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Quantum logic as superbraids of entangled qubit world lines

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Presented is a topological representation of quantum logic that views entangled qubit spacetime histories (or qubit world lines) as a generalized braid, referred to as a superbraid. The crossing of world lines can be quantum-mechanical in nature, most conveniently expressed analytically with ladder-operator-based quantum gates. At a crossing, independent world lines can become entangled. Complicated superbraids are systematically reduced by recursively applying quantum skein relations. If the superbraid is closed (e.g., representing quantum circuits with closed-loop feedback, quantum lattice gas algorithms, loop or vacuum diagrams in quantum field theory), then one can decompose the resulting superlink into an entangled superposition of classical links. Thus, one can compute a superlink invariant, for example, the Jones polynomial for the square root of a classical knot.

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I. INTRODUCTION

In topological quantum computing [1,2], a quantum gate operation derives from braiding quasiparticles, for example, two Majoranna zero-energy vortices made of entangled Cooper-pair states in a $p + ip$ superconductor where the vortex-vortex phase interaction has a non-Abelian SU(2) gauge group [3,4]. Dynamically braiding such quantum vortices (point defects in a planar cross section of the condensate) induces phase shifts in the quantum fluid's multiconnected wave function. Local nonlinear interactions between deflects (vortex-vortex straining) is otherwise neglected; that is, the separation distance δ of the zero-mode vortices is much greater then the vortex core size, which scales as the coherence length $\xi \ll \delta$ in quantum fluids. The braiding occurs adiabatically so the quantum fluid remains in local equilibrium and the number of deflects (qubits) remains fixed. For implementations, the usual question is how can quantum logic gates, and in turn quantum algorithms, be represented by braiding deflects, quasiparticles with a non-Abelian gauge group.

This article addresses the related fundamental question about the relationship between quantum entanglement, tangled strands, quantum logic, and quantum-information theory [5–11]. How can a quantum logic gate, and in turn a quantum algorithm, be decomposed into a linear combination (entangled superposition) of classical braid operators? The goal is to comprehend and categorize quantum-information topologically. This is done by first viewing a quantum gate as a braid of two qubit spacetime histories or world lines. Qubit-qubit interaction associated with a quantum gate is rendered as a tree-level scattering diagram, a form of ribbon graph. A quantum algorithm may be represented as a weave of such graphs, a superbraid of qubit world lines. Finally, one closes a superbraid to form a superlink. In fact, quantum lattice gas algorithms, for example, those employed for the simulation of superfluids themselves [12], are a good superlink archetype; hence the shared nomenclature.

With this technology we can calculate superlink invariants. In principle, each quantum circuit has its own unique invariant (associated with Laurent series); for example, two competing quantum circuit implementations of a particular algorithm can be judged equivalent, irrespective of circuit schemes and the placement of gates and wires. If two quantum algorithms, first- and second-order accurate, are topologically equivalent, then the simpler one can be used for analytical predictions of their common effective theory while the latter can be used for faster simulations with fewer resources.

In short, presented is a quantum generalization of the Temperley-Lieb algebra TL_Q and Artin braid group B_Q: a superbraid and its closure, a superlink, is formed out of the world lines of Q qubits (strands) undergoing dynamics generated by quantum gates. Furthermore, the superbraid representation of quantum dynamics works equally well for either bosonic or fermionic quantum simulations. There exists a classical limit where the generalized Temperley-Lieb algebra and the superbraid group, defined later in this article, reduce to the usual Temperley-Lieb algebra and braid group. There also exists a purely quantum-mechanical limit where superbraids reduce to conservative quantum logic operators. Thus, the superbraid is the progenitor of the braid operator and quantum gate operator.

This article is organized as follows. A condensed review of knot theory sufficient to define the Jones polynomial and a condensed review of qubit ladder operators sufficient to define quantum logic operators and superbraid operators are given in Sec. II. A diagrammatic representation of quantum logic, a hybrid between the usual quantum circuit diagrams and Feynman diagrams, is given in Sec. III. A generalization of the Temperley-Lieb algebra (that includes fermionic particle dynamics) is presented in Sec. IV. The superbraid group relations are given in terms of this generalized algebra in Sec. V, whereby the superbraid operator is cast in three different mathematical forms. The first (exponential form) illustrates how a superbraid is an amalgamation of a classical braid operator and a quantum gate. The second (knot theory form) illustrates the very close connection to the well-known classical braid operator. The third (product form) illustrates the physics of pairwise quantum entanglement as the braiding of two qubit world lines through a particular quantum gate Euler angle. Quantum skein relations are given in Sec. VI, and calculations of superlink invariants, for example, for the square root knots introduced here, are given in Sec. VII. Thus, using the superbraid formalism, quantum knots such as the square root of the unknot or the square root of the trefoil knot are well-defined topological objects and each have their own knot invariant. Finally, a brief summary

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and some final remarks are given in Sec. VIII. To help make the presentation more accessible to a wider audience, a primer on joint quantum logic is presented in the Appendix, which helps explain the origin of the superbraid's generator.

II. LINKS, LADDERS, AND LOGIC

Classical braid operators (nearest-neighbor permutations), represented in terms of Temperley-Lieb algebra [13], were originally discovered in six-vertex Potts models and statistical mechanical treatments of two-dimensional lattice systems [14,15]. The quantum algorithm to compute the Jones polynomial [16,17] employs unitary gate operators that are mapped to unitary representations of the braid group, that is, generated by Hermitian representations of the Temperley-Lieb algebra. To prepare for our presentation of superbraids as a topological representation of the quantum logic underlying quantum-information dynamics, let us first briefly review some basics of knot theory and some basic quantum gate technology using qubit ladder operators.

A link comprising Q strands, denoted by L, say, is the closure of a braid. The Jones polynomial $V_L(A)$ is an invariant of L [18], where A is a complex parameter associated with the link whose physical interpretation will be presented later in this article. $V_L(A)$ is a Laurent series in A. The Jones polynomial is defined for a link embedded in three space—an oriented link. One projects L onto a plane. In the projected image, in general crossing of strands occurs but is disambiguated by its sign ± 1; that is, one assigns overcrossings the sign of $+1$ and undercrossings -1. The writhe $w(L)$ is sum of the signs of all the crossings, that is, the net sign of a link's planar projection. The Jones polynomial is computed as follows:

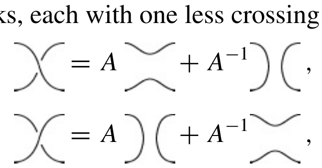
$$V_L(A) = (-A^3)^{-w(L)}\,\frac{K_L(A)}{d}, \tag{1}$$

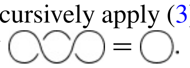
where $K_L(A)$ is the Kauffman bracket of the link. $K_L(A)$ is determined from a planar projection of L, for example, using the skein relations below. In the simplest case of an unknotted link (or unknot), the Kauffman bracket is

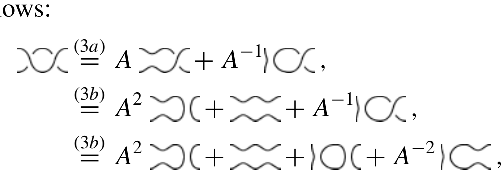
$$\bigcirc = K_{\bigcirc}(A) = d = -A^2 - A^{-2}. \tag{2}$$

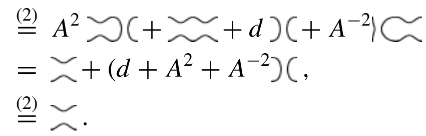
The Kauffman bracket of a disjoint union of n unknots has the value d^n, for example, $\bigcirc\bigcirc = d^2$.

