillators by $\mathcal{L}^{\mathrm{HP}}(n,m)$ and write $\mathcal{L}^{\mathrm{HP}}(n) = \cup_m \mathcal{L}^{\mathrm{HP}}(n,m)$. Systems $G \in \mathcal{L}^{\mathrm{HP}}(n,m)$ may be parameterized in several ways. In terms of the scattering matrix, S, vector of coupling operators L, and Hamiltonian H, we may write

$$G = (S, L, H). \tag{31}$$

Since these physical parameters are determined by the matrices given previously, we may also write

$$G = (S, \tilde{C}, \tilde{\Omega}). \tag{32}$$

Alternatively, we may use the matrices appearing in Eqs. (28),

$$G = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right], \tag{33}$$

a notation commonly used in linear systems and control theory. We remark that an arbitrary quadruple of matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ need not necessarily correspond to a quantum open system [15,16].

D. Stability

In linear systems theory, a system G of the form (33) is said to be *Hurwitz stable* if all eigenvalues in the matrix $\tilde{A}$ have strictly negative real parts. If the joint system-field state is one in which the inputs are mean zero, then $\frac{d}{dt}\langle \breve{a}(t)\rangle = \tilde{A}\langle \breve{a}(t)\rangle$, and so Hurwitz stability implies that $\langle \breve{a}(t)\rangle \to 0$ as $t \to \infty$.

When $A_+ = 0$, we have $\tilde{A} = \Delta(A_-, 0)$ with $A_- \equiv -\frac{1}{2}C_-^{\dagger}C_- - i\Omega_-$ and $\Omega_-^* = \Omega_-$. Since $X^{\dagger}X$ is non-negative definite, it is easy to determine whether A_- is Hurwitz. For instance, it is sufficient to have C_- invertible. However, expressions such as $X^{\flat}X$ are indefinite due to the presence of the matrix J. There may be nonpassive contributions to $\tilde{A}$ from both C_+ and Ω_+.

As an illustration, let us consider how the eigenvalues of $\tilde{A}$ depend on the physical parameters in the simplest case, $n = 1 = m$. We have seen that the most general parametrization is

$$\tilde{C} = \Delta(\sqrt{\gamma_-}e^{i\phi_-}, \sqrt{\gamma_+}e^{i\phi_+}) \quad \text{and} \quad \tilde{\Omega} = \Delta(\omega_-, \omega_+),$$

with $\gamma_\pm, \phi_\pm, \omega_-$ real, and $\omega_+ \in \mathbb{C}$. In this case,

$$\tilde{A} = -\Delta\left[\tfrac{1}{2}(\gamma_- - \gamma_+) + i\omega_-, i\omega_+\right],$$

which has eigenvalues $\frac{1}{2}(\gamma_- - \gamma_+) \pm \sqrt{\zeta}$ where we recall the parameter $\zeta = |\omega_+|^2 - \omega_-^2$ from (12). The plant is Hurwitz if

(i) $\zeta \leqslant 0$ and $\gamma_- > \gamma_+$; or
(ii) $\zeta > 0$ and $\sqrt{\zeta} < \frac{1}{2}(\gamma_- - \gamma_+)$.

In situation 1, the system has a passive Hamiltonian and the damping rate is greater that the pumping rate. However, situation 2 shows that if the damping is sufficiently large, then the system may still be stable even if the Hamiltonian is not passive. In general, as one expects, stability depends on the relative flows of energy into and out of the system.

E. Series connections

Open linear dynamical systems $G_1 = (S_1, L_1, H_1)$ and $G_2 = (S_2, L_2, H_2)$ in $\mathcal{L}^{\mathrm{HP}}(n)$ [recall the parametrization (31)] may be connected in series by passing the output of system G_1 into the input of system G_2 [2,17,18]. The system formed from this connection in the zero-delay limit is an open system, $G = G_2 \lhd G_1$, which in terms of the parameters (31) is given by

$$G_2 \lhd G_1 = [S_2S_1, L_2 + S_2L_1, H_1 + H_2 + \mathrm{Im}(L_2^{\dagger}S_2L_1)]. \tag{34}$$

We refer to $\lhd$ as the *series product* of G_1 and G_2 and give the cascaded system. The set $\mathcal{L}^{\mathrm{HP}}(n)$ forms a group with respect to the series product, with inverse $G^{-1} = (S^{\dagger}, -S^{\dagger}L, -H)$ [where $G = (S, L, H)$].

Given an open system $G = (S, \tilde{C}, \tilde{\Omega})$ [now we use the parametrization (31) in anticipation of later use], it follows from properties of the series product that

$$G = (I, \tilde{C}, \tilde{\Omega}) \lhd (S, 0, 0). \tag{35}$$

This factorization says that an open system with scattering S is equivalent (in the zero-delay limit) to a dynamic open system without scattering $(I, \tilde{C}, \tilde{\Omega})$ connected in series with a nondynamic or static open system $(S, 0, 0)$.

F. Input-output maps

In classical systems and control theory [19], the input-output map is a basic tool, and in the case of linear systems it may be expressed explicitly in the time and frequency domains. Input-output maps for the open quantum linear system may be defined in the same way; in terms of the doubled-up parameters (33), we have

$$\breve{b}_{\mathrm{out}}(t) = \tilde{C}e^{\tilde{A}t}\breve{a}(0) + \tilde{\Sigma}_G[t; \breve{b}_{\mathrm{in}}], \tag{36}$$

where

$$\tilde{\Sigma}_G[t; \breve{b}] = -\int_0^t \tilde{C}e^{\tilde{A}(t-r)}\tilde{C}^{\flat}\tilde{D}\breve{b}(r)dr + \tilde{D}\breve{b}(t). \tag{37}$$

Here, the input b_{in} is understood as the input field b, and we often use the subscript for emphasis. The impulse response associated with the term $\tilde{\Sigma}_G[t; \breve{b}]$ is $\tilde{\sigma}_G(t) = -\tilde{C}e^{\tilde{A}t}\tilde{C}^{\flat}\tilde{D} + \tilde{D}\delta(t)$, from which we have the transfer function (the Laplace transform of $\tilde{\Sigma}_G[t; \breve{b}]$, in which s is a complex variable):

$$\tilde{\Xi}_G(s) = \left[\begin{array}{c|c} \tilde{A} & -\tilde{C}^{\flat}\tilde{D} \\ \hline \tilde{C} & \tilde{D} \end{array}\right](s) = -\tilde{C}(sI - \tilde{A})^{-1}\tilde{C}^{\flat}\tilde{D} + \tilde{D}. \tag{38}$$

Let us introduce the transformed

$$\breve{b}_{\mathrm{in}}[s] \triangleq \int_0^{\infty} e^{-st}\breve{b}_{\mathrm{in}}(t)dt; \tag{39}$$

that is, $b_{\mathrm{in}}[s] = \int_0^{\infty} e^{-st}b_{\mathrm{in}}(t)dt$ and $b_{\mathrm{in}}^{\#}[s] = b_{\mathrm{in}}[s^*]^{\#} = \int_0^{\infty} e^{-st}b_{\mathrm{in}}^{\#}(t)dt$.

By adopting a similar convention for the outputs, we then obtain an input-output relation of the form

$b_{\rm out}[s] = \Xi_{G,-}(s)b_{\rm in}[s] + \Xi_{G,+}(s)b^*_{\rm in}[s]$, or

$$\breve{b}_{\rm out}[s] = \tilde{\Xi}_G(s)\breve{b}_{\rm in}[s], \quad (40)$$

where

$$\tilde{\Xi}_G(s) = \begin{bmatrix} \Xi_{G,-}(s) & \Xi_{G,+}(s) \\ \Xi_{G,+}(s^*)^\# & \Xi_{G,-}(s^*)^\# \end{bmatrix}, \quad (41)$$

and we have ignored the initial value contribution of the system modes.

Note that while the transfer function $\tilde{\Xi}_G(s)$ is uniquely determined by G, the transfer function does not uniquely determine the system G—many systems may have the same transfer function.

IV. EXAMPLES

A. Annihilation systems

A system $\tilde{G} = (S, \tilde{C}, \tilde{\Omega})$ with $C_+ = 0, \Omega_+ = 0$ has dynamics and output relations that depend only on the annihilation operators and annihilation fields [5,20,21]. For a physically motivated reason, since neither the Hamiltonian (passive) nor the coupling operator of the system contain terms that would require an external source of quanta (i.e., a classical pump beam) to implement (this follows from the synthesis theory of [16]; see also [21, Sec. 7] for a discussion), they are also referred to as passive systems [21]. This type of system often arises in applications and includes optical cavities. Transfer functions for this class of systems take a simpler form, as we now describe.

