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Dynamics of a Bose-Einstein condensate in a horizontally vibrating shallow optical lattice

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We consider a solitonic solution of the self-attractive Bose-Einstein condensate in a one-dimensional external potential of a shallow optical lattice with large periodicity when the lattice is horizontally shaken. We investigate the dynamics of the bright soliton through the properties of the fixed points. The special type of bifurcation results in a simple criterion for the stability of the fixed points depending only on the amplitude of the shaking lattice. Because of the similarity of the equations with those of an ac-driven Josephson junction, some results may find applications in other branches of physics.

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I. INTRODUCTION

Stabilization of the unstable fixed points by periodic forces has always been an interesting problem; a case in point is that of the inverted pendulum. [1,2] In such problems, the oscillating force, which is usually much faster than the natural evolution of the autonomous system, effectively averages to a nonzero stabilizing (or destabilizing) force. For the Bose-Einstein condensate (BEC), in the mean-field approximation, the dynamics is governed by the nonlinear Schrödinger equation (NLSE). With the attractive interaction between the atoms, localized matter waves known as the bright solitons have been observed in effectively one-dimensional BECs [3]. External potentials produced by magnetic fields or laser beams, are used to trap and manipulate the BEC matter waves [4].

The idea of the stabilizing forces due to the presence of time-dependent potentials has found numerous applications in BEC-related studies. For instance, investigation of the motion of a soliton in an inverted trap, when its strength is varied rapidly and periodically, shows the formation of stable equilibria for a range of amplitude to frequency ratios of the varying trap [5]. Following the seminal paper of Dunlap and Kenkre [6], a quantum-mechanical version of stabilizing effects by the fast oscillating forces has been characterized in the BEC context as *dynamic localization.* In dynamic localization, tunneling is suppressed as a consequence of shaking of the periodic potential, when the ratio of the magnitude and the frequency is a root of the zeroth-order Bessel function [7]. Time-dependent parameters have also been suggested to prevent the condensate from collapsing and to stabilize it in two and three dimensions [8].

Studies of the NLSE when perturbed by a periodic potential is not a new problem [9,10], but the experimental ground made available by the advent of optical lattices (OLs) has renewed interest in it as an important topic of study in BEC-related research [11,12]. Changing the parameters of the optical lattice, e.g., the depth of the potential, allows the control of the interaction among the atoms and the study of the predicted quantum phase transitions [13]. Motion of the lattice leads to the manipulation of the internal structure of the atom cloud and the formation of vortices and solitons [14].

Here, we study the soliton dynamics for a one-dimensional (1D) BEC in a horizontally shaken periodic potential. The parameters are chosen so that the soliton is stable in the time scales of interest: the potential of the OL is chosen to be smaller than the interaction energy of the BEC, and the wavelength of the OL is large compared to the length scale of the matter wave. We state our problem in terms of how the shaking of an OL influences the properties of the fixed points. The equation describing the soliton dynamics can be expressed as that of a periodically forced pendulum or a driven Josephson junction [15,16]. This is in the same spirit as that of the stability conditions for the inverted pendulum, where the pivot is harmonically displaced, and instead of an external periodic force, the system is parametrically driven.

II. THE MODEL

A Bose-Einstein condensate when confined to one dimension is described by the NLSE as follows:

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + NG_{1\mathrm{D}}|\phi|^2\phi + V_{\mathrm{ext}}(x,t)\phi, \quad (2.1)$$

where ϕ is the mean-field wave function, N is the total number of atoms in the condensate, and $G_{1\mathrm{D}}$ is the coefficient of the nonlinear term. $G_{1\mathrm{D}} = 2\hbar a_s\omega_\perp$ characterizes the interatomic interaction in the condensate, with a_s being the scattering length, and $\omega_\perp$ the confinement frequency in the transverse direction. To rewrite the equation in a dimensionless form, we scale the time and the coordinate by $\tau = 2\hbar/(N|G_{1\mathrm{D}}|)$ and $\xi = \hbar/\sqrt{mN|G_{1\mathrm{D}}|}$, respectively; we have

$$i\frac{\partial\phi}{\partial t} + \frac{\partial^2\phi}{\partial x^2} + 2|\phi|^2\phi = V(x,t)\phi. \quad (2.2)$$