$K_L(A)$ for a link with crossings can be computed recursively using a skein relation that equates it to the weighted sum of two links, each with one less crossing:


(3a)
(3b)

where A and its inverse are the weighting factors. As an example, let us recursively apply (3) to prove an intuitively obvious link identity . One reduces the relevant braid as follows:


(4a)
(4b)
(4c)


(4d)
(4e)
(4f)

A quantum gate represents the qubit-qubit coupling that occurs at the crossing of world lines of a pair of qubits, say, $|q_\alpha\rangle$ and $|q_\gamma\rangle$ in a system of Q qubits. Every quantum gate is generated by a Hermitian operator, $\mathcal{E}_{\alpha\gamma}$ say, and whose action on the quantum state may be expressed as

$$|\ldots q_\alpha \ldots q_\gamma \ldots\rangle' = e^{i\zeta\mathcal{E}_{\alpha\gamma}}|\ldots q_\alpha \ldots q_\gamma \ldots\rangle, \tag{5}$$

where ζ is a real parameter. The archetypal case considered here is $\mathcal{E}_{\alpha\gamma}^2 = \mathcal{E}_{\alpha\gamma}$; the generator is idempotent.

Suppose the system of qubits is employed to model the quantum dynamics of fermions or bosons. Is there an analytical form of the generator $\mathcal{E}_{\alpha\gamma}$ that allows one to easily distinguish between the two cases? It is natural to begin by treating fermion statistics. With the logical one state of a qubit $|1\rangle = \binom{0}{1}$, notice that $\sigma_z|1\rangle = -|1\rangle$, so one can count the number of preceding bits that contribute to the overall phase shift due to fermionic bit exchange involving the γth qubit with tensor product operator, $\sigma_z^{\otimes\gamma-1}|\psi\rangle = (-1)^{N_\gamma}|\psi\rangle$. The phase factor is determined by the number of bit crossings $N_\gamma = \sum_{k=1}^{\gamma-1} n_k$ in the state $|\psi\rangle$ and where the Boolean number variables are $n_k \in [0, 1]$. Hence, an annihilation operator is decomposed into a tensor product known as the Jordan-Wigner transformation [19],

$$a_\gamma = \sigma_z^{\otimes\gamma-1} \otimes a \otimes \mathbf{1}^{\otimes Q-\gamma}, \tag{6}$$

for integer $\gamma \in [1, Q]$ and here the singleton operator is $a = \frac{1}{2}(\sigma_x + i\sigma_y)$, where σ_i for $i = x, y, z$ are the Pauli matrices. See page 17 of Ref. [20] for a typical way of determining N_γ. Equation (6) and its transpose, the creation operator $a_\gamma^\dagger = a_\gamma^{\mathrm{T}}$, satisfy the anticommutation relations

$$\{a_\gamma, a_\beta^\dagger\} = \delta_{\gamma\beta}, \quad \{a_\gamma, a_\beta\} = 0, \quad \{a_\gamma^\dagger, a_\beta^\dagger\} = 0. \tag{7}$$

The Hermitian generator of a quantum gate can be analytically expressed in terms of qubit creation and annihilation operators. A novel generator that is manifestly Hermitian is the following:

$$\begin{aligned}\mathcal{E}_{\Delta\alpha\gamma} = d^{-1}[&-A^2 n_\alpha - A^{-2} n_\gamma - A\, a_\alpha^\dagger a_\gamma \\ &- A^{-1} a_\gamma^\dagger a_\alpha + d\,(\Delta - 1) n_\alpha n_\gamma],\end{aligned} \tag{8}$$

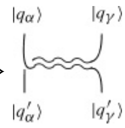
where $d = -A^2 - A^{-2}$ is real. The parameter Δ is Boolean, and it allows one to select between Fermi ($\Delta = 1$) or Bose ($\Delta = 0$) statistics of the modeled quantum particles. The operator generated by $\mathcal{E}_{\Delta\alpha\gamma}$ is

$$e^{z\,\mathcal{E}_{\Delta\alpha\gamma}} = \mathbf{1}^{\otimes Q} + (e^z - 1)\mathcal{E}_{\Delta\alpha\gamma}, \tag{9}$$

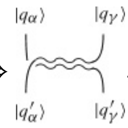
which is a quantum state interchange for the case when z is pure imaginary. That is, (9) is a conservative quantum logic gate for $z = i\zeta$. For the case when z is complex, (9) is proportional to a superbraid operator. The origin of the mathematical form of (8) is given in the Appendix.

III. DIAGRAMMATIC QUANTUM LOGIC

The state evolution (5) by the quantum logic gate (9) can be understood as scattering between two qubits

$$|\psi'\rangle = e^{i\zeta\,\mathcal{E}_{\Delta\alpha\gamma}}|\psi\rangle \Longleftrightarrow$$ 

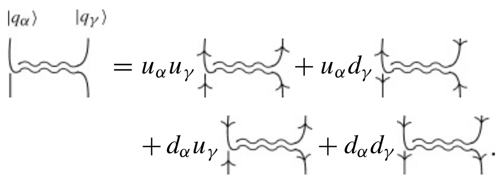
, (10a)

$$|\psi\rangle = e^{-i\zeta\,\mathcal{E}_{\Delta\alpha\gamma}}|\psi'\rangle \Longleftrightarrow$$ 

, (10b)

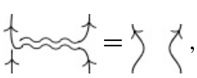
where the "gauge field" that couples the external qubit world lines is represented by an internal double wavy line (or ribbon). The external lines either overcross or undercross and are assigned $+1$ and -1 multiplying the action, that is, $\pm\zeta\mathcal{E}_\Delta$. This sign disambiguates between a quantum gate and its adjoint, respectively, as shown in (10a) and (10b). Let us denote a qubit graphically $|q_\alpha\rangle \equiv u_\alpha \uparrow + d_\alpha \downarrow$, with complex amplitudes constrained by conservation of probability $|u_\alpha|^2 + |d_\alpha|^2 = 1$.

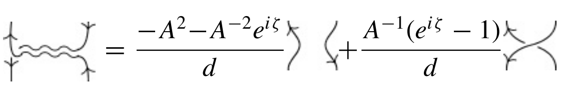
Starting, for example, with a separable input state $|\psi\rangle = |q_\alpha\rangle|q_\gamma\rangle$, a scattering diagram is a quantum superposition of four oriented graphs:



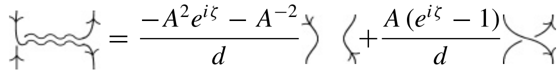
(11)

Each oriented scattering graph can be reduced to a quantum superposition of classical graphs, or just a single classical graph, as the case may be. There are four quantum skein relations representing dynamics generated by (8):

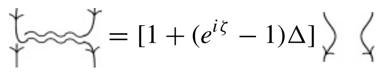

(12a)



(12b)



(12c)

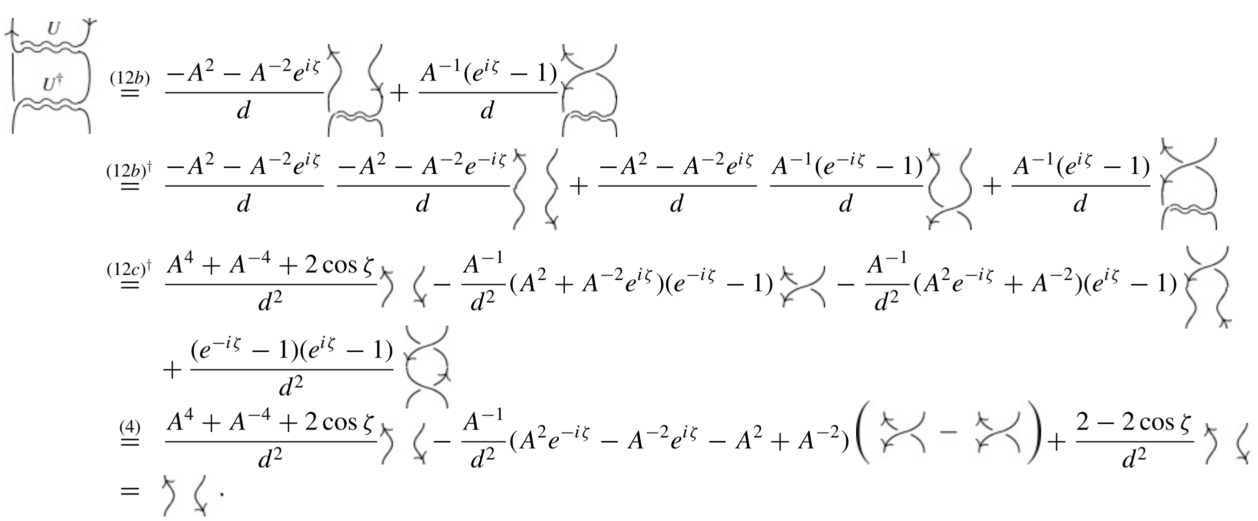


(12d)

These are the quantum analog of (3). Adjoint quantum skein relations are obtained simply by taking $\zeta \to -\zeta$ in the amplitudes in the diagrams in (12). All superbraids can be reduced to a quantum superposition of classical braids. The closure of a superbraid forms a superlink. Hence, a superlink can be reduced to a quantum superposition of classical links and, consequently, for each superlink one can compute an associated invariant, for example, a superposition of Jones polynomials. An example calculation of such invariants is presented in Sec. VII.

