Low-temperature coherence properties of Z_2 quantum memory

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We investigate low-temperature coherence properties of the Z_2 quantum memory which is capable of storing the information of a single logical qubit. We show that the memory has superposition of macroscopically distinct states for some values of a control parameter and at sufficiently low temperature and that the code states of this memory have no instability except for the inevitable one. However, we also see that the coherence power of this memory is limited by space and time. We also briefly discuss the resonating valence bond memory, which is an improvement of the Z_2 quantum memory, and the relations of our results to the obscured symmetry breaking in statistical physics.

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I. INTRODUCTION

Quantum memory [1,2], which stores the coherent information of logical qubits, is an essential ingredient of quantum information processing and plays crucial roles in almost all fields of quantum information science [3]. Basically, a quantum memory consists of macroscopically many physical qubits which encode the state of a few logical qubits. Each logical basis should be encoded on macroscopically distinct physical states since otherwise the indistinguishability of logical bases is easily destroyed by local errors. On the other hand, in order to store quantum coherent superposition of these logical bases, the memory must be able to have superposition of macroscopically distinct states of physical qubits. To maintain such superposition is very difficult. First, if the size of the memory is infinite, such macroscopic superposition is impossible since superposition of different phases is just a classical mixture of them in an infinite system [4–6]. Second, even in finite systems, such macroscopic superposition is usually very unstable [5]. Therefore, how to balance those two contradictory demands (i.e., classical information must be encoded on macroscopically distinct states but we also need superposition of them) is one of the most challenging problem in the implementation of quantum memory.

The code space of a quantum memory is often realized as the lowest-energy eigenspace of a many-body Hamiltonian. One of the most beautiful examples is Kitaev's toric code [1]. Kitaev introduced a four-body Hamiltonian which exhibits a topological phase transition. The ground states are degenerated and energetically isolated from the excited states. Each of these ground states corresponds to different topological phases in the thermodynamic limit, and therefore logical bases encoded on these degenerate ground states are immune to local errors. Logical qubit operations are performed by applying long strings of Pauli operators on physical qubits, which means that logical operations are nonlocal.

Although Kitaev's toric code is highly sophisticated and indeed has inspired plenty of successive studies [2,6–13], it is not easy to implement a scalable toric code in a laboratory since nonlocal operations are required. A complementary approach is therefore also important from the practical point of view.

In this article, we study the less elaborate but more feasible quantum memory, namely, the Z_2 quantum memory. Although the Z_2 quantum memory is too primitive to be a complete and universal quantum memory, it is still valuable to study it, since the simple structure of this memory means feasibility in a laboratory and the possibility of capturing the essence of theoretical aspects of quantum memory. In general, a quantum memory must have a large coherence for some values of a control parameter in order to store coherent information of logical qubits. Therefore, we analyze the coherent properties of the Z_2 quantum memory at low temperature by using the method, developed in Refs. [5,14–16], of detecting superposition of macroscopically distinct states. We show that the Z_2 quantum memory can have superposition of macroscopically distinct states for some values of a control parameter and at sufficiently low temperature and that the code states have no instability except for the inevitable one. However, we also see that the power of superposition of macroscopically distinct states in this memory is limited by space and time. These results suggest that the Z_2 quantum memory is of limited use as a prototype of a small quantum memory.

This article is organized as follows. In the next section, we briefly review the method of detecting superposition of macroscopically distinct states. In Sec. III, we study the zero-temperature case. We next study the finite-temperature case in Sec. IV. Finally, in Sec. V we briefly discuss the resonating valence bond (RVB) quantum memory, which is an improvement of the Z_2 quantum memory, and in Sec. VI we discuss the relation of our results to the concept of symmetry breaking in statistical physics.

II. INDEX p AND VARIANCE-COVARIANCE MATRIX

In this section, we briefly review the method of detecting superposition of macroscopically distinct states in quantum many-body states [5,14–16].