Here, we have assumed an attractive atom-atom interaction (i.e., negative scattering length). We take the normalized external potential $V(x,t) = 2V_{\mathrm{ext}}/(N|G_{1\mathrm{D}}|)$ as

$$V(x,t) = \Omega^2x^2 + \epsilon\cos[k(x - \Delta\sin\omega t)], \quad (2.3)$$

where the first term defines the longitudinal magnetic trap, with $\Omega = \hbar\omega_x/(N|G_{1\mathrm{D}}|)$, and ω_x the axial trap frequency. The second term characterizes the potential of the OL; $\epsilon = 2V_0/(N|G_{1\mathrm{D}}|)$, V_0 being the OL strength. k is the the wave number of the OL; the amplitude and frequency of vibration of the OL are denoted, respectively, by Δ and ω. All lengths are scaled by ξ, and all times are scaled by τ.

Equation (2.2) with $\Delta = \Omega = 0$ is studied by Scharf and Bishop [10] and by Kevrekidis *et al.* for $\Delta = 0$ [17]. In this paper, we assume that the longitudinal magnetic trap is absent,

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or that the axial frequency Ω is so small as to be neglected compared with ϵ. In the absence of any external potential, Eq. (2.2) has a stationary solitonic solution as

$$\phi(x,t) = \eta \frac{\exp(it/2\eta^2)}{\cosh[\eta(x-q)]}, \tag{2.4}$$

where η defines the soliton amplitude as well as its width, and q is the position of the soliton center. In the presence of the periodic potential, we treat q, the *particle coordinate*, as a variational parameter. Fixing η, and with no other variational parameter, we are assuming that the soliton is stable in the time scales of our study and that radiations are negligible. These conditions are satisfied when we keep the OL amplitude and wave number small; we will return to this point below. For our purposes, it is enough to write the equation of motion for q, which is independent of the phase [10], as

$$\frac{d^2q}{dt^2} = -\frac{1}{\eta}\frac{dV_{\text{eff}}}{dq}. \tag{2.5}$$

The effective potential is defined as

$$V_{\text{eff}}(q) = \epsilon \int_{-\infty}^{+\infty} dx |\phi(x-q)|^2 \cos[k(x-\Delta \sin \omega t)]. \tag{2.6}$$

Using Eq. (2.4) we find

$$V_{\text{eff}} = \frac{k\pi\epsilon}{\sinh(k\pi/2\eta)} \cos[k(q-\Delta \sin \omega t)]. \tag{2.7}$$

The equation of motion for the soliton is obtained from Eq. (2.5):

$$\frac{d^2q}{dt^2} + A \sin[\delta \sin(\omega t) - q] = 0. \tag{2.8}$$

Here, we have put $q \to kq$; $\delta = k\Delta$, and

$$A = \frac{k^3\pi\epsilon}{\eta \sinh(k\pi/2\eta)}. \tag{2.9}$$

The notion of a particle for the soliton is valid when the length scale of the soliton which is proportional to $1/\eta$ is much smaller than the wavelength of the OL $\lambda = 2\pi/k$ [10]. For all the results reported here, we keep this criterion by choosing $k/\eta \approx 1/2$, although our investigations show that we can keep the particle notion even when the two length scales are comparable in magnitude, i.e., $k \simeq \eta$.

Figure 1 demonstrates the comparison between the numerical integration of Eq. (2.2) and what results from the numerical integration of Eq. (2.8). It is not far from expectation to find the results matching each other. The idea is that the unstable point of the stationary potential changes to a (uniformly) stable point, here a center, when the OL is vibrated with a definite amplitude and frequency. Since there is no dissipation present in the system under study, the stable critical points of Eq. (2.8) will be centers. One can imagine a small amount of damping in Eq. (2.2), changing the centers to stable spirals which are asymptotically stable [18]. This handmade damping is different from what is usually introduced in the BEC context as the damped NLSE [19], where an imaginary term, reminding one of the imaginary potential in nuclear physics, is introduced causing the norm of the condensate to become nonconserved.