In the context of quantum-information dynamics, a physical interpretation of the parameter A can be rendered as follows. If the strands in L are considered closed spacetime histories of Q qubits (e.g., qubit states evolving in a quantum circuit with closed-loop feedback), then the left-hand side of (12) represents a trajectory configuration within a piece of the superlink where entanglement is generated by a qubit-qubit coupling that occurs at a quantum gate (i.e., generalized crossing point). For the one-body cases (12b) and (12c), the right-hand side represents classical alternatives in quantum superposition: $d^{-1}e^{i\frac{\zeta}{2}}(-A^2e^{\mp i\frac{\zeta}{2}} - A^{-2}e^{\pm i\frac{\zeta}{2}})$ is the amplitude for no interaction (nonswapping of qubit states), whereas the amplitude of a SWAP interaction (interchanging of qubit states) goes as $d^{-1}A^{\mp 1}(e^{i\zeta}-1)$.

As a first example of reducing a superbraid, let us recursively apply (12) to prove an obvious evolution identity: the composition of a quantum gate with its adjoint is the identity operator, that is, $UU^\dagger = 1$. For simplicity, we start with $|q_\alpha\rangle = \uparrow$ and $|q_\gamma\rangle = \downarrow$, so the initially oriented superbraid is reduced to a superposition of classical braids as follows:



(13)

It is easy to verify that the same result occurs for inputs $|q_\alpha\rangle = \downarrow$ and $|q_\gamma\rangle = \uparrow$. Furthermore, the identity trivially follows for $|q_\alpha\rangle = \uparrow$ and $|q_\gamma\rangle = \uparrow$ and for $|q_\alpha\rangle = \downarrow$ and $|q_\gamma\rangle = \downarrow$ since Δ is Boolean.

IV. GENERALIZATION OF TL$_Q(d)$

With adjacent indices, for example, $\gamma = \alpha + 1$ in (9), we need write the first index only (i.e., suppress the second indice), $\mathcal{E}_{\Delta\alpha} \equiv \mathcal{E}_{\Delta\alpha,\alpha+1}$. Using this compressed notation, (8) satisfies the following generalized Temperley-Lieb algebra:

$$\mathcal{E}_{\Delta\alpha}^2 = \mathcal{E}_{\Delta\alpha}, \qquad \alpha = 1, 2, \ldots, Q-1, \tag{14a}$$

$$\mathcal{E}_{\Delta\alpha}\mathcal{E}_{\Delta\alpha\pm1}\mathcal{E}_{\Delta\alpha} - \mathcal{E}_{\Delta\alpha\pm1}\mathcal{E}_{\Delta\alpha}\mathcal{E}_{\Delta\alpha\pm1} = d^{-2}\mathcal{E}_{\Delta\alpha} - d^{-2}\mathcal{E}_{\Delta\alpha\pm1}, \tag{14b}$$

$$\mathcal{E}_{\Delta\alpha}\mathcal{E}_{\Delta\beta} = \mathcal{E}_{\Delta\beta}\mathcal{E}_{\Delta\alpha}, \qquad |\alpha-\beta| \geqslant 2. \tag{14c}$$

To help understand this algebra, we may write (14b) as follows:

$$\mathcal{E}_{\Delta\alpha}\mathcal{E}_{\Delta\alpha+1}\mathcal{E}_{\Delta\alpha} - d^{-2}\mathcal{E}_{\Delta\alpha} = d^{-2}X_{\alpha,\alpha+1}, \tag{15a}$$

$$\mathcal{E}_{\Delta\alpha+1}\mathcal{E}_{\Delta\alpha}\mathcal{E}_{\Delta\alpha+1} - d^{-2}\mathcal{E}_{\Delta\alpha+1} = d^{-2}Y_{\alpha,\alpha+1}, \tag{15b}$$

where $X_{\alpha,\alpha+1}$ and $Y_{\alpha,\alpha+1}$ are introduced solely for the purpose of separating (14b) into two equations. For (15) to be equivalent to (14b), one must demonstrate that $X_{\alpha,\alpha+1} = Y_{\alpha,\alpha+1}$. Inserting (8) into the left-hand side of (15), after considerable ladder operator algebra, one finds that the difference of the right-hand side of (15) is

$$X_{\alpha,\alpha+1} - Y_{\alpha,\alpha+1} = \Delta(\Delta-1)[(A^4 - A^{-4})n_\alpha n_{\alpha+1} n_{\alpha+2} - A^4 n_\alpha n_{\alpha+1} + A^{-4} n_{\alpha+1} n_{\alpha+2}], \tag{16}$$

vanishing for Boolean Δ. Thus, (14b) follows from (8).

One finds X and Y are proportional to Δ, so a remarkable reduction of (14) occurs for the $\Delta = 0$ case:

$$\mathcal{E}_{0\alpha}^2 = \mathcal{E}_{0\alpha}, \qquad \alpha = 1, 2, \ldots, Q-1, \tag{17a}$$

$$\mathcal{E}_{0\alpha}\mathcal{E}_{0\alpha\pm1}\mathcal{E}_{0\alpha} \overset{(15)}{=} d^{-2}\mathcal{E}_{0\alpha}, \tag{17b}$$

$$\mathcal{E}_{0\alpha}\mathcal{E}_{0\beta} = \mathcal{E}_{0\beta}\mathcal{E}_{0\alpha}, \qquad |\alpha-\beta| \geqslant 2. \tag{17c}$$

This is the Temperley-Lieb algebra over a system of Q qubits (TL$_Q$). Thus, entangled bosonic states generated by $\mathcal{E}_{0\alpha}$ are isomorphic to links generated by $\mathcal{E}_{0\alpha}$. So (14) is a generalization of TL$_Q$. We now consider the generalized braid that it generates: a superbraid.

V. SUPERBRAID GROUP

A general superbraid operator is an amalgamation of both a classical braid operator and a quantum gate,

$$b^{\mathrm{s}}_{\Delta\alpha\beta} \equiv A\, e^{z\,\mathcal{E}_{\Delta\alpha\beta}}, \tag{18}$$

where A and z are complex parameters. Equation (18) can be applied to any two qubits, α and β, in a system of qubits (i.e., we do not impose a restriction to the adjacency case when $\beta = \alpha + 1$). Equation (18) can be written in several different ways, each way useful in its own right.