Let us consider an N-site lattice ($1 \ll N < \infty$) where the dimension of the Hilbert space on each site is an N-independent constant, such as a chain of N spin-1/2 particles. Throughout this article, $f(N) = O(N^k)$ means

$$\lim_{N\to\infty} \frac{f(N)}{N^k} = \text{const.} \neq 0.$$

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For a given pure state $|\psi\rangle$, the index p $(1 \leqslant p \leqslant 2)$ is defined by

$$\max_{\hat{A}}[\langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2] = O(N^p),$$

where the maximum is taken over all Hermitian additive operators $\hat{A}$. Here, an additive operator,

$$\hat{A} = \sum_{l=1}^{N} \hat{a}(l),$$

is a sum of local operators $\{\hat{a}(l)\}_{l=1}^{N}$, where $\hat{a}(l)$ is a local operator acting on site l. For example, if the system is a chain of N spin-1/2 particles, $\hat{a}(l)$ is a linear combination of three Pauli operators, $\hat{\sigma}_x(l)$, $\hat{\sigma}_y(l)$, $\hat{\sigma}_z(l)$, and the identity operator $\hat{1}(l)$ acting on site l. In this case, the x component of the total magnetization,

$$\hat{M}_x \equiv \sum_{l=1}^{N} \hat{\sigma}_x(l),$$

and the z component of the total staggered magnetization,

$$\hat{M}_z^{\mathrm{st}} \equiv \sum_{l=1}^{N} (-1)^l \hat{\sigma}_z(l),$$

are, for example, additive operators. The index p takes the minimum value 1 for any product state

$$\bigotimes_{l=1}^{N} |\phi_l\rangle,$$

where $|\phi_l\rangle$ is a state of site l (this means that $p > 1$ is an entanglement witness for pure states). If p takes the maximum value 2, the state contains superposition of macroscopically distinct states because in this case the relative fluctuation of an additive operator does not vanish in the thermodynamic limit,

$$\lim_{N\to\infty} \frac{\sqrt{\langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2}}{N} \neq 0,$$

and because the fluctuation of an observable in a pure state means the existence of a superposition of eigenvectors of that observable corresponding to different eigenvalues.

For example, the N-qubit Greenberger-Horne-Zeilinger (GHZ) state

$$|\mathrm{GHZ}\rangle \equiv \frac{1}{\sqrt{2}}(|0^{\otimes N}\rangle + |1^{\otimes N}\rangle),$$

which obviously contains superposition of macroscopically distinct states, has $p = 2$, since

$$\langle\mathrm{GHZ}|\hat{M}_z^2|\mathrm{GHZ}\rangle - \langle\mathrm{GHZ}|\hat{M}_z|\mathrm{GHZ}\rangle^2 = O(N^2).$$

It was shown in Ref. [5] that a state having $p = 2$ is unstable against a local noise from the environment and a local measurement, whereas a state having $p = 1$ is stable.

There is an efficient method of calculating index p [14,15]. For simplicity, we assume that the Hilbert space on each site is a two-dimensional one. Generalizations to higher dimensional cases are immediate.

For a given pure state $|\psi\rangle$, let us define the $3N \times 3N$ Hermitian matrix called the variance-covariance matrix (VCM) by

$$V_{\alpha l,\beta l'} \equiv \langle\psi|\hat{\sigma}_\alpha(l)\hat{\sigma}_\beta(l')|\psi\rangle - \langle\psi|\hat{\sigma}_\alpha(l)|\psi\rangle\langle\psi|\hat{\sigma}_\beta(l')|\psi\rangle,$$

where $\alpha, \beta = x, y, z; l, l' = 1, 2, \ldots, N$; and $\hat{\sigma}_x(l)$, $\hat{\sigma}_y(l)$, and $\hat{\sigma}_z(l)$ are Pauli operators on site l. Since the VCM is Hermitian, all eigenvalues are real. Let e_1 be the largest eigenvalue of the VCM. Then

$$e_1 = O(N^{p-1})$$

is satisfied [14,15], which means that we have only to calculate e_1 to obtain the value of index p. Since a matrix of a polynomial size can be diagonalized within a polynomial step, e_1 is obtained efficiently by numerical calculations.