We have $A_- \equiv -\frac{1}{2}C_-^\dagger C_- + i\Omega_-$ and $A_+ = 0$. Then the matrices $\tilde{C} = \Delta(C_-, 0)$ and $\tilde{A} = \Delta(A_-, 0)$ are block diagonal, and the transfer function takes the form

$$\tilde{\Xi}_G(s) = \begin{bmatrix} \Xi_{G,-}(s) & 0 \\ 0 & \Xi_{G,-}(s^*)^\# \end{bmatrix}, \quad (42)$$

with

$$\Xi_{G,-}(s) = \left[\begin{array}{c|c} A_- & -C_-^\dagger S \\ \hline C_- & S \end{array}\right](s) = -C_-(sI - A_-)^{-1}C_-^\dagger S + S. \quad (43)$$

In this situation, we have the input-output relation $b_{\rm out}(t) = Ce^{At}a(0) + \Sigma_G[t;b]$ with $\Sigma_G[t;b] = -\int_0^t Ce^{A(t-r)}C^\flat Db(r)dr + Db(t)$. In comparison with (36) and (37), the output field depends affinely on b but not the conjugate $b^\sharp$.

B. Cavity

In a rotating reference frame, a model for a detuned cavity is characterized by the parameters $G_{\rm cav} = (1, \sqrt{\gamma}a, \omega a^* a)$; that is, $\Omega_- = \omega$, $\Omega_+ = 0$, $C_- = \sqrt{\gamma}$, $C_+ = 0$, $S = I$.

This corresponds to an annihilation-form system

$$\dot{a} = -\left(\frac{\gamma}{2} + i\omega\right)a - \sqrt{\gamma}b_{\rm in}, \qquad b_{\rm out} = \sqrt{\gamma}a + b_{\rm in} \quad (44)$$

when driven by vacuum input b. The transfer function for this system may readily be computed to be

$$\Xi_{{\rm cav},-}(s) = \frac{s - (\gamma/2) + i\omega}{s + (\gamma/2) + i\omega}, \quad (45)$$

which in doubled-up form is

$$\tilde{\Xi}_{\rm cav}(s) = \begin{bmatrix} \dfrac{s - \frac{\gamma}{2} + i\omega}{s + \frac{\gamma}{2} + i\omega} & 0 \\ 0 & \dfrac{s - \frac{\gamma}{2} - i\omega}{s + \frac{\gamma}{2} - i\omega} \end{bmatrix}. \quad (46)$$

Thus this system is

$$G_{\rm cav} = (I, \Delta(\sqrt{\gamma}I, 0), -i\Delta(i\omega, 0)) \in \mathcal{L}^{\rm HP}(1).$$

C. Degenerate parametric amplifier

We consider the model for a degenerate parametric amplifier (DPA) [22, Sec. 7.2], which corresponds to a single oscillator G coupled to a single field with $S = I$, $\omega_- = 0$, $\omega_+ = (i/2)\epsilon$, $\epsilon > 0$, $C_- = \sqrt{\kappa}$, and $C_+ = 0$. The Hamiltonian is not passive; however, the system is stable in the sense of Hurwitz if we take $\epsilon \leqslant \kappa$, as we do from now on. By using (38), we find that the doubled-up transfer function, in agreement with [22], is

$$\tilde{\Xi}_{\rm DPA}(s) = \frac{1}{P(s)}\begin{bmatrix} s^2 - \frac{\kappa^2+\epsilon^2}{4} & -\frac{1}{2}\epsilon\kappa \\ -\frac{1}{2}\epsilon\kappa & s^2 - \frac{\kappa^2+\epsilon^2}{4} \end{bmatrix},$$

where $P(s) = (s + \frac{1}{2}\kappa)^2 - \frac{1}{4}\epsilon^2$. The poles of the transfer function therefore occur at the zeros of P, namely $s = \pm(\epsilon/2) - (\kappa/2)$. In the frequency domain, the output field is

$$b_{\rm out}(s) = \frac{1}{P(s)}\left(s^2 - \frac{\kappa^2 + \epsilon^2}{4}\right)b(s) - \frac{1}{2P(s)}\epsilon\kappa b^*(s).$$

(Here we ignore the initial condition contribution, which is justified by the stability of the system.) In terms of quadratures $b^x = \frac{1}{2}(b + b^*)$ and $b^y = \frac{1}{2i}(b - b^*)$, we find that

$$b^x_{\rm out}(s) = \Xi^x_{\rm DPA}(s)b^x(s), \qquad b^y_{\rm out}(s) = \Xi^y_{\rm DPA}(s)b^y(s),$$

where (in agreement with [22, Eq. (7.2.26)])

$$\Xi^x_{\rm DPA}(s) = \frac{s - [(\kappa + \epsilon)/2]}{s + [(\kappa - \epsilon)/2]} = \frac{1}{\Xi^y_{\rm DPA}(s)}.$$

The DPA can be implemented in a single-ended cavity and in a case that is our main interest in this article, the idealized one (for a full discussion, see [22, Sec. 10.2.1.g]) where $\kappa, \epsilon \to \infty$ (in practice to be taken large) such that the ratio ϵ/κ is constant. Rescaling $\kappa = k\kappa_0$ and $\epsilon = k\epsilon_0$ is equivalent to replacing κ by κ_0 and ϵ by ϵ_0 and rescaling s as s/k:

$$\tilde{\Xi}_{\rm DPA}(s, \kappa = k\kappa_0, \epsilon = k\epsilon_0) = \tilde{\Xi}_{\rm DPA}\left(\frac{s}{k}, \kappa_0, \epsilon_0\right).$$

The limit $k \to \infty$ is the appropriate limit, and here the cavity has an instantaneous response. The internal cavity dynamics are essentially eliminated by adiabatic elimination. This results in $b_{\rm out}(s)$ being given as the following Bogoliubov transformation of the input:

$$b_{\rm b}(s) = -\cosh(r_0)b(s) - \sinh(r_0)b^\dagger(s),$$

where

$$r_0 = \ln\frac{\kappa_0 + \epsilon_0}{\kappa_0 - \epsilon_0}.$$

The output is then an ideal squeezed white-noise process satisfying the quantum Itō rule discussed in Sec. III A, where

$$N = \sinh^2 r_0 = \frac{4\kappa_0\epsilon_0}{\left(\kappa_0^2 - \epsilon_0^2\right)^2},$$
$$M = \cosh r_0 \sinh r_0 = \frac{2\kappa_0\epsilon_0\left(\kappa_0^2 - \epsilon_0^2\right)}{\left(\kappa_0^2 - \epsilon_0^2\right)^2}.$$

Note here that M and N satisfy the relation $|M|^2 = N(N+1)$. In this limit, the DPA device behaves like a static device that instantaneously outputs a squeezed white-noise field from a vacuum white-noise field source, and the transfer function has a constant Bogoliubov matrix value across *all* frequencies. That is,

$$\tilde{\Xi}_{\text{DPA static}}(s) = \lim_{k\to\infty} \tilde{\Xi}_{\text{DPA}}\left(\frac{s}{k}, \kappa_0, \epsilon_0\right) = -\Delta(\cosh r_0, \sinh r_0), \forall s \in \mathbb{C},$$

and the quadrature transfer functions are

$$\Xi^x_{\text{DPA static}}(s) = -e^{r_0}, \quad \Xi^y_{\text{DPA static}}(s) = -e^{-r_0}.$$

For a DPA device with a sufficiently wide bandwidth, one may approximately model it as a static DPA device with the ideal characteristics described previously. Clearly, $\mathcal{L}^{\text{HP}}(n)$ is not closed with respect to this type of approximation.

V. COMPONENTS INVOLVING BOGOLIUBOV TRANSFORMATIONS

In Sec. IV C, we obtained the constant transfer function $\tilde{\Xi}_{\text{DPA static}} \in \text{Sp}(\mathbb{C})$ for a static approximation to a DPA. Such static approximations afford useful simplifications, though in reality the DPA is a dynamical physical device. The idealized DPA therefore yields outputs that are a squeezing of the inputs.

Motivated by this, we consider in Sec. V A Bogoliubov matrices acting on boson fields, thereby extending the class of static components beyond unitary scattering devices. These components are combined with linear dynamics in Sec. V C to form a general class of quantum linear systems; such models may be useful when the time scales of the dynamical parts are slower than the time scales of the systems represented by static Bogoliubov matrices. These components are combined with linear dynamics in Sec. V C to form a general class of quantum linear systems; such models may be useful when the time scales of the dynamical parts are slower than the time scales of the systems represented by static Bogoliubov matrices.

A. Bogoliubov static components

More generally, we could consider a static component that performs a Bogoliubov transformation of the input field $b_{\text{in}} \in \mathcal{F}(n)$:

$$\breve{b}_{\text{out}}(t) = \tilde{S}\breve{b}_{\text{in}}(t), \tag{47}$$

where now $\tilde{S} \in \text{Sp}(\mathbb{C}^n)$. This transformation, of course, preserves the canonical commutation relations so that $b_{\text{out}} \in \mathcal{F}(n)$.