Letting $z \equiv i\zeta + \ln\tau$, the superbraid operator has the following exponential form

$$b^{\mathrm{s}}_{\Delta\alpha\beta} \equiv \tau^{-\frac{1}{4}}\, e^{(i\zeta+\ln\tau)\,\mathcal{E}_{\Delta\alpha\beta}} = \tau^{-\frac{1}{4}}(e^{i\zeta}\tau)^{\mathcal{E}_{\Delta\alpha\beta}}, \tag{19}$$

where $A \equiv \tau^{-\frac{1}{4}}$ (and so $\tau = A^{-4}$). The superbraid operator can be written linearly in its generator,

$$b^{\mathrm{s}}_{\Delta\alpha\beta} = A[\mathbf{1}_Q + (A^{-4}\, e^{i\zeta} - 1)\mathcal{E}_{\Delta\alpha\beta}]. \tag{20}$$

Thus, the superbraid operator and its inverse can be expressed in knot theory form,

$$b^{\mathrm{s}}_{\Delta\alpha\beta} = A\,\mathbf{1}_Q + A^{-1}\left(\frac{1 - e^{i\zeta}\tau}{1+\tau}\right) d\,\mathcal{E}_{\Delta\alpha\beta}, \tag{21a}$$

$$\left(b^{\mathrm{s}}_{\Delta\alpha\beta}\right)^{-1} = A^{-1}\,\mathbf{1}_Q + A\left(\frac{-e^{-i\zeta} + \tau}{1+\tau}\right) d\,\mathcal{E}_{\Delta\alpha\beta}. \tag{21b}$$

A nontrivial classical limit of quantum logic gates represented as (9) occurs at $\zeta = \pi$ (SWAP operator). Consequently, the superbraid operator in product form is

$$b^{\mathrm{s}}_{\Delta\alpha\beta} \equiv \tau^{-\frac{1}{4}}\, e^{(\ln\tau + i\pi)\,\mathcal{E}_{\Delta\alpha\beta}}\, e^{(i\zeta - i\pi)\,\mathcal{E}_{\Delta\alpha\beta}}, \tag{22a}$$

$$= b_{\Delta\alpha\beta}\, e^{i(\zeta-\pi)\,\mathcal{E}_{\Delta\alpha\beta}}, \tag{22b}$$

where $b_{\Delta\alpha\beta} = \tau^{-\frac{1}{4}}\, e^{(\ln\tau + i\pi)\,\mathcal{E}_{\Delta\alpha\beta}}$ is the conventional braid operator. Equation (22b) is useful for comprehending the physical behavior of the superbraid operator. It classically braids world lines α and β and quantum-mechanically entangles these world lines according to the deficit angle $\zeta - \pi$.

The superbraid group is defined by

$$b^{\mathrm{s}}_\alpha\, b^{\mathrm{s}}_\beta = b^{\mathrm{s}}_\beta\, b^{\mathrm{s}}_\alpha \quad \text{for} \quad |\alpha-\beta| > 1, \tag{23a}$$

$$b^{\mathrm{s}}_\alpha\, b^{\mathrm{s}}_{\alpha+1}\, b^{\mathrm{s}}_\alpha + \gamma\, b^{\mathrm{s}}_\alpha = b^{\mathrm{s}}_{\alpha+1} b^{\mathrm{s}}_\alpha\, b^{\mathrm{s}}_{\alpha+1} + \gamma\, b^{\mathrm{s}}_{\alpha+1} \quad \text{for} \quad 1 \leqslant \alpha < Q, \tag{23b}$$

where γ is a constant that depends on the representation. For (8), we have $\gamma = (A^4 + A^{-4}e^{i\zeta})(1 + e^{i\zeta})A^{-2}d^{-2}$.

In the classical limit $\zeta = \pi$, the superbraid operator reduces to the classical braid operator, $b_\alpha \equiv b^{\mathrm{s}}_\alpha(\pi, \tau)$, and (23) reduces to the Artin braid group,

$$b_\alpha\, b_\beta = b_\beta\, b_\alpha \quad \text{for} \quad |\alpha-\beta| > 1, \tag{24a}$$

$$b_\alpha\, b_{\alpha+1}\, b_\alpha = b_{\alpha+1} b_\alpha b_{\alpha+1} \quad \text{for} \quad 1 \leqslant \alpha < Q. \tag{24b}$$

Equation (24) follows from Eq. (23) because $\gamma = 0$ for $\zeta = \pi$. Also, in this classical limit, Eq. (20) reduces to the braid operator

$$b_\alpha = A\,\mathbf{1}_Q + A^{-1} d\,\mathcal{E}_{\Delta\alpha\beta}, \tag{25}$$

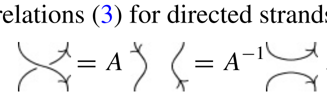
for $\alpha = 1, 2, \ldots, Q-1$ (and technically the standard braid operator when $\beta = \alpha + 1$). After some ladder operator manipulations using (8), one finds that

$$b_\alpha b_{\alpha+1} b_\alpha - b_{\alpha+1} b_\alpha b_{\alpha+1} \overset{(25)}{=} A^{-1}(A^4 - A^{-4})d^{-2}\Delta(\Delta-1) \times (1 - n_\alpha)n_{\alpha+1}n_{\alpha+2}, \tag{26}$$

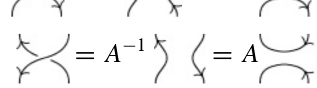
where $n_\alpha \equiv a^\dagger_\alpha a_\alpha$. Since Δ is Boolean, the right-hand side vanishes, and this is just (24b).

VI. QUANTUM SKEIN RELATIONS

The skein relations (3) for directed strands are

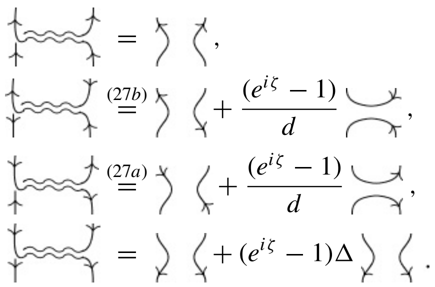


(27a)



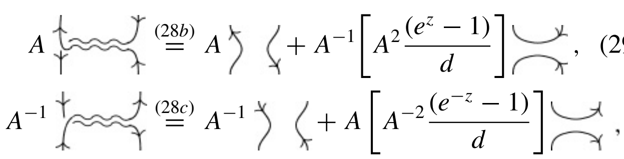
(27b)

The writhe of a braid, $+1$ for (27a) and -1 for (27b), is determined by applying the right-hand rule at the crossing point (viz., considering the strands as vectors), rotating the strand above toward the one below with the axis of rotation either out of the plane $(+1)$ or into the plane (-1). By inserting (27) into (12), one arrives at a remarkably simple form of the quantum skein relations:



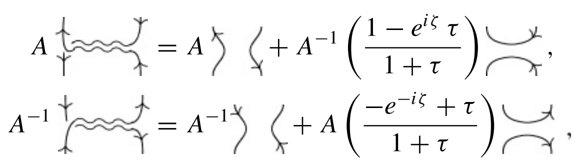
(28a) (28b) (28c) (28d)

Equation (28) is most useful for reducing a closed quantum circuit into a superposition of oriented links. The relations in the one-body sector can be written as



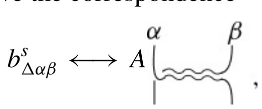
(29a) (29b)

for the case of a complex rotation angle (i.e., $i\zeta \to z$) and where we have multiplied through by A and A^{-1}, respectively, and taking the adjoint of the latter relation. Now with $e^{z} = A^{-4}e^{i\zeta}$ and $A^{-4} = \tau$, we arrive at



(30a) (30b)

the diagrammatic representation of the superbraid operators (21). Thus, we have the correspondence

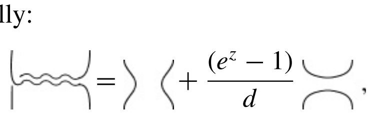


(31)

which is just (18) in graphical form. The superbraid operator (31) is unitary when $A = \sqrt[4]{1}$, in which case it reduces to a conservative quantum logic gate.[1]

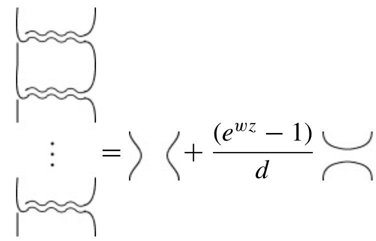
VII. SUPERLINK INVARIANTS

Let us begin by writing the conservative quantum logic gate (9) graphically:



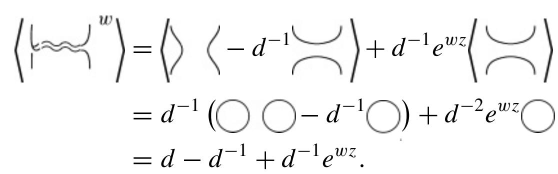
(32)

where we choose to use the complex time parameter z. Now w number of successive superbraids is



(33)

where the right-hand side follows from (32) by taking $z \to wz$. Letting $\langle b\rangle \equiv d^{-1}K_L(A)$, where L is the closure of b, the Markov trace closure of (33), here denoted with angled brackets, is



(34a) (34b)

Multiplying through by A^{w}, we then have the trace closure of w superbraids:

$$\langle (b^{s})^{w}\rangle \overset{(31)}{=} A^{w}(d - d^{-1}) + d^{-1}(A\, e^{z})^{w}. \tag{35}$$

In the classical limit $\zeta = \pi$ and $e^{z} = -A^{-4}$, then (35) becomes a standard w braid:

$$\langle b^{w}\rangle \overset{(31)}{=} A^{w}(d - d^{-1}) + d^{-1}(-A^{-3})^{w} \tag{36a}$$

$$= (-A^{3})^{-w}d^{-1}[1 + (-A^{4})^{w}(d^{2} - 1)]. \tag{36b}$$

The Jones polynomial is $V_L(A) \overset{(1)}{=} (-A^{3})^{-w}\,\langle b^{w}\rangle$, so

$$V_L(A) = (A^{6})^{-w}d^{-1}[1 + (-A^{4})^{w}(d^{2} - 1)]. \tag{37}$$

For example, considering classical links formed from the closure of two strands braided an integer number of times with $w = 0, 1, 2, 3, 4, 5\ldots$, (37) gives

$$V_{\bigcirc\bigcirc}(A) = -A^{-2} - A^{2}\,, \tag{38a}$$

$$V_{\bigcirc\!\bigcirc}(A) = -A^{3}\,, \tag{38b}$$

$$V_{\bigcirc\!\bigcirc\!\bigcirc}(A) = -A^{-4} - A^{4}\,, \tag{38c}$$

$$V_{\bigcirc\!\bigcirc\!\bigcirc\!\bigcirc}(A) = A^{-7} - A^{-3} - A^{5}\,, \tag{38d}$$

$$V_{\bigcirc\!\bigcirc\!\bigcirc\!\bigcirc\!\bigcirc}(A) = -A^{-10} + A^{-6} - A^{-2} - A^{6}\,, \tag{38e}$$

$$V_{\bigcirc\!\bigcirc\!\bigcirc\!\bigcirc\!\bigcirc\!\bigcirc}(A) = A^{13} - A^{9} + A^{5} - A^{-1} - A^{7}\,, \tag{38f}$$

$$\vdots$$

Equation (38b) is the Jones polynomial invariant for the unknot, (38c) for the Hopf link, (38d) for the trefoil knot, and so forth, and these Laurent series are well known.

Yet the formula (37) follows from the quantum logic gate relation (34), where w is a time scaling factor. Since time is a continuous variable in quantum logic, we are free to take w to be a real-valued parameter whereby the formula for the invariant Jones polynomial remains physically well defined. Thus, for example, we can evaluate (37) for half-integer w, and calculate Jones polynomial invariants for quantum links that are “halfway” between classical links, which is to say in

[1] Since flipping over a braid preserves the writhe of its crossing, we may flip over each diagram in (30) to have yet another way to specify operative quantum skein relations. Using this specification, if we take $\zeta = \pi$, then the flipped-over quantum skein relations just reduce to the classical skein relations (27).

equal superposition of two classical links. For quantum links formed from the closure of two strands braided an half-integer number of times with $w = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \ldots$, (37) then gives

$$V_{\sqrt{\infty}}(A) = \frac{-i - A^{-2} - A^2 - A^6}{A^{-\frac{3}{2}}(A^{-2} + A^2)} \quad (39a)$$

$$V_{\sqrt{\infty\infty}}(A) = \frac{iA^{-3} - A^{-1} - A^3 - A^7}{A^{-\frac{3}{2}}(A^{-2} + A^2)} \quad (39b)$$

$$V_{\sqrt{\infty\infty\infty}}(A) = \frac{-1 - iA^{-6} - A^4 - A^8}{A^{-\frac{3}{2}}(A^{-2} + A^2)} \quad (39c)$$

$$V_{\sqrt{\infty\infty\infty\infty}}(A) = \frac{iA^{-9} - A - A^5 - A^9}{A^{-\frac{3}{2}}(A^{-2} + A^2)} \quad (39d)$$

$$\vdots$$

Equation (39a) is the superlink invariant for the square root of the unknot ($\sqrt{\infty}$), (39b) for the square root of the trefoil knot ($\sqrt{\infty\infty}$), and so forth. The square root of unknot and trefoil knots are examples of quantum knots, a special class of superlinks.

VIII. CONCLUSION

Einstein, Podolsky, and Rosen discovered nonlocal quantum entanglement over three quarters of a century ago [21]. Although this inscrutable seminal result is the most cited one in the physics literature, quantum entanglement still remains one of the most mysterious properties of quantum physics. Here we have strived to unravel some of the mystery behind this important physical effect by rendering quantum entanglement geometrically as tangles of the most basic of strands: quantum-informational spacetime trajectories (or qubit world lines). The advantage of this approach is that it allows us to represent quantum entangled states in terms of intuitive constructs borrowed from the mathematics developed to understand knots.

In knot theory, the most fundamental construct is braiding (or crossing) two adjacent strands. In quantum-information theory, a fundamental construct is entangling two qubits.[2] In general, the braiding operation is nonunitary, whereas an entangling operation (two-qubit universal quantum gate) is manifestly unitary. Yet, these two operations are not entirely unrelated—they are in fact special aspects of a general operation, termed a superbraid. The superbraid covers both nonunitary and unitary fundamental physical operations. It both braids qubit world lines and entangles the qubits and in this way mathematically disambiguates these most basic physical processes.

Analytical defining relations for a superbraid operator were presented, as was the algebra for its generator. For Q number of qubits, the generator of a superbraid was found to be a Hermitian operator that is a generalization of the usual generators in knot theory satisfying the Temperley-Lieb algebra, $\mathrm{TL}_Q(d)$. The generalization presented here handles both the fermionic and the bosonic cases of quantum-information dynamics; the quantum particles (whose motion defines the quantum-informational strands) can obey either Fermi or Bose statistics. In the bosonic case, the operative generators are the usual ones that satisfy $\mathrm{TL}_Q(d)$ and they serve as a Hermitian representation. The generalization of $\mathrm{TL}_Q(d)$ was actually needed to handle the case of entangled fermionic world lines. These generators, and in turn their respective superbraid operators, are analytically expressed in terms of the qubit ladder operators: qubit anticommuting creation and destruction operators and number operators.

With the technology presented, one can topologically classify closed-loop quantum circuits with various schemes for quantum wires crossed (braided) at some locations and coupled together at some locations via quantum logic gates, for example, a sequence of braid operations (particle motion) and quantum gate operations (particle-particle interactions) that specify the local dynamics of a quantum lattice gas used for a computational physics application. The closure of a sequence of superbraids is called a superlink. We have demonstrated how a superlink invariant may be computed, for example, as a generalized Jones polynomial invariant. Invariants were calculated for two-stranded superlinks, and extending this to Q strands is straightforward. The approach is to reduce a superbraid with n crossings to a simpler superbraid with $n-1$ crossings by applying a quantum skein relation, a straightforward generalization of the skein relations of knot theory. The quantum skein relations are summarized as follows.