It is worth mentioning here that a change of variable according to $\theta = q - \delta \sin \omega t$ transforms Eq. (2.8) to the equation describing a driven undamped pendulum:

$$\frac{d^2\theta}{dt^2} + A \sin\theta = -\delta\omega^2 \sin \omega t. \tag{2.10}$$

This equation also describes an undamped Josephson junction which is periodically excited, in which case θ is the superconducting phase difference across the junction and A plays the role of the critical current [16]. There is vast literature on this topic [20], and we will make contact with it in the next section, when we derive an analytic condition for the stability of the fixed points for Eq. (2.8).

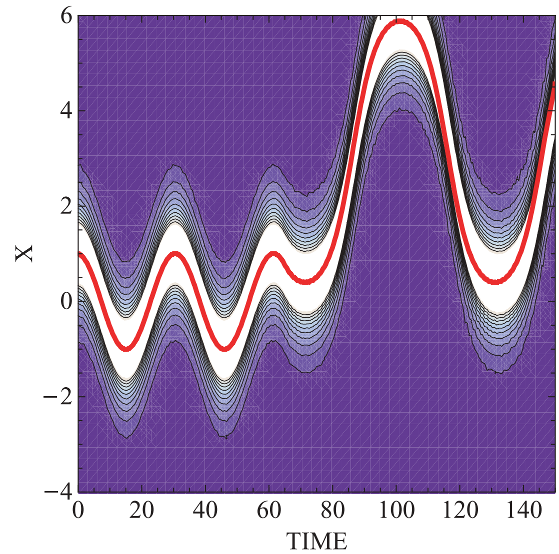

FIG. 1. (Color online) A contour plot of the soliton dynamics in the presence of the OL potential, resulting from the direct integration of Eq. (2.2). At $t = 65$, the oscillation amplitude is switched from $\delta = 3$ to $\delta = 2$. The other parameters are $\omega = 5$, $\epsilon = 0.2$, and $k = 1$, and as the initial conditions we have assumed a stationary soliton with $q(0) = 1$ and $\eta = 2$. The red thick line shows the result of the integration of Eq. (2.8) with the coordinate defined as $x = kq$ to match Eq. (2.2). All the quantities in the figures are dimensionless.

Figure 1 and all other plots have been sketched for the value of the OL strength $\epsilon = 0.2$. Returning to the question of the limits of stability of the soliton, as ϵ increases past 0.2, we observe considerably more radiation, although we repeated the calculations for the OL strengths of up to $\epsilon \lesssim 1$, where the soliton survives in the time scales of our experiments, and found very similar results. Since in Eq. (2.3) energies are normalized by $N|G_{1\text{D}}|$, this criterion means that the OL strength V_0 should remain smaller than the condensate interaction energy. The maximum number of the atoms in a 1D condensate is limited by its size [11]. Here we need the size of the BEC to be less than the OL periodicity. For an optical lattice with the periodicity of 20μ [21], an estimate for the number of atoms and the depth of the OL potential would be $N \sim 20\,000$ and $V_0 \lesssim 10^{-5}\,\hbar\omega_\perp$, respectively.

III. SLOW AND FAST DYNAMICS

With $\sqrt{A} \ll \omega$, two time scales will be present in Eq. (2.8), and it is reasonable to consider the dynamical variable as the sum of a slow component, v, and a fast component, u;

namely, $q = v + u$. Equation (2.8) can then be separated into two equations for fast and slow variables (we have scaled time by $\sqrt{A}$), that is,

$$\ddot{u} + J_0(\delta)u\cos v + S_{2n-1}\cos v - S_{2n}\sin v = 0, \tag{3.1}$$

$$\ddot{v} - \langle S_{2n-1}u\rangle_T \sin v - \langle S_{2n}u\rangle_T \cos v - J_0(\delta)\sin v = 0, \tag{3.2}$$

with

$$S_{2n} = 2\sum_{n=1}^{\infty} J_{2n}(\delta)\sin 2n\omega t, \tag{3.3}$$

$$S_{2n-1} = 2\sum_{n=1}^{\infty} J_{2n-1}(\delta)\sin(2n-1)\omega t, \tag{3.4}$$