III. ZERO TEMPERATURE

Let us first study the zero-temperature case. We consider the one-dimensional periodic chain of N qubits. The code space of the Z_2 quantum memory is stabilized by the "bond operators" $(l = 1, 2, \ldots, N)$ [8]

$$\hat{B}_l \equiv \hat{\sigma}_z(l)\hat{\sigma}_z(l+1),$$

which act on the "virtual qubits" (or "dual qubits") embedded on bonds, where $\hat{\sigma}_z(N+1) = \hat{\sigma}_z(1)$. Since

$$\hat{B}_N = \prod_{l=1}^{N-1} \hat{B}_l,$$

stabilizers of the Z_2 quantum memory are generated by $N-1$ bond operators $\{\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_{N-1}\}$, which means that the code space is $2^{N-(N-1)} = 2^1$ dimensional subspace. The centralizers of these stabilizers are generated by

$$\hat{X} \equiv \prod_{l=1}^{N} \hat{\sigma}_x(l), \tag{1}$$

and

$$\hat{Z} \equiv \hat{\sigma}_z(1).$$

They work as the logical bit flip and logical phase, respectively. The code space is also specified as the lowest-energy eigenspace of the two-body Hamiltonian

$$\hat{H}_0 = -\sum_{l=1}^{N} \hat{B}_l.$$

It is obvious that two degenerate separable ground states of this Hamiltonian are macroscopically distinct from each other, and therefore the indistinguishability of logical bases, $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$, is not destroyed by local errors.

Ideally, the logical Hadamard operation

$$|\tilde{0}\rangle \to \frac{1}{\sqrt{2}}(|\tilde{0}\rangle + |\tilde{1}\rangle),$$

$$|\tilde{1}\rangle \to \frac{1}{\sqrt{2}}(|\tilde{0}\rangle - |\tilde{1}\rangle),$$

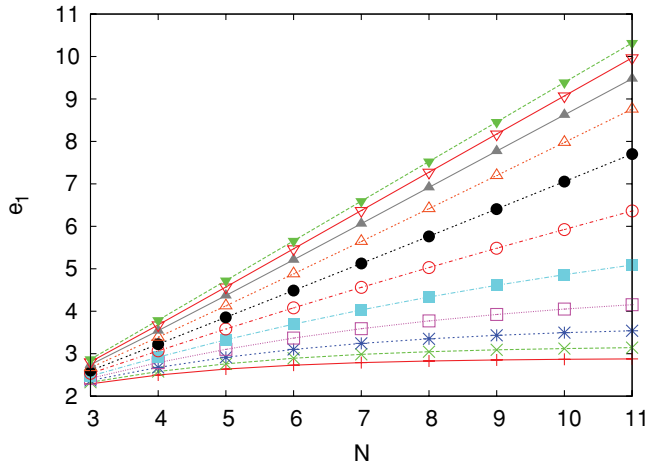

FIG. 1. (Color online) e_1 versus N for $|E_0(\lambda)\rangle$ with various λ. From the bottom, λ =1.5, 1.4, 1.3, 1.2, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, and 0.5, respectively. Lines are guides to the eye.

which is the essential ingredient of various quantum information processing, is realized by using the logical $\hat{X}$ operation, Eq. (1). However, such nonlocal operation is not easy to experimentally implement. Therefore, it is reasonable to try to manipulate the quantum memory in the local way:

$$\hat{H} = \hat{H}_0 + \lambda \sum_{l=1}^{N} \hat{\sigma}_x(l), \tag{2}$$

where λ is an external control parameter.

From the Perron-Frobenius theorem [17], the ground state of this Hamiltonian is nondegenerate if $\lambda \neq 0$. It is also known that this Hamiltonian exhibits the quantum phase transition at $\lambda = 1$ [18]. Let us denote the exact ground state of $\hat{H}$ corresponding to the external parameter λ by $|E_0(\lambda)\rangle$ and evaluate index p of $|E_0(\lambda)\rangle$ for various λ. In Fig. 1, the largest eigenvalue e_1 of the VCM versus N is plotted by changing the value of λ. This figure shows that

(i) $|E_0(\lambda \geqslant 1)\rangle$ has $p < 2$,
(ii) $|E_0(\lambda < 1)\rangle$ has $p = 2$,

which means that the Z_2 quantum memory has superposition of macroscopically distinct states for values $\lambda < 1$ of the control parameter λ.

In order to see the structure of the superposition, the probability distribution $P(M_z)$ of $\hat{M}_z$ in $|E_0(0.5)\rangle$ is plotted in Fig. 2 for $N = 13$. This figure suggests that the ground state is, in a rough picture, a superposition of two macroscopically distinct states:

$$|E_0(0.5)\rangle \simeq |\phi_+\rangle + |\phi_-\rangle, \tag{3}$$

where $|\phi_\pm\rangle$ are some states satisfying $\langle\phi_\pm|\hat{M}_z|\phi_\pm\rangle \simeq \pm N$, respectively.