Some caution should be applied here as we are now using the symbol $\tilde{S}$ in (47) in a purely algebraic manner as an element of $\text{Sp}(\mathbb{C}^n)$ when we strictly mean the second quantization of the Bogoliubov matrix as an operator on the fields. Despite its formal similarity to (6), the relation (47) is of a different character as the fields carry a continuous time variable. Moreover, since such a transformation in general form is linear combinations of field annihilation operator and creation operators, the transformation $\breve{b}_{\text{out}}(t) = \tilde{S}\breve{b}_{\text{in}}(t)$ cannot be described by the usual Hudson-Parthasarathy quantum stochastic differential equation (QSDE) for open Markov systems (cf. Sec. III C). Such a QSDE can only model linear combinations of field annihilation operators of the form $\breve{b}_{\text{out}}(t) = \Delta(S,0)\breve{b}_{\text{in}}(t)$ for a unitary matrix S that appears as one of the parameters of the QSDE (here we set the other parameters to $L = 0$ and $H = 0$). As such, in the transformation of fields with a nonunitary Bogoliubov matrix, we do not have an analog of (9) in the form of $\breve{b}_{\text{out}}(t) = U(t)^*\breve{b}_{\text{in}}(t)U(t)$ for some unitary process $U(t)$ on the system and noise Hilbert space. At present, we do not know whether a unitary transformation exists and, if it exists, what kind of dynamical equations it would satisfy. Since unitary evolution is a fundamental postulate of quantum mechanics, the situation is somewhat unsatisfactory and is the subject of continuing research. However, the relation (47) is nevertheless a useful idealization for certain devices used in quantum optics, such as what we have seen with the static DPA in Sec. IV C, and has formally been employed up to now (see, for instance, the discussion in Chapter 7 of [22] on various quantum optical amplifiers). The physical meaning of the Bogoliubov transformation (47) is correctly interpreted as a limiting situation.

B. Bogoliubov static components as limits of dynamical components

The class of linear dynamical components described in Sec. III C is not closed under input-output convergence. We now show how arbitrary static Bogoliubov components may arise as limits of unitary models. The idea is to exploit the Shale decomposition (7). Thus, any given Bogoliubov matrix $\tilde{S}$ has the decomposition $\tilde{S} = \Delta(\tilde{S}^\dagger_{\text{out}}, 0)\Delta(\cosh R, \sinh R)\Delta(\tilde{S}_{\text{in}}, 0)$, where $\tilde{S}_{\text{in}}$ and $\tilde{S}_{\text{out}}$ are some unitary matrices and R is some real diagonal matrix. We note that the end terms $\Delta(\tilde{S}^\dagger_{\text{out}}, 0)$ and $\Delta(\tilde{S}_{\text{in}}, 0)$ can each be realized as a static passive network made of beam splitters, mirrors, and phase shifters. The middle term, of course, describes squeezing, but this arises from a straightforward construction involving n independent static-limit DPAs acting as ideal squeezing devices. [Each DPA corresponding to a diagonal entry of R provides a degree of squeezing (cf. Sec. IV C) as determined by that entry.] Then we note that we may approximate each DPA with a corresponding dynamic (nonideal) DPA with appropriate parameters (see the discussion of the DPA in Sec. IV C).

C. Dynamical Bogoliubov components

We introduce an extension of the class of dynamical linear models $\mathcal{L}^{\text{HP}}(n)$ considered until now to accommodate the notion of squeezing. This extension is inspired by the factorization (35) for open linear systems of $\mathcal{L}^{\text{HP}}(n)$ type, suggesting that we consider a new class of dynamical components of the form

$$G = (\tilde{S}, \tilde{C}, \tilde{\Omega}) \triangleq (I, \tilde{C}, \tilde{\Omega}) \lhd \tilde{S}, \quad (48)$$

where $(I, \tilde{C}, \tilde{\Omega}) \in \mathcal{L}^{\mathrm{HP}}(n, m)$ and $\tilde{S} = \Delta(S_-, S_+) \in \mathrm{Sp}(\mathbb{C}^n)$.

A system $G = (\tilde{S}, \tilde{C}, \tilde{\Omega})$ is defined by Eqs. (28), where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are as before (Sec. III C), but now $\tilde{D} = \tilde{S}$. We use the notation $\mathcal{L}^{\mathrm{Bog}}(n, m)$ to denote this class of systems and write $\mathcal{L}^{\mathrm{Bog}}(n) = \cup_m \mathcal{L}^{\mathrm{Bog}}(n, m)$. The class $\mathcal{L}^{\mathrm{Bog}}(n)$ includes $\mathcal{L}^{\mathrm{HP}}(n)$ as a special case [with $\tilde{S} = \Delta(S, 0)$]. The justification for the cascade expression (48) is given in Sec. VI F, where we extend the series product for cascaded systems in $\mathcal{L}^{\mathrm{Bog}}(n)$.

The doubled up input-output map is of the form (36), where now $\tilde{D} = \tilde{S}$. The transfer function is explicitly

$$\Xi_{\tilde{G}}(s) = \left[\begin{array}{c|c} \tilde{A} & -\tilde{C}^\flat \tilde{S} \\ \hline \tilde{C} & \tilde{S} \end{array}\right](s) = -\tilde{C}(sI - \tilde{A})^{-1}\tilde{C}^\flat \tilde{S} + \tilde{S}. \quad (49)$$

The transfer function has the following properties:

(i) $\Xi_G \equiv \left[\begin{array}{c|c} \tilde{A} & -\tilde{C}^\flat \\ \hline \tilde{C} & I \end{array}\right] \tilde{S}$.

(ii) Whenever its value exists, we have $\Xi_G(i\omega) \in \mathrm{Sp}(\mathbb{C}^n)$, for $\omega \in \mathbb{R}$.

Property (i) follows directly from (49) whereas property (ii) follows mutatis mutandis from the proof of [5], Lemma 2] by the replacing † with $^\flat$ and replacing (A, B, C, D) with $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$.

Physically, (48) means that to process an input signal, the Bogoliubov transformation $\tilde{S}$ is applied and the result is fed into the dynamical subsystem. As we have argued in Sec. V A, Shale's theorem precludes a unitary stochastic dynamical model giving rise to a system $\tilde{G} \in \mathcal{L}^{\mathrm{Bog}}(n)$. Nevertheless, as we have also seen, a sequence $\tilde{G}_L \in \mathcal{L}^{\mathrm{HP}}(n)$ exists such that pointwise

$$\lim_{L\to\infty} \Xi_{\tilde{G}_L}(s) = \Xi_{\tilde{G}}(s),$$

with $\tilde{G} \in \mathcal{L}^{\mathrm{Bog}}(n)$ but not in $\mathcal{L}^{\mathrm{HP}}(n)$. One might envisage other modes of convergence of transfer function; however, we restrict this article to pointwise convergence. It is interesting to note that the class of Hudson-Parthasarathy models is not closed in the sense of convergence in the input-output sense but may be extended to include Bogoliubov transformations.

We remark that in many cases where a boson field is in a squeezed state (recall Sec. III B), this field may be regarded as the output of a static Bogoliubov component $\tilde{S}$ driven by vacuum inputs. This means, for example, that a dynamical component $(1, \tilde{C}, \tilde{\Omega})$ driven by squeezed fields may be represented as a dynamical Bogoliubov component $(\tilde{S}, \tilde{C}, \tilde{\Omega})$.

D. Example: cavity with squeezed input

Consider the cavity discussed in Sec. IV B, where now we suppose that the cavity input is given by the output of a squeezer G_{sq}, described by the Bogoliubov transformation

$$\tilde{S}_{\mathrm{sq}} = \Delta(\cosh\lambda, \sinh\lambda) = \begin{bmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{bmatrix}. \quad (50)$$

That is,

$$G_{\mathrm{sq}} = [\Delta(\cosh\lambda, \sinh\lambda), 0, 0]).$$

The squeezed-input cavity $G_{\mathrm{cav,sq}} = \tilde{G}_{\mathrm{cav}} \lhd \tilde{G}_{\mathrm{sq}}$ has transfer function

$$\begin{aligned} &\Xi_{\mathrm{cav,sq}}(s) \\ &= \Xi_{\mathrm{cav}}(s)\tilde{S}_{\mathrm{sq}} \\ &= \begin{bmatrix} \frac{s-(\gamma/2)+i\omega}{s+(\gamma/2)+i\omega}\cosh r & \frac{s-(\gamma/2)+i\omega}{s+(\gamma/2)+i\omega}\sinh r \\ \frac{s-(\gamma/2)-i\omega}{s+(\gamma/2)-i\omega}\sinh r & \frac{s-(\gamma/2)-i\omega}{s+(\gamma/2)-i\omega}\cosh r \end{bmatrix}. \end{aligned} \quad (51)$$

This corresponds to the equations

$$\begin{aligned} \begin{bmatrix} \dot{a} \\ \dot{a}^* \end{bmatrix} &= \begin{bmatrix} -\left(\frac{\gamma}{2}+i\omega\right) & 0 \\ 0 & -\left(\frac{\gamma}{2}-i\omega\right) \end{bmatrix}\begin{bmatrix} a \\ a^* \end{bmatrix} \\ &\quad + \begin{bmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{bmatrix}\begin{bmatrix} b \\ b^* \end{bmatrix} \end{aligned} \quad (52)$$

$$\begin{bmatrix} b_{\mathrm{out}} \\ b^*_{\mathrm{out}} \end{bmatrix} = \begin{bmatrix} \sqrt{\gamma} & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix}\begin{bmatrix} a \\ a^* \end{bmatrix} + \begin{bmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{bmatrix}\begin{bmatrix} b \\ b^* \end{bmatrix}.$$

The physical parameters for the squeezed-input cavity are $\Omega_{\mathrm{cav,sq}-} = \omega$, $\Omega_{\mathrm{cav,sq}+} = 0$, $C_{\mathrm{cav,sq}-} = \sqrt{\gamma}$, $C_{\mathrm{cav,sq}+} = 0$, and $\tilde{S}_{\mathrm{cav,sq}-} = \tilde{S}_{\mathrm{sq}} = \Delta(\cosh r, \sinh r)$, and so

$$G_{\mathrm{cav,sq}} = [\Delta(\cosh\lambda, \sinh\lambda), \Delta(\sqrt{\gamma}I, 0), -i\Delta(i\omega, 0)].$$

This system is a member of $\mathcal{L}^{\mathrm{Bog}}(1)$ but not of $\mathcal{L}^{\mathrm{HP}}(1)$.