where J_n is the Bessel function of nth order, and $\langle\,\rangle_T$ denotes time averaging of the fast variable over $T = 2\pi/\omega$. Assuming the slow component v to be constant over the range of this fast time scale, Eq. (3.1) can be solved, and $\langle S_{2n-1}u\rangle_T$ and $\langle S_{2n}u\rangle_T$ calculated. Putting the results into Eq. (3.2), we have

$$\ddot{v} - J_0(\delta)\sin v + 2f(v)\sin 2v = 0, \tag{3.5}$$

with

$$f(v) = -\sum_{n=1}^{\infty}\frac{(-1)^n J_n^2(\delta)}{\Lambda^2 - n^2\omega^2}. \tag{3.6}$$

Λ is a function of v as follows:

$$\Lambda^2 = J_0(\delta)\cos v. \tag{3.7}$$

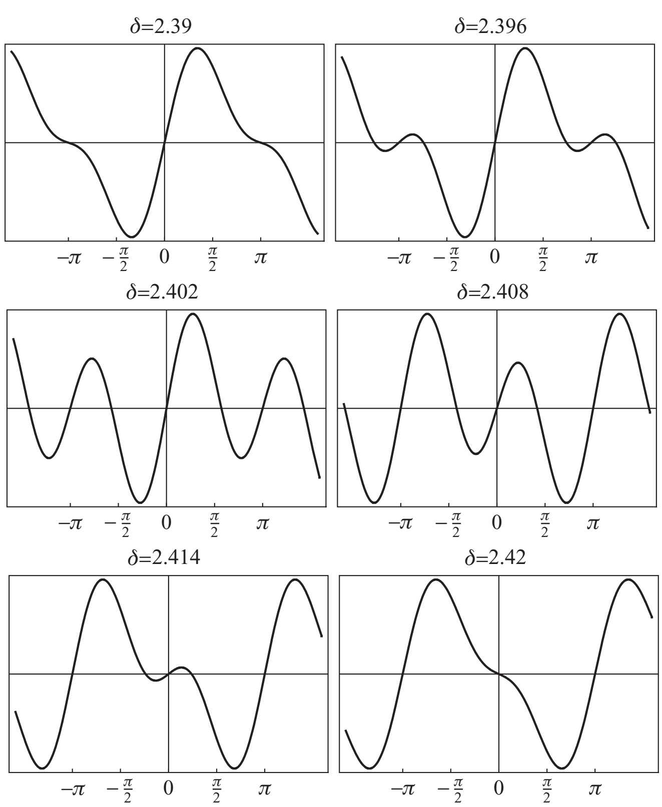

FIG. 2. Six plots show $F(v,\delta)$ (arbitrary units) vs. v for various values of δ in the vicinity of the bifurcation point. Other parameters are $\omega = 5$ and $\epsilon = 0.2$.

With $|\Lambda| \leqslant 1$, we have $\omega \gg |\Lambda|$. Now, factoring out $1/\omega^2$, the absolute value of the summation can be shown to be smaller than $\pi^2/12$. So, $f(v) \sim O(1/\omega^2)$, and the coefficient of $\sin 2v$ in Eq. (3.5) can be neglected, once δ is not close to one of the zeros of the J_0.

In this approximation, the stability of the fixed points just depends on the sign of the coefficient of $\sin v$; i.e., $J_0(\delta)$. So for $J_0(\delta) < 0$, $v = 2n\pi$ are stable fixed points; whereas for $J_0(\delta) > 0$, they are no longer stable, and $v = (2n+1)\pi$ become stable. This seems like a transcritical bifurcation, but since the critical value of bifurcation parameter δ is for $J_0(\delta) = 0$ where neglecting the harmonic term is not valid, a more precise look is needed near the bifurcation point.

In Fig. 2, we have sketched $F(v,\delta) = J_0(\delta)\sin v - 2f(v)\sin 2v$, with smoothly increasing δ in the vicinity of the first root of the J_0, $\delta_0 \simeq 2.405$. It can be seen that first $q = \pi$ (or $-\pi$) loses stability via a supercritical pitchfork bifurcation, and then $q = 0$ becomes stable via a subcritical one. Note that these two successive bifurcations occur in very narrow windows for the bifurcation parameter δ [actually, the bifurcations we talk about here are defined for the damped systems; for the definitions to be exact, we may imagine a small damping term is added to Eqs. (2.8) and (3.5) as we stated before]. Numerical results show that the width of these windows grows with decreasing frequency of shaking; this point is justified below.