By seeing the eigenvector of the VCM corresponding to the largest eigenvalue e_1, we can also know that the additive operator which gives the maximum fluctuation in $|E_0(\lambda < 1)\rangle$ is

$$\hat{M}_z \equiv \sum_{l=1}^{N} \hat{\sigma}_z(l).$$

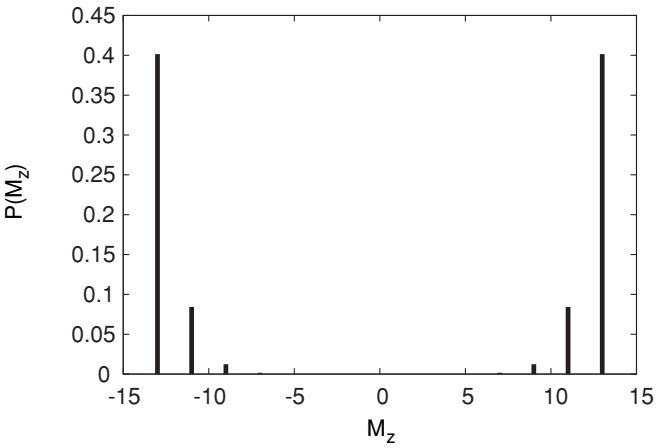

FIG. 2. The probability distribution $P(M_z)$ of $\hat{M}_z$ in $|E_0(0.5)\rangle$ with $N = 13$. Similar structures are obtained for other values of N.

According to Ref. [5], this means that $|E_0(\lambda < 1)\rangle$ is unstable against the local noise described by the interaction Hamiltonian

$$\hat{H}_{\rm int} \equiv \sum_{l=1}^{N} f(l)\hat{\sigma}_z(l), \tag{4}$$

where $f(l)$ is a noise parameter of a long wavelength. Although this noise is inevitable since we need the superposition of macroscopically distinct logical bases, we can still show that $|E_0(\lambda < 1)\rangle$ has no other instability than this inevitable one. In Fig. 3, we plot the second largest eigenvalue e_2 of the VCM versus N for $|E_0(0.5)\rangle$. From this figure we obtain

$$e_2 \leqslant O(N^0),$$

which means that $|E_0(0.5)\rangle$ is stable against all noises of the type

$$\hat{H}_{\rm int} \equiv \sum_{l=1}^{N} f(l)\hat{a}(l)$$

except for the case of Eq. (4).

In short, we have seen that the Z_2 quantum memory can have superposition of macroscopically distinct states and it

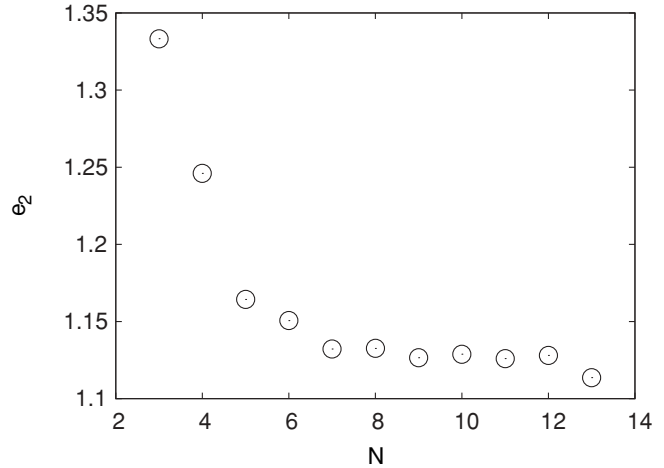

FIG. 3. e_2 versus N for $|E_0(0.5)\rangle$.