The A parameter commonly used in the classical skein relation of knot theory, $\times = A\,)(\, + A^{-1} \asymp$, a version of (27a) that is flipped over, may be understood in the context of quantum-information processing as representing the two alternatives for the exchange of a pair of bits for configurations with one bit up (logical zero) and the other bit down (logical one). The diagrammatic convention has information flowing from top to bottom, entering in the top leads of the diagram and exiting from the bottom leads. So if the initial state is $|\ldots\uparrow\downarrow\ldots\rangle$, then the final-state alternatives are: (a) no interaction (identity operation) $|\ldots\uparrow\downarrow\ldots\rangle$ or (b) an exchange interaction (classical SWAP) $|\ldots\downarrow\uparrow\ldots\rangle$. A is the amplitude for the identity transition $|\ldots\uparrow\downarrow\ldots\rangle \Rightarrow |\ldots\uparrow\downarrow\ldots\rangle$, whereas A^{-1} represents the amplitude for the exchange transition $|\ldots\uparrow\downarrow\ldots\rangle \Rightarrow |\ldots\downarrow\uparrow\ldots\rangle$. Since the braid operator conserves bit number and A is an amplitude, we generalized the classical skein relation by allowing the interaction alternative (b) to represent a conservative quantum-mechanical exchange—one defined by a bit-conserving, two-qubit entangling gate operation. This resulted in the following quantum skein relation for a superbraid operator:

$$A \,\times\!\!\!\sim = A\,)(\, + A^{-1}[(e^z - 1)/(A^{-2}d)] \asymp,$$

[2]Quantum computing algorithms can be specified in terms of entangling quantum gates that act only between adjacent qubits—two-qubit entangling gate operations between nonadjacent qubits (customarily used in specifying quantum algorithms in the quantum computing literature) can each be represented as a sequence of two-qubit gate operations acting on adjacent qubit pairs. Thus, local braid and quantum gate operations are both universal operations in their respective contexts.

where $d = -A^2 - A^{-2}$. Thus, a classical point occurs for any value of z that causes the quantity in the square bracket to become unity, $e^z - 1 = A^{-2}d$. That is, a superbraid reduces to a braid when $e^z = -A^{-4}$.

In the case when $e^z = e^{i\zeta} A^{-4}$ and A is complex unimodular, the phase of A physically acts as an internal ebit phase angle; that is, $A = e^{i(\xi - \frac{\pi}{2})}$ as is discussed in the Appendix. In (22b) we wrote the superbraid operator as the product of a braid and conservative quantum gate. It braids two-qubit world lines and entangles them according to the deficit angle $\Delta\zeta = \zeta - \zeta_\circ \neq 0$, where ζ is the real-valued time (related to the imaginary part of z) parametrizing the operation and $\zeta_\circ = \pi$ is the classical point. Entangled qubit world lines and tangled strands are related through their respective skein relations sharing a common A parameter. The approach of representing quantum-information topologically in terms of tangled strands [5–11,22], and that we have explored here, offers insights about quantum entanglement as the quantum skein relation just mentioned, for purely imaginary z, is an entangling conservative quantum gate.

The aforementioned considerations naturally lead to some relevant further outlooks, for example the observability of quantum knots such as the square root of a knot, which of course has no classical counterpart. Just how the square root of a knot may be physically realized in an experimental setup is not presently known for certain, but according to Gell-Mann's totalitarian principle it should be experimentally compulsory as a physical phenomenon. Here is one possibility: a superbraid could exist within a Bose-Einstein condensate (BEC) superfluid as a superposition of quantum vortex loops. Furthermore, such vortex loops could represent a topologically protected qubit, and entangled states of such qubits could exist within a spinor BEC.[3] In a BEC, all the vorticity in the flow is pinned to filamentary topological defects in the phase of the condensate, that is, quantum vortices with integer winding numbers [25]. In a spinor BEC with two or more components in the condensate, each component may have its own quantum vortices. In the dilute vortex limit for each component, these quantum vortices act like strands that may be braided and entangled as the spinor superfluid flow evolves in time.

Consider a configuration of quantum vortices in a superfluid comprising two unlinked closed loops ○○.[4] Since nearby quantum vortex segments spontaneously undergo reconnection, one would expect that ○○ ⇒ ∞ ⇒ ○○ and so on. To form a qubit, one could identify the logical states with the two basic quantum vortex configurations: $|0\rangle \equiv |○○\rangle$ and $|1\rangle \equiv |∞\rangle$. So a topologically protected qubit (superbraided qubit) could be represented as a superposition of quantum vortex solitons,

$$|q\rangle = a|○○\rangle + b|∞\rangle, \tag{40}$$

with amplitudes constrained by $|a|^2 + |b|^2 = 1$.[5] These amplitudes are time-dependent quantities, like the amplitudes of a half-integer spin precessing about a uniform magnetic field, but in this case the effective Rabi frequency is set by the inverse of two reconnection times.

In a two-component BEC, a topologically protected (perpendicular) pairwise entangled state could be formed by coupling two superbraided qubits:

$$|\psi\rangle = \psi_{01}\,|○○, ∞\rangle + \psi_{10}\,|∞, ○○\rangle. \tag{41}$$

Allowing for spatial overlap of the quantum vortices in each component, the two-qubit quantum vortex configuration $|○○, ∞\rangle$ could have its two-loop configuration ○○ simultaneously occupy the same location as its unknot quantum vortex configuration ∞ in the spinor superfluid's second component. So, $|○○, ∞\rangle$ might physically occur as ○○ (overlapped). If the quantum vortex solitons can be spatially correlated in this way, then the quantum particles comprising the condensate can become physically entangled across their respective vortex cores, and in turn so too the vortex solitons themselves—a physical pathway whereby linkage may be related to entanglement.[6]

Superbraid solitons such as (40) could store topologically protected qubits, and superlinks such as (41) could process topologically protected e-bits. If many superbraided qubits were coupled together to realize controllable quantum logic operations, then topological quantum computation may be directly achievable within spinor superfluids. This offers us a potential alternative to exotic non-Abelian vortices (Fibonacci anyons) recently proposed for thin-film superconductor-based topological quantum-information processing. This proposition will be explored in future work on topological quantum computing with superbraids.

Finally, since a superbraid can be a nonunitary operation for certain values of the complex time parameter z, in a future study one could also explore if, and if so how, this particular feature might be akin to other behaviors of quantum systems in the real world. Quantum dynamics is effectively nonunitary because of decoherence (loss of phase coherency between previously correlated qubits) as well as the nonunitarity of projective measurement (collapse of qubit states onto a logical basis). So, for example, the behavior of the nonunitary dynamics of superbraids could also be explored regarding its potential connection to measurement-based quantum computation—either ruling out or elucidating this connection.

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[3] Spinor BECs have been realized in a confined cold spinor atomic BEC with several hyperfine states [23,24].

[4] The vorticity points along a quantum vortex line, so these act like directed strands. For simplicity, we omit arrow labels. Furthermore, these loops are not necessarily coplanar, for example, with the first loop in the plane of the paper, the second one could be rotated about the axis along its horizontal diameter, and perpendicularly oriented after a 90° rotation.

[5] The square root of unknot $\sqrt{∞}$ is realized in this qubit when $a = b = 1/\sqrt{2}$, or 45° polarization in the logical basis.

[6] This is allowed so long as the separation δ between the component vortices is much less than the coherence length ξ. This particular condition $\delta \ll \xi$ for interspinor superflow with quantum vortex solitons of unit winding number is different than the requirement $\xi \ll \delta$ in topological quantum computing with quantum logic represented in terms of braided zero-mode vortices in a $p + ip$ superconductor (as discussed in the Introduction).

APPENDIX: JOINT QUANTUM LOGIC

The coefficients in (8) can be parametrized by a real angle ϑ: $\mathcal{E}_{\Delta\alpha\gamma} = \mathcal{E}_{\Delta\alpha\gamma}(\vartheta)$ with

$$A^2 = -\frac{\cos\vartheta + 1}{\sin\vartheta}, \tag{A1}$$

and $d = -A^2 - A^{-2} = 2\csc\vartheta$. Then, (8) takes the form

$$\mathcal{E}_{\Delta\alpha\gamma} = \cos^2\frac{\vartheta}{2}\, n_\alpha + \sin^2\frac{\vartheta}{2}\, n_\gamma - \frac{1}{2}\sin\vartheta\,(A\, a_\alpha^\dagger a_\gamma + A^{-1} a_\gamma^\dagger a_\alpha) + (\Delta - 1) n_\alpha n_\gamma. \tag{A2}$$

The purpose of this appendix is to explain the origin of (A2).