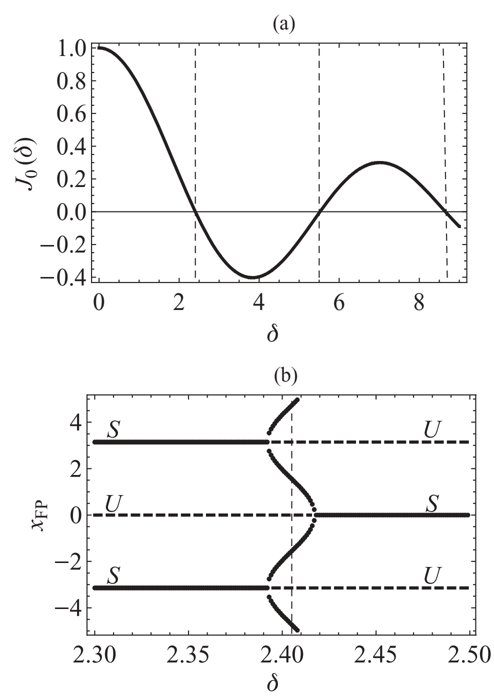

FIG. 3. (a) $J_0(\delta)$ has been plotted. Its zeros are bifurcation points, and its sign determines the stability of the fixed points. (b) The bifurcation diagram has been plotted in the vicinity of the first zero of the J_0. The evolution of fixed points shows two successive pitchfork bifurcations [stable (S) and unstable (U) fixed points have been represented by solid and dashed lines, respectively]. Parameters are the same as in Fig. 2 and the vertical dashed line shows the position of the first root of the Bessel function.

We can find an approximate value for the width of the bifurcation window around the roots of the Bessel function. Approaching one of the roots of J_0, say, the first root, i.e., $\delta_0 \simeq 2.405$, the second and third terms of the Eq. (3.5) will be of comparable magnitude. Keeping the first term in the summation of Eq. (3.6) and ignoring Λ^2 in the denominator, we find the following criterion for $(2n+1)\pi$ to be unstable fixed points:

$$\frac{2J_1(\delta)^2}{\omega^2} > J_0(\delta). \tag{3.8}$$

A similar condition can be found for the $2n\pi$ to change to stable points. Together, we arrive at the approximate limits of the first bifurcation window:

$$\delta_0 - \frac{2J_1(\delta)}{\omega^2} > \delta > \delta_0 + \frac{2J_1(\delta)}{\omega^2}. \tag{3.9}$$

Within this window, none of the $n\pi$ fixed points is stable, as can be seen from Fig. 3. This equation shows that the width of the window quadratically decreases with increasing frequency. We again emphasize that out of the bifurcation window, the stability is set only by the amplitude of the vibrating OL and not its frequency. This contradicts the common concept of the classical problem of stabilizing the inverted pendulum by the vibrating support [1,2].

To see to what extent the slow-fast approximation gives reliable results, we have compared, in Fig. 4, the results of the direct solution of NLSE [Eq. (2.2)], the equation of motion for center of mass [Eq. (2.8)], and the equation for the slow dynamics [Eq. (3.5)], for different values of δ and ω. Note that far from bifurcation points, the result of the averaged equation fits those of the other two quite remarkably, even for such a moderate value of frequency as $\omega = 1$. It is only near the bifurcation point that the averaged equation fails for moderate values of frequency.