VI. LINEAR QUANTUM FEEDBACK NETWORKS

We are now in a position to described feedback networks constructed from Bogoliubov dynamical components as nodes and boson fields as links. The general form of such an LQFN is shown in Fig. 2. The fundamental algebraic tool for describing such networks in subsequent sections is the LFT.

A. Linear fractional transformations

LFTs arise naturally when dealing with feedback networks, and a formal notation has been developed in classical linear

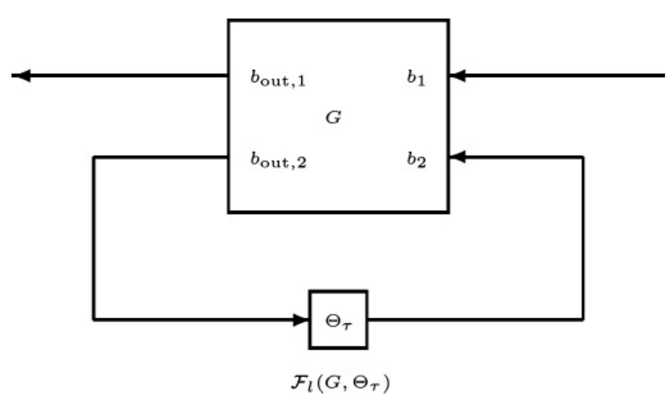

FIG. 2. General form of an LQFN, with the time delay due to the spatial extent of the feedback connection represented by Θ_τ (see text).

systems theory [19]. Consider a classical transfer function $\Xi(s)$ partitioned as

$$\Xi(s) = \begin{bmatrix} \Xi_{11}(s) & \Xi_{12}(s) \\ \Xi_{21}(s) & \Xi_{22}(s) \end{bmatrix}$$

corresponding to a partition $u = (u_1, u_2)^\top$, $y = (y_1, y_2)^\top$ of the input and output signals. If the system is placed in a feedback arrangement defined by $u_2 = K(s)y_2$, then the closed-loop system is described by the transfer function

$$\mathfrak{F}(\Xi(s), K(s)) = \Xi_{11}(s) + \Xi_{12}(s)K(s)[I - \Xi_{22}(s)K(s)]^{-1}\Xi_{21}(s).$$

The arrangement is said to be well posed whenever the inverse $[I - \Xi_{22}(s)K(s)]^{-1}$ exists. This transfer function is obtained by eliminating the in-loop variables.

In what follows, we generalize this type of representation to our class of LQFNs (see also [2,5]).

B. Fractional linear transformations

In this section, we provide some technical results needed for the network theory described in subsequent sections.

Lemma 1. Let $\tilde{S} = \Delta(S_-, S_+) \in \mathrm{Sp}(\mathbb{C}^{n_1+n_2})$ with block decomposition

$$S_\mp = \begin{bmatrix} S_{11}^\mp & S_{12}^\mp \\ S_{21}^\mp & S_{22}^\mp \end{bmatrix},$$

where $S_{jk}^\mp \in \mathbb{C}^{n_j \times n_k}$. By setting $\hat{S}_{jk} = \Delta(S_{jk}^-, S_{jk}^+) \in \mathbb{C}^{2n_j \times 2n_k}$, we have

$$\sum_{k=1,2} \hat{S}_{ki}^\flat \hat{S}_{kj} = \sum_{k=1,2} \hat{S}_{ik} \hat{S}_{jk}^\flat = \delta_{ij}. \qquad (53)$$

Proof. The relation $\tilde{S}^\flat \tilde{S} = I$ implies that $\Delta(S_-^\dagger, -S_+^\top)\Delta(S_-, S_+) = \Delta(I, 0)$, and so $S_-^\dagger S_- - S_+^\top S_+^\# = I$, $S_-^\dagger S_+ - S_+^\top S_-^\# = 0$. These may be written as $\sum_{k=1,2}(S_{ki}^{-\dagger} S_{kj}^- - S_{ki}^{+\top} S_{kj}^{+\#}) = \delta_{ij}$, and $\sum_{k=1,2}(S_{ki}^{-\dagger} S_{kj}^+ - S_{ki}^{+\top} S_{kj}^{-\#}) = 0$. Therefore

$$\sum_{k=1,2} \hat{S}_{ki}^\flat \hat{S}_{kj} = \sum_{k=1,2} \Delta(S_{ki}^{-\dagger}, -S_{ki}^{+\top})\Delta(S_{kj}^-, S_{kj}^+)$$
$$= \sum_{k=1,2} \Delta(S_{ki}^{-\dagger} S_{kj}^- - S_{ki}^{+\top} S_{kj}^{+\#}, S_{ki}^{-\dagger} S_{kj}^+ - S_{ki}^{+\top} S_{kj}^{-\#}),$$

which equals $\Delta(\delta_{ij}, 0) = \delta_{ij}$. The second identity similarly follows from $\tilde{S}\tilde{S}^\flat = I$. ∎

Theorem 2. Let $\tilde{S} \in \mathrm{Sp}(\mathbb{C}^{n_1+n_2})$ and define the fractional linear (Möbius) transformation $\Psi_{\tilde{S}}^{2\to1} : \mathrm{dom}(\Psi_{\tilde{S}}^{2\to1}) \in \mathbb{C}^{n_2\times n_2} \mapsto \mathbb{C}^{n_1\times n_1}$ by

$$\Psi_{\tilde{S}}^{2\to1}(X) \triangleq \hat{S}_{11} + \hat{S}_{12}X(I - \hat{S}_{22}X)^{-1}\hat{S}_{21}, \qquad (54)$$

with $X \in \mathrm{dom}(\Psi_{\tilde{S}}^{2\to1})$ if and only if the inverse $(1 - \hat{S}_{22}X)^{-1}$ exists. Then $\Psi_{\tilde{S}}^{2\to1}$ maps $\mathrm{Sp}(\mathbb{C}^{n_2}) \cap \mathrm{dom}(\Psi_{\tilde{S}}^{2\to1})$ into $\mathrm{Sp}(\mathbb{C}^{n_1})$.

Proof. We first note the Siegel-type identities

$$\Psi_{\tilde{S}}^{2\to1}(X)^\flat \Psi_{\tilde{S}}^{2\to1}(Y) = I - \hat{S}_{21}^\flat\left(I - X^\flat \hat{S}_{22}^\flat\right)^{-1}(I - X^\flat Y) \times (I - \hat{S}_{22}Y)^{-1}\hat{S}_{21},$$
$$\Psi_{\tilde{S}}^{2\to1}(X)\Psi_{\tilde{S}}^{2\to1}(Y)^\flat = I - \hat{S}_{12}(I - X\hat{S}_{22})^{-1}(I - XY^\flat) \times \left(I - \hat{S}_{22}^\flat Y^\flat\right)^{-1}\hat{S}_{12}^\flat. \qquad (55)$$

These are structurally the same as the standard Siegel identities based on partitioning a unitary $\hat{S}$, but the involution $\flat$ replaces the usual Hermitian involution †; see Theorem 21.16 and Corollary 21.17 of Ref. [23]. The identities rely on the unitary analog of the identities (53) and so follow mutatis mutandis. Evidently, if $X \in \mathrm{Sp}(\mathbb{C}^{n_2}) \cap \mathrm{dom}(\Psi_{\tilde{S}}^{2\to1})$, then $\Psi_{\tilde{S}}^{2\to1}(X)^\flat \Psi_{\tilde{S}}^{2\to1}(X) = \Psi_{\tilde{S}}^{2\to1}(X)\Psi_{\tilde{S}}^{2\to1}(X)^\flat = I$. ∎

Corollary 3. If $\tilde{K}(i\omega)$ is an $\mathrm{Sp}(\mathbb{C}^{n_2})$-valued transfer matrix function taking values in $\mathrm{dom}(\Psi_{\tilde{S}}^{2\to1})$ for all ω real, then the LFT $\Psi_{\tilde{S}}^{2\to1}[\tilde{K}(i\omega)]$ is a $\mathrm{Sp}(\mathbb{C}^{n_1})$-valued function of ω.