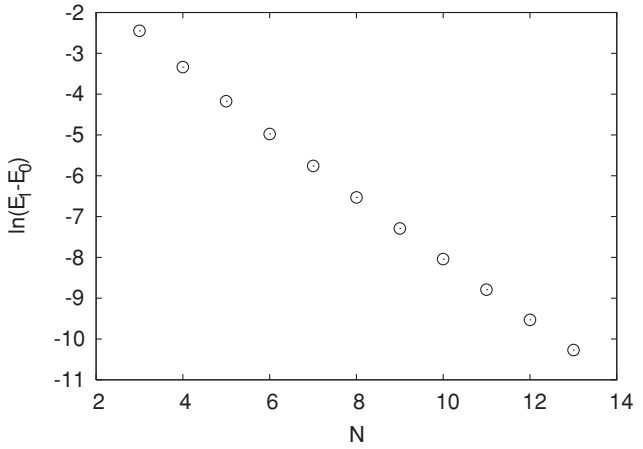

FIG. 4. $\ln \Delta E$ versus N for $\lambda = 0.5$.

is stable against any local noise except for the inevitable one.

As is seen in Fig. 4, the energy gap $\Delta E \equiv E_1 - E_0$ between the exact ground state and the first excited state for $\lambda = 0.5$ decays exponentially fast as $N \to \infty$. Therefore, it is physically allowed to take a linear combination of the exact ground state and the first excited state for sufficiently large N.

From Eq. (3) and the analogy with the physics of the single two-level atom system, the rough picture of the first excited state $|E_1(0.5)\rangle$ is expected to be

$$|E_1(0.5)\rangle \simeq |\phi_+\rangle - |\phi_-\rangle, \tag{5}$$

and therefore the superposition

$$|E_0'\rangle \equiv \frac{1}{\sqrt{2}}(|E_0(0.5)\rangle + |E_1(0.5)\rangle)$$

of the exact ground state and the first excited state is expected to have no superposition of macroscopically distinct states.

Indeed, we plot e_1 versus N for $|E_0'\rangle$ in Fig. 5. This figure shows that $e_1 = O(N^0)$, which means that the superposition of macroscopically distinct states in quantum memory disappears for sufficiently large system size. Therefore, the

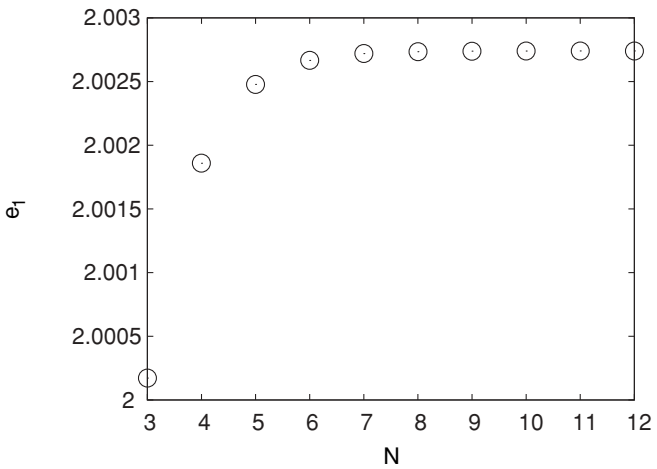

FIG. 5. e_1 versus N for $|E_0'\rangle$.

Z_2 quantum memory is of limited use as a small quantum memory.

It is worth mentioning that such a fast decay of the energy gap is a natural consequence of the locality of the Hamiltonian. According to Eqs. (3) and (5), the difference between the exact ground state and the first excited state is the relative phase of $|\phi_\pm\rangle$. If the energy gap does not go to 0 for large N, we can distinguish the exact ground state and the first excited state by measuring the energy. If the Hamiltonian is a local one, that is, it does not contain many-point correlations, this means that we can know the relative phase by measuring only few-point correlations, which obviously contradicts the common sense of decoherence [19].

In addition to the limit of the space, the exponential decay of the energy gap also gives the limit to the operation time. Assume that we perform the logical Hadamard gate by adiabatically increasing the control parameter λ. Then, according to the theory of adiabatic quantum computation [20,21], the exponential decay of the energy gap means an exponentially long operation time:

$$T \simeq \frac{1}{(\Delta E)^2},$$

where ΔE is the minimum energy gap. Therefore, the large Z_2 quantum memory also has the limit of the operation time.

IV. FINITE TEMPERATURE

Next let us study the Z_2 quantum memory at finite temperature.

At finite temperature T, a system is generally in the equilibrium state:

$$\hat{\rho} = \frac{e^{-\hat{H}/kT}}{\mathrm{Tr}(e^{-\hat{H}/kT})}.$$

If the state is not necessarily pure, index p cannot detect superposition of macroscopically distinct states since a fluctuation is not necessarily equivalent to the coherence in mixed states.