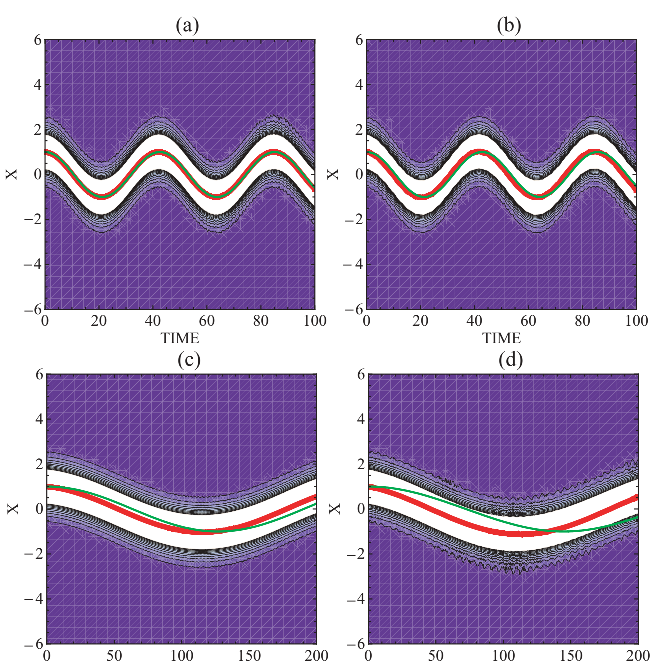

FIG. 4. (Color online) Results of the direct integration of the NLSE equation (contour plot), soliton center coordinate (red thick line), and the averaged equation (green line) plotted for several values of δ and ω. The parameters in (a) to (d) are, respectively, for $\delta = 3, 3, 2.42, 2.42$ and for $\omega = 10, 2, 10, 5$. The only figure in which the result of the averaged equation does not match with the two other results is (d), where it is close to the bifurcation point and the frequency is moderate.

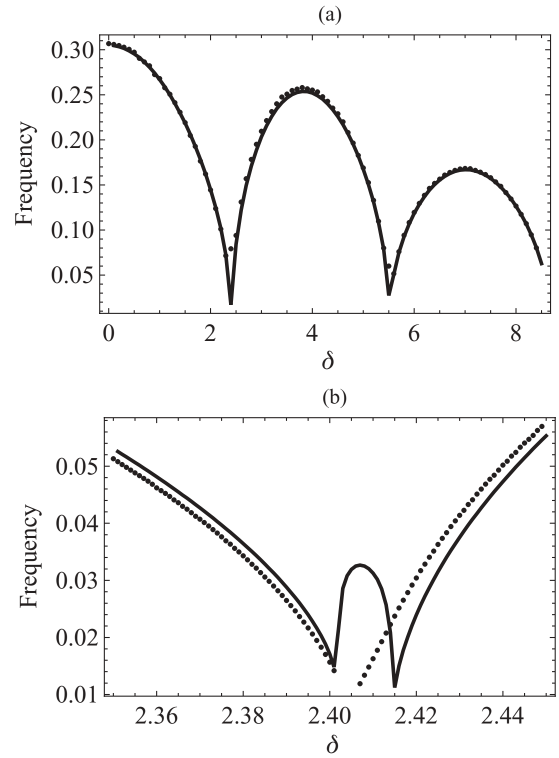

FIG. 5. (a) Frequency of the oscillation of the soliton center of mass vs. amplitude of the shaken OL resulting from integration of Eqs. (2.8) and (3.5); the latter points have been joined to be more distinguishable. (b) A narrow window around the first bifurcation point highlighted to show the invalidity of the approximation in this region. Other parameters are the same as Figs. 2 and 3.

In Fig. 5, a full comparison of the periods of the oscillations of Eqs. (2.8) and (3.5) is exhibited. It can be seen that for the values of the vibration amplitude not close to the bifurcation point, the results fit each other well; this is true even for the moderate values of the vibration frequency (not shown). So, it is only in the vicinity of the bifurcation points that the results differ substantially, both qualitatively and quantitatively.

Before ending this section, we would like to express the results of when the equations describe a Josephson junction as in Eq. (2.10). In this case, the stability of the fixed points will not be a matter of interest. Instead, adding a constant term to the right-hand side of Eq. (2.10), a dc-current feed, will result in a constant term appearing on the right-hand side of Eq. (3.5), with $J_0(\delta)$ having the role of an effective critical current, reflecting the effect of the high-frequency ac current. Therefore, the width of the zero-voltage Shapiro step of the current-voltage characteristic of the junction [16] shows a Bessel function dependence on the amplitude of the ac current, as has been noted before [16,22] and has an analytic support for the voltage-driven junctions [23].