Let us consider entangling two quantum bits, say, $|q_\alpha\rangle$ and $|q_\beta\rangle$, in a system comprised of $Q \geqslant 2$ qubits and where the integers α and $\beta \in [1, Q]$ are not equal, $\alpha \neq \beta$. Joint pair creation and annihilation operators [26] act on a qubit pair,

$$a_{\alpha\beta}^\dagger \equiv \frac{1}{\sqrt{2}}(a_\alpha^\dagger - e^{-i\xi} a_\beta^\dagger), \quad a_{\alpha\beta} \equiv \frac{1}{\sqrt{2}}(a_\alpha - e^{i\xi} a_\beta), \tag{A3}$$

and are defined in terms of the fermionic ladder operators $a_\alpha^\dagger$ and a_α. The joint number operator corresponding to (A3) is

$$n_{\alpha\beta} \equiv a_{\alpha\beta}^\dagger a_{\alpha\beta} \tag{A4a}$$

$$= \tfrac{1}{2}(n_\alpha + n_\beta - e^{i\xi} a_\alpha^\dagger a_\beta - e^{-i\xi} a_\beta^\dagger a_\alpha), \tag{A4b}$$

where the usual qubit number operator is $n_\alpha \equiv a_\alpha^\dagger a_\alpha$. Finally, I introduce an entanglement number operator (Hamiltonian) with a simple idempotent form

$$\epsilon_{\Delta\alpha\beta} \equiv n_{\alpha\beta} + (\Delta - 1) n_\alpha n_\beta, \tag{A5}$$

where Δ is a Boolean variable (0 for the bosonic case and 1 for the fermionic case). Eq. (A5) acts on some state $|\ldots q_\alpha \ldots q'_\beta \ldots\rangle$, with a qubit of interest located at α and another at β.

For convenience, I will use a shorthand for writing a state, specifying only two qubit locations as subscripts

$$|qq'\rangle_{\alpha\beta} \equiv |\ldots q_\alpha \ldots q'_\beta \ldots\rangle,$$

since the operators act on a qubit pair, regardless of the respective pair's location within the system of Q qubits. Then, the entangled "singlet" substate is $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{\alpha\beta}$ and the entangled "triplet" substates are $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)_{\alpha\beta}$ and $\frac{1}{\sqrt{2}}(|11\rangle \pm |00\rangle)_{\alpha\beta}$. I will refer to the ket $|qq'\rangle_{\alpha\beta}$ as perpendicular with respect to its constituent qubits $|q\rangle_\alpha$ and $|q'\rangle_\beta$ when $q \neq q'$ and as parallel when $q = q'$. Examples of perpendicular and parallel two-qubit (Fock) states are depicted in Fig. 1.

Neglecting normalization factors, the action of (A5) on the entangled $\alpha\beta$ substates is

$$\epsilon_\Delta(|01\rangle \pm |10\rangle)_{\alpha\beta} = \tfrac{1}{2}(1 \mp e^{-i\xi})|01\rangle - \tfrac{1}{2}(1 \pm e^{i\xi})|10\rangle, \tag{A6}$$

$$\epsilon_\Delta(|11\rangle \pm |00\rangle)_{\alpha\beta} = \Delta|11\rangle.$$

As a matter of convention, the $\alpha\beta$ indices are moved from ϵ_Δ to the state ket upon which the entanglement number operator acts and also the $\alpha\beta$ indices are not repeated on the right-hand side of an equation if avoiding redundancy does not introduce any ambiguity.

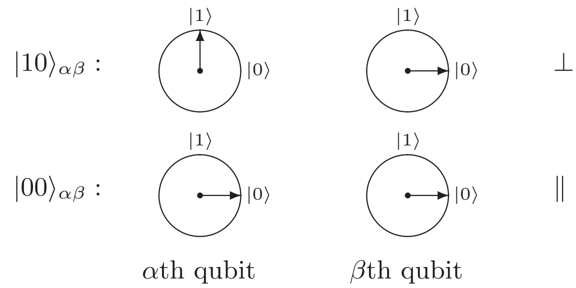

FIG. 1. Example of perpendicular and parallel two-qubit substates. The perpendicular substate $|10\rangle$ (top pair) and the parallel substate $|00\rangle$ (bottom pair) are depicted with qubits as unit vectors on the complex circle.

Two special cases of interest are $\xi = \pi$ and $\xi = 0$. For convenience, let us denote these particular angles with plus and minus symbols ($+ = \pi$ and $- = 0$):

$$n_{\alpha\beta}^{\pm} \equiv \tfrac{1}{2}(a_\alpha^\dagger \pm a_\beta^\dagger)(a_\alpha \pm a_\beta), \quad \epsilon_{\Delta\alpha\beta}^{\pm} = n_{\alpha\beta}^{\pm} + (\Delta - 1) n_\alpha n_\beta. \tag{A4$'$}$$

Eq. (A4′) measures entanglement between qubits that are perpendicularly oriented in the four-dimensional $\alpha\beta$ subspace of the 2^Q-dimensional Hilbert space.

The sum of these perpendicular joint number operators is related to the qubit number operators as follows:

$$\epsilon_{\Delta\alpha\beta}^{+} + \epsilon_{\Delta\alpha\beta}^{-} = n_\alpha + n_\beta + 2(\Delta - 1) n_\alpha n_\beta. \tag{A7}$$

The left-hand side of (A7) represents the total information in the $\alpha\beta$ subspace spanned by two qubits in the form of quantum-mechanical e-bits, whereas the right-hand side represents the equivalent amount of information in classical bits. Suppressing indices, ϵ_Δ^- has unity eigenvalue for the singlet state,

$$\epsilon_\Delta^-(|01\rangle - |10\rangle)_{\alpha\beta} = |01\rangle - |10\rangle, \tag{A8a}$$

$$\epsilon_\Delta^-(|01\rangle + |10\rangle)_{\alpha\beta} = 0,$$

$$\epsilon_\Delta^-(|11\rangle \pm |00\rangle)_{\alpha\beta} = \Delta|11\rangle, \tag{A8b}$$

but it has eigenvalue 0 for the triplet state, $|01\rangle + |10\rangle$. Conversely, ϵ_Δ^+ has a zero eigenvalue for the singlet state,

$$\epsilon_\Delta^+(|01\rangle - |10\rangle)_{\alpha\beta} = 0, \tag{A9a}$$

$$\epsilon_\Delta^+(|01\rangle + |10\rangle)_{\alpha\beta} = |01\rangle + |10\rangle,$$

$$\epsilon_\Delta^+(|11\rangle \pm |00\rangle)_{\alpha\beta} = \Delta|11\rangle, \tag{A9b}$$

but it has eigenvalue 1 for the first triplet state.

To avoid any contribution arising from parallel entanglement from the triplet states in (A8b) and (A9b), we must take $\Delta = 0$ in the Hermitian operator (A4′) to ensure we count only one bit of perpendicular pairwise entanglement. Hence, denoting the pairwise entangled states with qubits of interest at locations α and β as $\psi_\perp^{\pm} \equiv |\ldots 0 \ldots 1 \ldots\rangle \pm |\ldots 1 \ldots 0 \ldots\rangle$, then these entangled states are eigenvectors of the $\Delta = 0$ joint number operators (with unit eigenvalue): $\epsilon_0^{\pm}\psi_\perp^{\pm} = \psi_\perp^{\pm}$. The parallel joint operators can be defined in terms of the perpendicular joint operators as

$$\varepsilon_0^{\pm} \equiv \tfrac{1}{2}(1 \mp \epsilon_0^+ \pm \epsilon_0^- \pm \{\sigma_x\sigma_x, \epsilon_1^{\mp}\}), \tag{A10}$$

suppressing qubit indices, and $\sigma_x\sigma_x \equiv \sigma_x^{(\alpha)} \otimes \sigma_x^{(\beta)}$ is shorthand for a tensor product. The sum

$$\epsilon_0^+ + \epsilon_0^- + \varepsilon_0^+ + \varepsilon_0^- = \mathbf{1} \tag{A11}$$

is as fundamental to two-qubit substates as the singleton number operator identity $n + \bar{n} = \mathbf{1}$ is to one-qubit states.