In order to detect superposition of macroscopically distinct states in mixed states, index q was proposed in Ref. [22]. For a given many-body state $\hat{\rho}$, index q ($1 \leqslant q \leqslant 2$) is defined by

$$\max\left(N, \max_{\hat{A}} \|[\hat{A}, [\hat{A}, \hat{\rho}]]\|_1\right) = O(N^q),$$

where $\|\hat{X}\|_1 \equiv \mathrm{Tr}\sqrt{\hat{X}^\dagger \hat{X}}$ is the 1-norm and $\max_{\hat{A}}$ means the maximum over all Hermitian additive operators $\hat{A}$. As detailed in Ref. [22], q takes the minimum value 1 for any separable state,

$$\sum_i \lambda_i \bigotimes_{l=1}^{N} |\phi_l^{(i)}\rangle\langle\phi_l^{(i)}|,$$

where $|\phi_l^{(i)}\rangle$ is a state of site l. On the other hand, if q takes the maximum value 2, the state contains superposition of macroscopically distinct states. In particular, for pure states, $p = 2 \iff q = 2$.

Unlike the case of index p, there is no method of efficiently calculating index q at the time of writing. However, we can calculate a lower bound of the value of q. Indeed, let us note

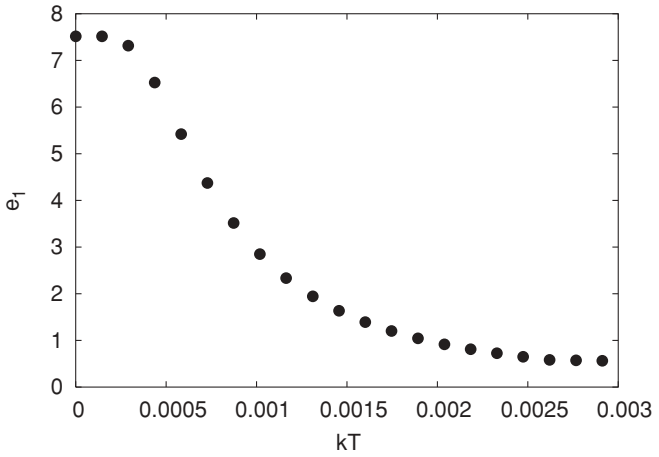

FIG. 6. e_1 versus kT/J for $N = 8$ ($J = 1$).

that

$$\begin{aligned}\|[\hat{A},\hat{\rho}]\|_2^2 &= \mathrm{Tr}([\hat{A},\hat{\rho}]^\dagger[\hat{A},\hat{\rho}]) \\ &= \mathrm{Tr}(\hat{\rho}[\hat{A},[\hat{A},\hat{\rho}]]) \\ &\leqslant \frac{\mathrm{Tr}(\hat{\rho}[\hat{A},[\hat{A},\hat{\rho}]])}{\|\hat{\rho}\|_\infty} \\ &\leqslant \|[\hat{A},[\hat{A},\hat{\rho}]]\|_1,\end{aligned}$$

where $\|\hat{X}\|_2 \equiv \sqrt{\mathrm{Tr}(\hat{X}^\dagger\hat{X})}$ is the 2-norm and $\|\hat{X}\|_\infty$ is the oper ator norm. If we define the $3N \times 3N$ Hermitian matrix W by

$$W_{\alpha,l,\beta,l'} \equiv \mathrm{Tr}\{[\hat{\rho},\hat{\sigma}_\alpha(l)][\hat{\sigma}_\beta(l'),\hat{\rho}]\}, \tag{6}$$

where $\alpha,\beta = x,y,z$ and $l,l' = 1,2,\ldots,N$, it is easy to see that the order of

$$\max_{\hat{A}} \|[\hat{A},\hat{\rho}]\|_2^2$$

with respect to N is equal to that of $e_1 N$, where e_1 is the largest eigenvalue of W.

In Fig. 6, we plot the largest eigenvalue e_1 of the matrix W, Eq. (6), versus kT for the equilibrium state of the Hamiltonian, Eq. (2), for $\lambda = 0.5$ and $N = 8$. This figure shows that superposition of macroscopically distinct states at zero temperature persists at sufficiently low temperature.