The center of mass of a BEC, when put in a tilted periodic potential [24], obeys an equation of a dc-biased Josephson junction. Shaking the periodic potential mimics the ac bias as in Eq. (2.10): now zero voltage means an oscillating soliton, while a nonzero voltage means a *running* soliton. The above argument about the effective critical current of a Josephson junction can define the minimum value of the constant force exerted by the linear potential which can depin the soliton, giving it a nonzero average velocity. Since the equations of motion are nondissipative, an initial momentum can also depin the soliton and give it a nonzero average velocity. It turns out that for such a *kicked* soliton, the minimum initial momentum for depinning the condensate is greater for more localized solitons [25]. Shaking the periodic potential effectively reduces the height of the potential, lowering the threshold momentum for depinning [26].

IV. CONCLUSION

To summarize, we have studied the dynamics of a 1D BEC soliton in a horizontally vibrating periodic potential. While normally for time periodic potentials the various criteria depend on the ratio of amplitude and frequency [5–7], here the results show that for a considerable range of the parameters, the stability of the fixed points is governed solely by the amplitude of the vibrations. Although the approximation assumes a large vibration frequency of the OL, the numerical results show good agreement even for the moderate values of the frequency. This property arises from the periodic nature of the potential and from the equations which show two successive fold bifurcations for the adjacent fixed points. The idea of stabilizing an unstable fixed point by horizontal vibration can also be exploited to trap a soliton by a vibrating repulsive potential created by, say, a laser beam [27]. But the simple criterion for the stability, deduced above, is dependent on the *periodic* nature of the potential.

The equation of the motion of the soliton center of mass for this arrangement, not surprisingly with some variations, finds application in other branches of physics, such as the classical problem of a pendulum, Josephson junctions, charge-density waves [28], and nano-machines proposed by using Casimir forces [29]. In this way, our results may find a wider range of applications. For example, one can expect that the symmetry-breaking running modes [20,29], which here mean a forward-moving soliton with an oscillating OL, can occur for the small values of the vibration frequency (not considered here). Also Shapiro steps, which in the Josephson junctions define the ranges of the dc input current, where the phase is (on average) locked to the external periodic input and voltage remains constant, can inspire an experiment where in a tilted shaken OL, the mean velocity of the soliton coincides with a multiple of the frequency [30].