In particular, if $I \in \mathrm{dom}(\Psi_{\tilde{S}}^{2\to1})$, then

$$\Psi_{\tilde{S}}^{2\to1}(I) = \hat{S}_{11} + \hat{S}_{12}(1 - \hat{S}_{22})^{-1}\hat{S}_{21} \in \mathrm{Sp}(\mathbb{C}^{n_1}). \qquad (56)$$

C. Finite time-delay LQFNs

A general LQFN is a network of linear quantum components $G_v \in \mathcal{L}^{\mathrm{Bog}}(n)$, labeled by the vertices v of the network, with quantum fields traveling along the edges. The edges are directed so that we distinguish inputs and outputs, and the multiplicity of input fields equals the multiplicity of outputs for each component.

In a physical LQFN, we have time delays associated with each internal edge due to the finite time taken by light to travel from an output port to an input port. In fact, we may lump the individual components as one single global component $\tilde{G}$ with all external inputs going into a collective input port 1 and coming out from a collective output port 1, as in Fig. 2. Likewise, all the internal fields can be viewed as traveling from the collective output port 2 to the collective input port 2. The effect of the (multichannel) time delay can be described by the operator Θ_τ defined by

$$\Theta_\tau[f_1(t), \ldots, f_n(t)]^\top = [f_1(t - \tau_1), \ldots, f_n(t - \tau_n)]^\top,$$

where $\tau_1 > 0, \ldots, \tau_n > 0$ are the time delays of each channel. Here, $f_k(t)$ denotes the quantum stochastic process propagating along channel k in doubled-up form. For instance, $f_k(t)$ could be $\breve{y}_k(t)$, the doubled-up output quantum output processes propagating along channel k. In a slight abuse of notation, we also occasionally overload the notation Θ_τ to denote the delayed version of a quantum process that is not in doubled-up form, such as when $f_k(t)$ is taken to be $y_k(t)$ for all k. Note that $[\Theta_\tau(i\omega)]_{jk} = e^{i\omega\tau_j}\delta_{jk}$. By extending the standard notation (recalled from Sec. VI A), we denote this by

$$\tilde{N}_\tau = \mathfrak{F}(\tilde{G}, \Theta_\tau).$$

A Hamiltonian for an LQFN with squeezing components could be constructed approximately by replacing Bogoliubov components $\tilde{S}$ with dynamical components $G_{\tilde{S}}^\epsilon$. This would then fit into the QFN framework of [1].

D. Parameters for network model

We now suppose that the LQFN of Fig. 2 is described by field channels $b_1, b_{\mathrm{out},1}$ and $b_2, b_{\mathrm{out},2}$ having lengths n_1 and n_2, respectively, so that the total number is $n_1 + n_2 = n$. The system is parameterized by $G = (\tilde{S}, L, H)$, with

$\tilde{S} = \Delta(S_-, S_+)$, and we partition the matrices as

$$C_\mp = \begin{bmatrix} C_1^\mp \\ C_2^\mp \end{bmatrix}, \quad S_\mp = \begin{bmatrix} S_{11}^\mp & S_{12}^\mp \\ S_{21}^\mp & S_{22}^\mp \end{bmatrix}.$$

The field-field component of the input-output relations can be now written as

$$\breve{b}_{\mathrm{out},i} = \sum_{j=1,2} \hat{G}_{ij}(s)\breve{b}_j,$$

with transfer matrix function

$$\hat{\Xi}_G(s) = \left[\begin{array}{c|c} \tilde{A} & -[\tilde{C}_1^\flat, \tilde{C}_2^\flat]\hat{S} \\ \hline \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix} & \hat{S} \end{array}\right](s)$$
$$= -\begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix}(sI - \tilde{A})^{-1}[\tilde{C}_1^\flat, \tilde{C}_2^\flat]\hat{S} + \hat{S} \quad (57)$$

where

$$\hat{S}_{jk} = \Delta(S_{jk}^-, S_{jk}^+), \quad \tilde{C}_j = \Delta(C_j^-, C_j^+).$$

The network N_τ is given by the linear fractional transformation

$$\tilde{N}_\tau = \mathfrak{F}(\tilde{G}, \Theta_\tau) = \left[\begin{array}{c|c} \tilde{A}_\tau & -\tilde{C}_\tau^\flat \tilde{S}_\tau \\ \hline \tilde{C}_\tau & \tilde{S}_\tau \end{array}\right], \quad (58)$$

where

$$\tilde{S}_\tau = \hat{S}_{11} + \hat{S}_{12}\Theta_\tau(I - \hat{S}_{22}\Theta_\tau)^{-1}\hat{S}_{21}, \quad (59)$$
$$\tilde{C}_\tau = \tilde{C}_1 + \hat{S}_{12}\Theta_\tau(I - \hat{S}_{22}\Theta_\tau)^{-1}\tilde{C}_2, \quad (60)$$
$$\tilde{A}_\tau = \tilde{A} - \sum_{j=1,2} \tilde{C}_j^\flat \hat{S}_{j2}\Theta_\tau(I - \hat{S}_{22}\Theta_\tau)^{-1}\tilde{C}_2. \quad (61)$$

Due to the nonzero delay, the network model N_τ is non-Markovian.

E. Zero-delay-limit models

Of particular interest are the simpler models that arise in the zero-delay limit $\Theta_\tau \to I$ ($\tau \to 0$). Assume that $I - \hat{S}_{11}$ is invertible. From this, we have

$$\tilde{N}_0 = \mathfrak{F}(\tilde{G}, I) = \left[\begin{array}{c|c} \tilde{A}_0 & -\tilde{C}_0^\flat \tilde{S}_0 \\ \hline \tilde{C}_0 & \tilde{S}_0 \end{array}\right], \quad (62)$$

where

$$\tilde{S}_0 = \hat{S}_{11} + \hat{S}_{12}(I - \hat{S}_{22})^{-1}\hat{S}_{21}, \quad (63)$$
$$\tilde{C}_0 = \tilde{C}_1 + \hat{S}_{12}(I - \hat{S}_{22})^{-1}\tilde{C}_2, \quad (64)$$
$$\tilde{A}_0 = \tilde{A} - \sum_{j=1,2} \tilde{C}_j^\flat \hat{S}_{j2}(I - \hat{S}_{22})^{-1}\tilde{C}_2. \quad (65)$$

We note that

$$\tilde{A}_0 = -\tfrac{1}{2}\tilde{C}_0^\flat \tilde{C}_0 - i\tilde{\Omega}_0, \quad (66)$$

where

$$\tilde{\Omega}_0 = \tilde{\Omega} + \mathrm{Im}_\flat \sum_{j=1,2} \tilde{C}_j^\flat \hat{S}_{j2}(I - \hat{S}_{22})^{-1}\tilde{C}_2. \quad (67)$$

Here, $\mathrm{Im}_\flat X$ means $\frac{1}{2i}(X - X^\flat)$. The matrix $\tilde{S}_0$ defined by (63) is a Bogoliubov matrix, as it corresponds to the matrix in (56).

Therefore, the zero-delay limit N_0 is a Markovian system belonging to $\mathcal{L}^{\mathrm{Bog}}(n)$ with parameters

$$N_0 = \Big[\hat{S}_{11} + \hat{S}_{12}(I - \hat{S}_{22})^{-1}\hat{S}_{21}, \tilde{C}_1 + \hat{S}_{12}(I - \hat{S}_{22})^{-1}\tilde{C}_2,$$
$$\tilde{\Omega} + \mathrm{Im}_\flat \sum_{j=1,2} \tilde{C}_j^\flat \hat{S}_{j2}(I - \hat{S}_{22})^{-1}\tilde{C}_2\Big]. \quad (68)$$

Thus, $\mathcal{L}^{\mathrm{Bog}}(n)$ is closed with respect to this zero-delay-limit network construction.

Other types of limits are also considered in applications (see, e.g. [24, Sec. 2.3]). Suppose that the system G^ϵ and the delay τ^ϵ depend on a small parameter $\epsilon > 0$, defining a physical regime of operation. Then one may obtain a limit model $\lim_{\epsilon\to 0} \mathfrak{F}(G^\epsilon, \Theta_{\tau^\epsilon})$. An example of this is considered in Sec. VII B.

F. Series product

The series product $G_2 \lhd G_1$ of two systems $\tilde{G}_1 = (\tilde{S}_1, \tilde{C}_1, \tilde{\Omega}_1)$ and $\tilde{G}_1 = (\tilde{S}_2, \tilde{C}_2, \tilde{\Omega}_2)$ follows from the zero-delay limit (62). For the series product, we interchange the index 1 and 2 (this simply means interchanging the role of G_1 and G_2 in the LQFN) and then set $\hat{S}_{12} = \tilde{S}_1$, $\hat{S}_{21} = \tilde{S}_2$, $\hat{S}_{22} = 0$, and $\hat{S}_{11} = 0$. (Note that without the interchange we would be computing $G_1 \lhd G_2$ instead $G_2 \lhd G_1$.)