If the other pairwise entangled states with qubits at α and β are $\psi_\parallel^\pm \equiv |\ldots 0 \ldots 0 \ldots\rangle \pm |\ldots 1 \ldots 1 \ldots\rangle$, then these entangled states are eigenvectors of the joint number operators (with unit eigenvalue): $\varepsilon_0^\pm \psi_\parallel^\pm = \psi_\parallel^\pm$. In summary, the operators $\epsilon_0^\pm$ and $\varepsilon_0^\pm$ are number operators for the Bell states.

A consistent framework for dealing with quantum logic gates using the quantum circuit model of quantum computation was introduced over a dozen years ago by DiVincenzo *et al.* [27]. Here we consider an alternative: an analytical approach to quantum computation based on generalized second quantized operators, which is also practical for numerical implementations. Let us now consider quantum gates that induce entangled states from previously independent qubits. The basic approach employs a conservative quantum logic gate generated by (A5):

$$e^{i\vartheta\,\epsilon_\Delta} = 1 + (e^{i\vartheta} - 1)\epsilon_\Delta. \tag{A12}$$

Thus, a similarity transformation of the number operators n_α and n_β yields generalized joint number operators

$$n'_\alpha(\vartheta,\xi) \equiv e^{i\vartheta\,\epsilon_{\Delta\alpha\beta}} n_\alpha e^{-i\vartheta\,\epsilon_{\Delta\alpha\beta}} \tag{A13a}$$

$$= \cos^2\left(\frac{\vartheta}{2}\right) n_\alpha + \sin^2\left(\frac{\vartheta}{2}\right) n_\beta + \frac{i\sin\vartheta}{2}(e^{i\xi}a_\alpha^\dagger a_\beta - e^{-i\xi}a_\beta^\dagger a_\alpha) \tag{A13b}$$

and

$$n'_\beta(\vartheta,\xi) \equiv e^{i\vartheta\,\epsilon_{\Delta\alpha\beta}} n_\beta e^{-i\vartheta\,\epsilon_{\Delta\alpha\beta}} \tag{A14a}$$

$$= \sin^2\left(\frac{\vartheta}{2}\right) n_\alpha + \cos^2\left(\frac{\vartheta}{2}\right) n_\beta - \frac{i\sin\vartheta}{2}(e^{i\xi}a_\alpha^\dagger a_\beta - e^{-i\xi}a_\beta^\dagger a_\alpha). \tag{A14b}$$

Thus, the generalized joint number operators rotate continuously from $n_\alpha(0,\xi) = n_\alpha$ and $n_\beta(0,\xi) = n_\beta$ as ϑ ranges from 0 to π to the number operators $n_\alpha(\pi,\xi) = n_\beta$ and $n_\beta(\pi,\xi) = n_\alpha$.

That information is conserved by this similarity transformation is readily expressed by the number conservation identity obtained by adding (A13a) and (A14):

$$n'_\alpha(\vartheta,\xi) + n'_\beta(\vartheta,\xi) = n_\alpha + n_\beta. \tag{A15}$$

The left-hand side counts the information in its quantum-mechanical (entangled) form, whereas the right-hand side counts information in its classical form (separable) form. In any case, the total information content in the $\alpha\beta$ subspace is conserved. This is why the quantum logic gate operation (A12) is referred to as a *conservative* operation.

Comparing the generalized joint number operator (A13a) to the joint number operator (A4a), we obtain the useful identity in the special case of maximal entanglement,

$$n_\alpha\left(\frac{\pi}{2}, \xi + \frac{\pi}{2}\right) = n_\beta\left(\frac{\pi}{2}, \xi - \frac{\pi}{2}\right) = \frac{1}{2}(n_\alpha + n_\beta - e^{i\xi}a_\alpha^\dagger a_\beta - e^{-i\xi}a_\beta^\dagger a_\alpha) = n_{\alpha\beta}. \tag{A16}$$

The tensor product $n_\alpha n_\beta$ is invariant under similarity transformation:

$$n_\alpha n_\beta = e^{i\vartheta\,\epsilon_{\Delta\alpha\beta}}\, n_\alpha n_\beta\, e^{-i\vartheta\,\epsilon_{\Delta\alpha\beta}}. \tag{A17}$$

Let us also construct generalized joint ladder operators:

$$a'_\alpha = e^{i\vartheta\,\epsilon_{\Delta\alpha\beta}}\, c_\alpha\, e^{-i\vartheta\,\epsilon_{\Delta\alpha\beta}}, \tag{A18a}$$

$$a'^\dagger_\alpha = e^{i\vartheta\,\epsilon_{\Delta\alpha\beta}}\, c_\alpha^\dagger\, e^{-i\vartheta\,\epsilon_{\Delta\alpha\beta}}. \tag{A18b}$$

After some algebraic manipulation, these can be expressed explicitly just in terms of the original ladder operators,

$$a'_\alpha = e^{-i\frac{\vartheta}{2}}\left[\cos\left(\frac{\vartheta}{2}\right)a_\alpha + i e^{i\xi}\sin\left(\frac{\vartheta}{2}\right)a_\beta\right], \tag{A19a}$$

$$a'^\dagger_\alpha = e^{i\frac{\vartheta}{2}}\left[\cos\left(\frac{\vartheta}{2}\right)a_\alpha^\dagger - i e^{-i\xi}\sin\left(\frac{\vartheta}{2}\right)a_\beta^\dagger\right]. \tag{A19b}$$

The product of the generalized joint ladder operators (A19),

$$n'_\alpha = a'^\dagger_\alpha a'_\alpha, \tag{A20}$$

yields the generalized joint number operator (A13a), as expected.

An important special case of (A12) for half angles $\xi = \frac{\pi}{2}$ and $\vartheta = \frac{\pi}{2}$ is an antisymmetric square root of swap gate,

$$U_\Delta \equiv e^{i\frac{\pi}{2}\epsilon_{\Delta,\pi/2}}. \tag{A21}$$

Using (A21) as a similarity transformation, we find the identities

$$\epsilon_{\Delta\alpha\beta}^+ = U_\Delta{}^\dagger n_\beta\, U_\Delta \quad \text{and} \quad \epsilon_{\Delta\alpha\beta}^- = U_\Delta{}^\dagger n_\alpha\, U_\Delta, \tag{A22}$$

illustrating how the entanglement operators are related to the standard number operators.

Noting that the joint number operator is related to the generalized joint number operator according to (A16), and also noting that $n_\alpha n_\beta$ is invariant under the similarity transformation (A17), we can also write a generalization of (A5) as follows:

$$\mathcal{E}_{\Delta\alpha\beta} \equiv e^{i\vartheta\,\epsilon_{\Delta\alpha\beta}}[n_\alpha + (\Delta - 1)n_\alpha n_\beta]e^{-i\vartheta\,\epsilon_{\Delta\alpha\beta}} \tag{A23a}$$

$$\overset{(A13b)}{=} \cos^2\left(\frac{\vartheta}{2}\right)n_\alpha + \sin^2\left(\frac{\vartheta}{2}\right)n_\beta + \frac{i\sin\vartheta}{2}(e^{i\xi}a_\alpha^\dagger a_\beta - e^{-i\xi}a_\beta^\dagger a_\alpha) + (\Delta - 1)n_\alpha n_\beta. \tag{A23b}$$

We have the freedom to choose $A = -ie^{i\xi}$, and thus we have recovered the mathematical form of (A2).

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