ACKNOWLEDGMENTS

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[1] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, Oxford, 1960).
[2] E. I. Butikov, Am. J. Phys. **69**, 755 (2001).
[3] K. E. Strecker, G. B. Partridge, and R. G. Hulet, Nature (London) **417**, 150 (2002); L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, Science **296**, 1290 (2002).
[4] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[5] F. Kh. Abdullaev and R. Galimzyanov, J. Phys. B: At. Mol. Opt. Phys. **36**, 1099 (2003); Yu. V. Bludov and V. V. Konotop, Phys. Rev. A **75**, 053614 (2007).
[6] D. H. Dunlap and V. M. Kenkre, Phys. Rev. B **34**, 3625 (1986).
[7] A. Eckardt, C. Weiss, and M. Holthaus, Phys. Rev. Lett. **95**, 260404 (2005); A. Eckardt, M. Holthaus, H. Lignier, A. Zenesini, D. Ciampini, O. Morsch, and E. Arimondo, Phys. Rev. A **79**, 013611 (2009).
[8] F. Kh. Abdullaev, J. G. Caputo, R. A. Kraenkel, and B. A. Malomed, Phys. Rev. A **67**, 013605 (2003); H. Saito and M. Ueda, Phys. Rev. Lett. **90**, 040403 (2003).
[9] R. Scharf and A. R. Bishop, Phys. Rev. A **46**, R2973 (1992).
[10] R. Scharf and A. R. Bishop, Phys. Rev. E **47**, 1375 (1993).
[11] O. Morsch and M. Oberthaler, Rev. Mod. Phys. **78**, 179 (2006).
[12] Th. Anker, M. Albiez, R. Gati, S. Hunsmann, B. Eiermann, A. Trombettoni, and M. K. Oberthaler, Phys. Rev. Lett. **94**, 020403 (2005); K. Winkler, G. Thalhammer, F. Lang, R. Grimm, J. H. Denschlag, A. J. Daley, A. Kantian, H. P. Buechler, and P. Zoller, Nature (London) **441**, 853 (2006); T. Roscilde and J. I. Cirac, Phys. Rev. Lett. **98**, 190402 (2007).
[13] M. Greiner, I. Bloch, O. Mandel, T. W. Hänsch, and T. Esslinger, Phys. Rev. Lett. **87**, 160405 (2001); M. Greiner, O. Mandel, T. Slinger, T. W. Hänsch, and I. Bloch, Nature (London) **415**, 39 (2002).
[14] R. G. Scott, A. M. Martin, T. M. Fromhold, S. Bujkiewicz, F. W. Sheard, and M. Leadbeater, Phys. Rev. Lett. **90**, 110404 (2003).
[15] D. E. McCumber, J. Appl. Phys. **39**, 3113 (1968).
[16] K. K. Likharev, *Dynamics of Josephson Junctions and Circuits* (Gordon and Breach, New York, 1986).
[17] P. G. Kevrekidis, D. J. Frantzeskakis, R. Carretero-González, B. A. Malomed, G. Herring, and A. R. Bishop, Phys. Rev. A **71**, 023614 (2005).
[18] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations* (Oxford University, New York, 1999).
[19] M. Kollmann, H. W. Capel, and T. Bountis, Phys. Rev. E **60**, 1195 (1999); K. Kasamatsu, M. Machida, N. Sasa, and M. Tsuboa, Phys. Rev. A **71**, 063616 (2005).
[20] M. Iansiti, Q. Hu, R. M. Westervelt, and M. Tinkham, Phys. Rev. Lett. **55**, 746 (1985); A. H. MacDonald and M. Plischke, Phys. Rev. B **27**, 201 (1983); M. Octavio, *ibid.* **29**, 1231 (1984).

[21] L. Fallani, C. Fort, J. E. Lye, and M. Inguscio, Opt. Express **13**, 4303 (2005).

[22] R. L. Kautz, J. Appl. Phys. **52**, 3528 (1981); R. L. Kautz and J. C. Macfarlane, Phys. Rev. A **33**, 498 (1986).

[23] Sidney Shapiro, Phys. Rev. Lett. **11**, 80 (1963).

[24] S. Wimberger, R. Mannella, O. Morsch, E. Arimondo, A. R. Kolovsky, and A. Buchleitner, Phys. Rev. A **72**, 063610 (2005); D. Witthaut, M. Werder, S. Mossmann, and H. J. Korsch, Phys. Rev. E **71**, 036625 (2005).

[25] A. Cetoli, L. Salasnich, B. A. Malomed, and F. Toigo, Physica D **238**, 1388 (2009).

[26] In terms of the ground-state effective mass m^* for a particle in a periodic potential, we have $m^* = m/(1 - U^2/E_B^2)$, where E_B is the energy at the Brillouin zone and U is the main Fourier component of the potential, taken to be small compared to E_B. Reduction of the height of the potential amounts to reduction of the effective mass. See also Ref. [25].

[27] G. Herring, P. G. Kevrekidis, R. Carretero-Gonzáles, B. A. Malomed, D. J. Frantzeskakis, and A. R. Bishop, Phys. Lett. **A345**, 144 (2005).

[28] G. Grüner, Rev. Mod. Phys. **60**, 1129 (1988).

[29] A. Ashourvan, M. F. Miri, and R. Golestanian, Phys. Rev. Lett. **98**, 140801 (2007).

[30] Following the idea of the Josephson effect for two coupled BECs studied by A. Smerzi *et al.*, Phys. Rev. Lett. **79**, 4950 (1997), a counterpart for Shapiro steps in BEC Josephson junctions has been introduced by S. Raghavan, A. Smerzi, S. Fantoni, and S. R. Shenoy, Phys. Rev. A **59**, 620 (1999). In our study, a Shapiro step is interpreted as the entrainment of the soliton movement with the external periodic force.