By substituting into (63, 64, 65), we find

$$\tilde{\Xi}_{\mathrm{series}} = \left[\begin{array}{c|c} \tilde{A} - \tilde{C}_2^\flat \tilde{S}_2 \tilde{C}_1 & -(\tilde{C}_2^\flat \tilde{S}_2 + \tilde{C}_1^\flat)\tilde{S}_1 \\ \hline \tilde{C}_2 + \tilde{S}_2\tilde{C}_1 & \tilde{S}_2\tilde{S}_1 \end{array}\right]. \quad (69)$$

The matrices for $G_2 \lhd G_1 = (\tilde{S}_{\mathrm{series}}, \tilde{C}_{\mathrm{series}}, \tilde{\Omega}_{\mathrm{series}})$ are given by

$$\tilde{S}_{\mathrm{series}} = \tilde{S}_2\tilde{S}_1,$$
$$\tilde{C}_{\mathrm{series}} = \tilde{C}_2 + \tilde{S}_2\tilde{C}_1 = \Delta(C_{\mathrm{series}-}, C_{\mathrm{series}+}),$$
$$\tilde{\Omega}_{\mathrm{series}} = \tilde{\Omega}_1 + \tilde{\Omega}_2 + \mathrm{Im}_\flat \tilde{C}_2^\flat \tilde{S}_2 \tilde{C}_1$$
$$\equiv \Delta(\Omega_{\mathrm{series}-}, \Omega_{\mathrm{series}+}),$$

where

$$C_{\mathrm{series}-} = C_{2-} + S_{2-}C_{1-} + S_{2+}C_{1+}^\#,$$
$$C_{\mathrm{series}+} = C_{2+} + S_{2-}C_{1+} + S_{2+}C_{1-}^\#.$$

From

$$\tilde{C}_2^\flat \tilde{S}_2 \tilde{C}_1 = \Delta(C_{2-}^\dagger, -C_{2+}^\top)\Delta(S_{2-}, S_{2+})\Delta(C_{1-}, C_{1+})$$
$$= \Delta(X_-, X_+)$$

with

$$X_\mp = (C_{2-}^\dagger S_{2-} - C_{2+}^\top S_{2+}^\#)C_{1\mp} + (C_{2-}^\dagger S_{2+} - C_{2+}^\top S_{2-}^\#)C_{1\pm}^\#,$$

we see that

$$\Omega_{\mathrm{series}-} = \Omega_{1-} + \Omega_{2-} + \frac{1}{2i}(X_- - X_-^\dagger),$$
$$\Omega_{\mathrm{series}+} = \Omega_{1+} + \Omega_{2+} + \frac{1}{2i}(X_+ + X_+^\top).$$

Succinctly, the series product in $\mathcal{L}^{\mathrm{Bog}}(n)$ is given by

$$G_2 \lhd G_1 = [\tilde{S}_2\tilde{S}_1, \tilde{C}_2 + \tilde{S}_2\tilde{C}_1, \tilde{\Omega}_1 + \tilde{\Omega}_2 + \mathrm{Im}_\flat(\tilde{C}_2^\flat \tilde{S}_2 \tilde{C}_1)]; \quad (70)$$

cf. (34).

Clearly, $\mathcal{L}^{\rm Bog}(n)$ is a group with respect to the series product, and the classes of components $\mathcal{L}^{\rm HP}(n)$, $\mathrm{Sp}(\mathbb{C}^n)$, and $U(n)$ are subgroups. If $G = (\tilde{S}, \tilde{C}, \tilde{\Omega})$, the inverse is given by

$$G^{-1} = (\tilde{S}^\flat, -\tilde{S}^\flat \tilde{C}, -\tilde{\Omega}), \tag{71}$$

with transfer function given by

$$\tilde{\Xi}_{G^{-1}}(s) \equiv \tilde{\Xi}_G(s^*)^\flat. \tag{72}$$

The series product is not limited simply to feedforward, and this formula applies to the case where the two systems have one or more modes in common. The series product is therefore highly nontrivial [1,2,5].

G. Series product and cascaded transfer functions

In classical linear systems theory, the transfer function of a cascade of two separate systems is obtained by multiplying the transfer functions (see, e.g., [19]). We need to emphasize here that two systems are distinct if they consist of different oscillator modes. Specifically, let there be m_1 modes in the first system and m_2 in the second, and set

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathcal{S}(m_1 + m_2).$$

Then we consider $G_1 = (\tilde{S}_1, \tilde{C}_1, \tilde{\Omega}_1)$ and $G_2 = (\tilde{S}_2, \tilde{C}_2, \tilde{\Omega}_2)$ with

$$\begin{aligned} \tilde{C}_1 &= \Delta([C_{1-}, 0], [C_{1+}, 0]), \\ -i\tilde{\Omega}_1 &= -\Delta\left(\begin{bmatrix} i\Omega_{1-} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} i\Omega_{1+} & 0 \\ 0 & 0 \end{bmatrix}\right), \\ \tilde{C}_2 &= \Delta([0, C_{2-}], [0, C_{2+}]), \\ -i\tilde{\Omega}_2 &= -\Delta\left(\begin{bmatrix} 0 & 0 \\ 0 & i\Omega_{2-} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & i\Omega_{2+} \end{bmatrix}\right), \end{aligned}$$

with respect to the decomposition $\mathbb{C}^m = \mathbb{C}^{m_1} \oplus \mathbb{C}^{m_2}$. This is simply a statement that the dynamics of G_1 does not depend on the internal variables of G_2 and vice versa. By putting this particular form into (69), we then obtain

$$\begin{aligned} \tilde{\Xi}_{G_2 \lhd G_1}(s) &= \left[\begin{array}{c|c} \begin{bmatrix} \bar{A}_1 & 0 \\ -\bar{C}_2^\flat \tilde{S}_2 \bar{C}_1 & \bar{A}_2 \end{bmatrix} & \begin{bmatrix} -\bar{C}_1^\flat \tilde{S}_1 \\ -\bar{C}_2^\flat \tilde{S}_2 \tilde{S}_1 \end{bmatrix} \\ \hline [\tilde{S}_2 \bar{C}_1, \bar{C}_2] & \tilde{S}_2 \tilde{S}_1 \end{array}\right](s) \\ &= \tilde{\Xi}_{G_2}(s)\, \tilde{\Xi}_{G_1}(s), \end{aligned} \tag{73}$$

where $\bar{C}_j = \Delta(C_{j,-}, C_{j,+})$ and $\bar{A}_j = -\frac{1}{2}\bar{C}_j^\flat \bar{C}_j - i\Delta(\Omega_{j,-}, \Omega_{j,+})$. Here we use the unitary transformation

$$\breve{a} \mapsto \begin{bmatrix} \breve{a}_1 \\ \breve{a}_2 \end{bmatrix}$$

to present the transfer function in a more convenient form. The algebra is then similar to that in Sec. IV A in [5].

If the systems are not separate, then we do not expect such a factorization of the transfer function to hold. The series product [1,2,5] is defined quite generally in terms of physical parameters (Sec. III E), which may, for example, depend on the same oscillator mode variables. In the general case, the transfer function can be computed using the general formulas (62)–(65).

Let us remark that the series product inverse G^{-1} given by (71) may be realized in terms of a physical system that is *not* separate from the original system. Physically, if we pass input fields through a system with parameters G, then G^{-1} gives the parameters required to undo the effect by passing the output back through the *same* system for a second pass.

H. Inverse transfer functions

The input-output relation $\breve{b}_{\rm out} = \tilde{\Xi}\breve{b}_{\rm in} + \tilde{\xi}\breve{a}(0)$ may be inverted to yield

$$\breve{b}_{\rm in} = \tilde{\Xi}^{-1}\breve{b}_{\rm out} - \tilde{\Xi}^{-1}\tilde{\xi}\breve{a}(0).$$

In particular, we can give the following useful description of $\tilde{\Xi}^{-1}$. The linear equations (28) in the time domain may be rearranged algebraically to give

$$\begin{aligned} \frac{d}{dt}\breve{a} &= (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})\breve{a} + \tilde{B}\tilde{D}^{-1}\breve{b}_{\rm out}, \\ \breve{b}_{\rm in} &= -\tilde{D}^{-1}\tilde{C}\breve{a} + \tilde{D}^{-1}\breve{b}_{\rm out}, \end{aligned}$$

with $\tilde{D}$ invertible, and in the transform domain, we deduce that

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]^{-1} = \left[\begin{array}{c|c} \tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C} & \tilde{B}\tilde{D}^{-1} \\ \hline -\tilde{D}^{-1}\tilde{C} & \tilde{D}^{-1} \end{array}\right].$$

For the model with parameters $G = (\tilde{S}, \tilde{C}, \tilde{\Omega})$, we find

$$\begin{aligned} \tilde{\Xi}_G^{-1}(s) &= \left[\begin{array}{c|c} \tilde{A} & -\tilde{C}^\flat \tilde{S} \\ \hline \tilde{C} & \tilde{S} \end{array}\right]^{-1} \equiv \left[\begin{array}{c|c} -\tilde{A}^\flat & -\tilde{C}^\flat \\ \hline -\tilde{S}^\flat \tilde{C} & \tilde{S}^\flat \end{array}\right] \\ &= \tilde{S}^\flat \tilde{C}(sI + \tilde{A}^\flat)^{-1}\tilde{C}^\flat + \tilde{S}^\flat, \end{aligned}$$

or

$$\tilde{\Xi}_G(s)^{-1} \equiv \tilde{\Xi}_G(-s^*)^\flat. \tag{74}$$

We note that $\tilde{\Xi}_{G^{-1}}(s) = \tilde{\Xi}_G(-s)^{-1} \equiv \tilde{\Xi}_G(s^*)^\flat$ [recall (72)].

1. Example: separate cavity inverse

As a concrete example, consider the single mode cavity $G = G_{\rm cav}$ considered in Sec. IV B. The transfer function (45) and related functions are given by

$$\begin{aligned} \Xi_{G,-}(s) &= \frac{s - (\gamma/2) + i\omega}{s + (\gamma/2) + i\omega}, \\ \Xi_{G^{-1},-}(s) &= \frac{s - (\gamma/2) - i\omega}{s + (\gamma/2) - i\omega}, \\ \Xi_{G,-}^{-1}(s) &= \frac{s + (\gamma/2) + i\omega}{s - (\gamma/2) + i\omega}. \end{aligned}$$

That is, G^{-1} is obtained from G by keeping $C_- = \sqrt{\gamma}$, $C_+ = 0$, and replacing $\Omega_- = \omega$ by $-\omega$. In what follows, we obtain a physical realization $\hat{G}$ of the transfer function $\Xi_{G,-}^{-1}(s)$ that is a system distinct from G, so that

$$\tilde{\Xi}_{\hat{G}}(s) = \tilde{\Xi}_G^{-1}(s). \tag{75}$$

The system $\hat{G}$ is obtained by setting $\Omega_- = -\omega$, $\Omega_+ = 0$ and swapping $C_- = 0$ and $C_+ = \sqrt{\gamma}$.

In this example, we can take the two modes to be

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathcal{S}(2)$$

and write G as $(\tilde{S}_1 = I, \tilde{C}_1, \tilde{\Omega}_1)$, where

$$\tilde{C}_1 = \Delta([\sqrt{\gamma}, 0], [0, 0]),$$
$$-i\tilde{\Omega}_1 = \Delta\left(\begin{bmatrix} -i\omega & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right),$$

and $\hat{G}$ as $(\tilde{S}_2 = I, \tilde{C}_2, -\tilde{\Omega}_2)$, where

$$\tilde{C}_2 = \Delta([0, 0], [0, \sqrt{\gamma}]),$$
$$-i\tilde{\Omega}_2 = -\Delta\left(\begin{bmatrix} -i\omega & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right).$$

That is, in terms of scattering matrices, coupling operators, and Hamiltonians (31), we have

$$G = (I, \sqrt{\gamma} a_1, \omega a_1^* a_1) \quad \text{and} \quad \hat{G} = (I, \sqrt{\gamma} a_1^*, -\omega a_2^* a_2).$$

The series product $\hat{G} \lhd G$ is given by $[I, \tilde{C} = \tilde{C}_1 + \tilde{C}_2, \tilde{\Omega} = \tilde{\Omega}_1 + \tilde{\Omega}_2 + \mathrm{Im}_\flat(\tilde{C}_2^\flat \tilde{C}_1)]$ [recall (70)], and this corresponds to a system with a nontrivial dynamics, since, by (34),

$$\hat{G} \lhd G = [I, \sqrt{\gamma}(a_1 + a_2^*), \quad \omega(a_1^* a_1 - a_2^* a_2) + \mathrm{Im}(a_2 a_1)].$$

Nevertheless, some calculation shows that

$$\tilde{\Xi}_{\hat{G} \lhd G}(s) = I,$$

as required by (73) and (75). This continues to hold if there is a term $\Omega_+ = \omega_+$ added to the cavity Hamiltonian.

2. General separate system inverses

We now show how to obtain a physical realization for $\Xi_G^{-1}(s)$ in the case where $\tilde{\Omega} = 0$. We have

$$\tilde{\Xi}_G^{-1}(s) = \tilde{S}^\flat + \tilde{S}^\flat \tilde{C}\left(sI - \tfrac{1}{2}\tilde{C}^\flat \tilde{C}\right)^{-1} \tilde{C}^\flat$$
$$= \left[I + \tilde{S}^\flat\left(s - \tfrac{1}{2}\tilde{C}\tilde{C}^\flat\right)^{-1} \tilde{C}\tilde{C}^\flat\right]\tilde{S}^\flat.$$

Now, $\tilde{C}\tilde{C}^\flat$ is not definite, and in fact we have

$$\tilde{C}\tilde{C}^\flat = \Delta(C_- C_-^\dagger - C_+ C_+^\dagger, -C_- C_+^\top + C_+ C_-^\top)$$
$$= -\Delta(C_+, C_-)\Delta(C_+^\dagger, -C_-^\top)$$
$$\equiv -\tilde{K}\tilde{K}^\flat,$$

where

$$\tilde{K} = \Delta(C_+, C_-). \tag{76}$$

We therefore obtain

$$\tilde{\Xi}_G^{-1}(s) = \left[I - \tilde{S}^\flat\left(s + \tfrac{1}{2}\tilde{K}\tilde{K}^\flat\right)^{-1} \tilde{K}\tilde{K}^\flat\right]\tilde{S}^\flat$$
$$= \left[I - \tilde{S}^\flat \tilde{K}\left(s + \tfrac{1}{2}\tilde{K}^\flat \tilde{K}\right)^{-1} \tilde{K}^\flat\right]\tilde{S}^\flat$$
$$\equiv \tilde{\Xi}_{\hat{G}}(s),$$

where

$$\hat{G} = (\tilde{S}^\flat, \tilde{S}^\flat \tilde{K}, 0). \tag{77}$$

We note that this choice of $\hat{G}$ is not unique in producing a transfer function inverse to $\tilde{\Xi}_G$. Finding an inverse when $\tilde{\Omega} \neq 0$ is more involved.

3. Comments

We have the mapping $G \mapsto \tilde{\Xi}_G$ from the group of $\mathcal{L}^{\mathrm{Bog}}(n)$ of system parameters with series product to the group of matrix transfer functions. However, this mapping is not a group homomorphism. Indeed, we typically have

$$\tilde{\Xi}_{G_2 \lhd G_1}(s) \neq \tilde{\Xi}_{G_2}(s)\tilde{\Xi}_{G_1}(s),$$

though equality—the cascade formula (73)—holds when the systems are separate assemblies of oscillators.

We should also caution that, as we have seen in the example in Sec. VI H1, there are solutions for G other than the trivial $G = (I, 0, 0)$ to the equation $\tilde{\Xi}_G(s) = I$.

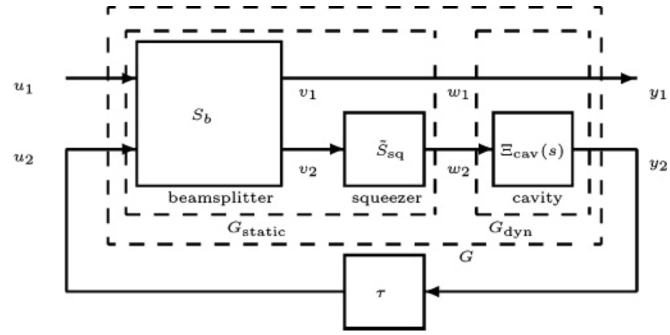

FIG. 3. LQFN of Fig. 1 redrawn in standard from Fig. 2.

VII. NETWORK EXAMPLES

A. In-loop squeezing and cavity as a feedback network

We now describe the LQFN of Fig. 1, which contains a cavity and squeezer in a feedback loop resulting from interconnection with a beam splitter. The total propagation delay around the loop is τ, which we take to be small, and send $\tau \to 0$. In order to determine an equivalent zero-delay-limit model, following the general approach in Sec. VI E, we redraw the network as shown in Fig. 3.

As indicated in Fig. 3, the in-loop system $G = G_{\mathrm{dyn}} \lhd G_{\mathrm{static}}$ is a dynamical Bogoliubov component obtained by cascading the beam splitter, the (augmented) squeezer (which together form G_{static}), and the (augmented) cavity G_{dyn}.

The static part, G_{static}, is described as follows. Because the beam splitter has two inputs and two outputs, we augment the squeezer S_{sq} [given by (50)] by including a direct feed through channel (v_1 to w_1 in Fig. 3). Because the squeezer is represented by a static Bogoliubov transformation expressed in doubled-up form, we express the beam splitter in doubled-up form: $\tilde{S}_b = \Delta(S_b, 0)$. To be clear, the beam splitter is described by

$$\begin{aligned} v_1 &= \alpha u_1 - \beta u_2, \\ v_2 &= \beta u_1 + \alpha u_2, \end{aligned} \tag{78}$$

where $|\alpha|^2 + |\beta|^2 = 1$, $\alpha^*\beta = \beta^*\alpha$. Thus, we have

$$S_b = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \tag{79}$$

and

$$\tilde{S}_b = \Delta(S_b, 0)$$
$$= \begin{bmatrix} \alpha & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha^* & -\beta^* \\ 0 & 0 & \beta^* & \alpha^* \end{bmatrix}. \tag{80}$$

The static component $G_{\rm static}$ has inputs $(u_1, u_2)^\top$ and outputs $(w_1, w_2)^\top$ and is given by the Bogoliubov matrix

$$\tilde{R} = \Delta\left(\begin{bmatrix}1 & 0\\ 0 & \cosh r\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & \sinh r\end{bmatrix}\right)\tilde{S}_b = \Delta(R_-, R_+), \tag{81}$$

where

$$R_- = \begin{bmatrix}\alpha & -\beta\\ \beta\cosh r & \alpha\cosh r\end{bmatrix}, \quad R_+ = \begin{bmatrix}0 & 0\\ \beta^*\sinh r & \alpha^*\sinh r\end{bmatrix}. \tag{82}$$

The dynamic component $G_{\rm dyn}$, with inputs $(w_1, w_2)^\top$ and outputs $(y_1, y_2)^\top$, is given by

$$\begin{bmatrix}\dot{a}\\ \dot{a}^*\end{bmatrix} = \begin{bmatrix}-(\frac{\gamma}{2}+i\omega) & 0\\ 0 & -(\frac{\gamma}{2}-i\omega)\end{bmatrix}\begin{bmatrix}a\\ a^*\end{bmatrix} - \begin{bmatrix}0 & \sqrt{\gamma} & 0 & 0\\ 0 & 0 & 0 & \sqrt{\gamma}\end{bmatrix}\begin{bmatrix}w_1\\ w_2\\ w_1^*\\ w_2^*\end{bmatrix}, \tag{83}$$

$$\begin{bmatrix}y_1\\ y_2\\ y_1^*\\ y_2^*\end{bmatrix} = \begin{bmatrix}0 & 0\\ \sqrt{\gamma} & 0\\ 0 & 0\\ 0 & \sqrt{\gamma}\end{bmatrix}\begin{bmatrix}a\\ a^*\end{bmatrix} + \begin{bmatrix}w_1\\ w_2\\ w_1^*\\ w_2^*\end{bmatrix}.$$

Thus, $A_- = -(\frac{\gamma}{2}+i\omega)$, $A_+ = 0$, $\tilde{A} = \Delta[-(\frac{\gamma}{2}+i\omega), 0]$,

$$C_- = \begin{bmatrix}0\\ \sqrt{\gamma}\end{bmatrix}, \quad C_+ = \begin{bmatrix}0\\ 0\end{bmatrix}, \quad \tilde{C} = \Delta\left(\begin{bmatrix}0\\ \sqrt{\gamma}\end{bmatrix}, 0\right). \tag{84}$$

Also, $\Omega_- = \omega$ and $\Omega_+ = 0$.

Now that we have a complete model for the in-loop system G, we may apply the formulas in Sec. VI E to obtain a zero-delay network model N_0. This involves first working out the Bogoliubov matrix in partitioned form:

$$\hat{S}_{11} = \Delta(\alpha, 0), \quad \hat{S}_{12} = \Delta(-\beta, 0), \quad \hat{S}_{21} = \Delta(\beta\cosh r, \beta^*\sinh r), \quad \hat{S}_{22} = \Delta(\alpha\cosh r, \alpha^*\sinh r). \tag{85}$$

We set $\alpha = \sqrt{\epsilon}$ and $\beta = \sqrt{1-\epsilon}$ to simplify some of the algebra. The network model is given as follows. We use Eqs. (63, 64, 65) to determine the network parameters. The equivalent network Bogoliubov matrix is

$$\tilde{S}_0 = \Delta\left(\sqrt{\epsilon} - \frac{1-\epsilon}{\mu}(\cosh r - \sqrt{\epsilon}), -\frac{(1-\epsilon)\sinh r}{\mu}\right), \tag{86}$$

where

$$\mu = 1 - 2\sqrt{\epsilon}\cosh r + \epsilon. \tag{87}$$

Next,

$$\tilde{C}_0 = -\frac{\sqrt{1-\epsilon}\sqrt{\gamma}}{\mu}\Delta(1-\sqrt{\epsilon}\cosh r, \sqrt{\epsilon}\sinh r), \tag{88}$$

so that

$$C_0^- = -\frac{\sqrt{1-\epsilon}\sqrt{\gamma}}{\mu}(1-\sqrt{\epsilon}\cosh r),$$
$$C_0^+ = -\frac{\sqrt{1-\epsilon}\sqrt{\gamma}}{\mu}\sqrt{\epsilon}\sinh r.$$

Now, $\tilde{A}_0 = \Delta(A_0^-, A_0^+)$, where

$$A_0^- = -\left(\frac{\gamma}{2}+i\omega\right) - \frac{\sqrt{\epsilon}\gamma}{\mu}(\cosh r - \sqrt{\epsilon}), \quad A_0^+ = \frac{\sqrt{\epsilon}\gamma}{\mu}\sinh r. \tag{89}$$

From this, we compute

$$\Omega_0^- = \omega, \quad \Omega_0^+ = i\frac{\sqrt{\epsilon}\gamma}{\mu}\sinh r. \tag{90}$$

We therefore see that not only does the network model $N_0 \in \mathcal{L}^{\rm Bog}(1)$ have a nontrivial static Bogoliubov term, it also has field couplings involving a creation operator a^* and Hamiltonian terms involving a^2 and $(a^*)^2$.

Stability of the feedback system may be analyzed using the methods in Sec. III D or the small gain theorem [19,25].

As a possible application, we note that the squeezing parameter of a DPA may be altered by placing it in-loop in a beam splitter arrangement of this type [26].

B. Dynamics from feedback

In this example, we give an illustration from quantum optics showing that LQFNs involving only static components may give rise to dynamical behavior. This dynamical behavior is due to a time delay in the feedback loop. We consider the network shown in Fig. 4 [1,7]. This is a special case of the LQFN network of Fig. 1 but with no squeezing and no cavity. The beam splitter S_b is given by (78) or (79), with $\alpha = \sqrt{\epsilon}$ and $\beta = \sqrt{1-\epsilon}$.

Feedback in the network is defined by the constraint $u_2(t) = \Theta_\tau y_2(t) = y_2(t-\tau)$, where $\tau > 0$ is the time taken for light to travel from the output to the input.

This network is a LQFN with $G^\epsilon = [\Delta(S_b, 0), 0, 0] \in U(2)$. With $\epsilon > 0$ fixed, the zero-delay network model $\tau \to 0$ is the system $N_0 = \mathfrak{F}_l(G^\epsilon, I) = (\Delta(-I, 0), 0, 0) \in U(1)$, with transfer function $N_0(s) = -1$, a trivial pass-through system with sign change (phase shift).

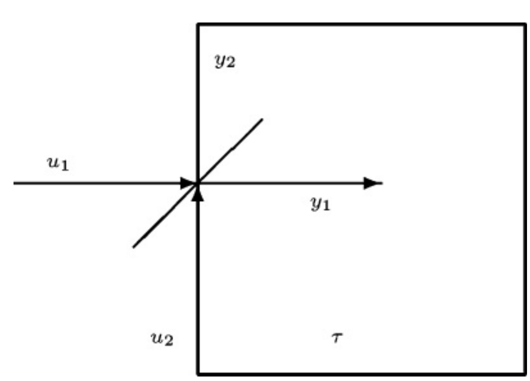

FIG. 4. Beam splitter feedback network with time delay.

If the reflectivity coefficient ϵ and the time delay are comparable, say $\tau = \epsilon/\gamma$, where $\gamma > 0$, then we obtain a dynamical model as $\epsilon \to 0$ [24, Sec. 2.3] (recall Sec. VI E). Indeed, by solving (79) and (80) in the frequency domain, we find that the transfer function is

$$N^{\epsilon}(s) = \mathfrak{F}_l(G^{\epsilon}, \Theta_{\tau^{\epsilon}})(s) = \sqrt{1-\epsilon} - \frac{\epsilon e^{-s\epsilon/\gamma}}{1-\sqrt{1-\epsilon}e^{-s\epsilon/\gamma}}. \tag{91}$$

By L'Hopital's rule, we find that the limit transfer function is

$$N(s) = \lim_{\epsilon\to 0} N^{\epsilon}(s) = 1 - \frac{\gamma}{s+\gamma/2} = \frac{s-\gamma/2}{s+\gamma/2}. \tag{92}$$

This transfer function corresponds to a cavity $N = (I, \sqrt{\gamma}a, 0) = (I, \sqrt{\gamma}I, 0) \in \mathcal{L}^{\mathrm{HP}}(1,1)$, where $a \in \mathcal{S}(1)$, [22, 24]. Here, γ plays the role of the coupling strength between the trapped cavity mode and the external free field.

This example shows that $U(2)$ is not closed under this type of physically natural approximation process [since the limit belongs to $\mathcal{L}^{\mathrm{HP}}(1)$, which is outside $U(2)$], while $\mathcal{L}^{\mathrm{Bog}}(2)$ is closed [since it contains both $U(2)$ and $\mathcal{L}^{\mathrm{HP}}(1)$].

VIII. CONCLUSION

We have shown how to extend linear quantum dynamical network theory to include static Bogoliubov components (such as squeezers). This unified framework accommodates squeezing components, which are important in quantum information applications. We provided tools for describing network connections and feedback using generalizations of linear fractional transformations and the series product [2,5,7,19]. We have also defined input-output maps and transfer functions within this linear quantum network theory and shown how they can be used in applications. Finally, we explained the natural group structure arising from the series product.

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