V. RVB MEMORY

In this article, we have seen that the Z_2 quantum memory can have superposition of macroscopically distinct states for values $\lambda < 1$ of the control parameter λ, and this superposition of macroscopically distinct states is stable against any local noise except for the inevitable one. Let us briefly discuss a possibility of avoiding this inevitable instability without introducing too many sophisticated methods.

One of the most simple ways would be to use RVB states:

$$\begin{aligned}|\Psi\rangle &\equiv \frac{1}{\sqrt{2+4\left(-\frac{1}{2}\right)^{N/2}}}\left[\bigotimes_{l=1}^{N/2}|2l-1,2l\rangle + \bigotimes_{l=1}^{N/2}|2l,2l+1\rangle\right] \\ &\simeq \frac{1}{\sqrt{2}}\bigotimes_{l=1}^{N/2}|2l-1,2l\rangle + \frac{1}{\sqrt{2}}\bigotimes_{l=1}^{N/2}|2l,2l+1\rangle \\ &\equiv \frac{1}{\sqrt{2}}|\mathrm{VB}_1\rangle + \frac{1}{\sqrt{2}}|\mathrm{VB}_2\rangle,\end{aligned}$$

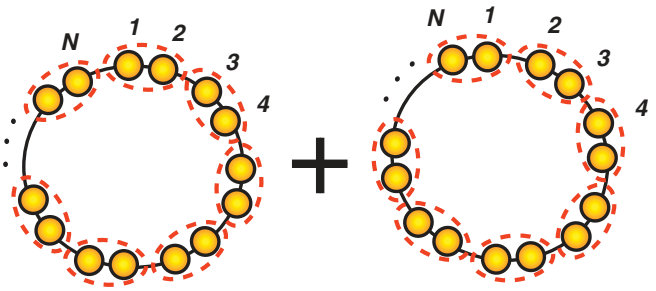

FIG. 7. (Color online) The nearest-neighbor RVB state on a one-dimensional periodic lattice. Red circles (dashed circles) represent singlet pairs. Two macroscopically distinct valence bond states are superposed.

where $|i,j\rangle$ represents the singlet pair between sites i and j and the periodic boundary condition is assumed (Fig. 7). This state is realized as a ground state of the one-dimensional spin ladder model or the Majumdar-Ghosh model [23].

$|\Psi\rangle$ is a superposition of two macroscopically distinct symmetry-broken states, $|\mathrm{VB}_1\rangle$ and $|\mathrm{VB}_2\rangle$. The advantage of this state is that, as is shown in the Appendix, there is no long-range two-point correlation in this state:

$$\langle\Psi|\hat{a}(l)\hat{a}(l')|\Psi\rangle - \langle\Psi|\hat{a}(l)|\Psi\rangle\langle\Psi|\hat{a}(l')|\Psi\rangle = 0, \tag{7}$$

for any pair of local operators $\hat{a}(l)$ and $\hat{a}(l')$ with $|l-l'| \geqslant 2$. Therefore, $|\Psi\rangle$ has $p = 1$ and this means that the state is stable against any local noise of the type

$$\hat{H}_{\mathrm{int}} = \sum_{l=1}^{N} f(l)\hat{a}(l).$$

In order to detect superposition of macroscopically distinct states in $|\Psi\rangle$, we must consider the sum of bilocal operators. Indeed, let us consider the operator

$$\hat{T} \equiv \sum_{l=1}^{N}(-1)^l\hat{t}_{l,l+1},$$

where

$$\hat{t}_{l,l+1} \equiv |l,l+1\rangle\langle l,l+1|$$

is the projection operator on the singlet state of sites l and $l+1$. Then, we can show that the state $|\Psi\rangle$ has superposition of macroscopically distinct states in the sense of

$$\langle\Psi|\hat{T}^2|\Psi\rangle - \langle\Psi|\hat{T}|\Psi\rangle^2 = O(N^2). \tag{8}$$

A proof is given in the Appendix.

In summary, we have seen that the stability of the Z_2 quantum memory can be improved by introducing local entanglement between nearest neighbor sites. The RVB state thus created still has superposition of macroscopically distinct states.

VI. CONCLUSION AND DISCUSSION

In this article, we have investigated the low-temperature coherence properties of the Z_2 quantum memory. We have shown

that (i) the memory can have superposition of macroscopically distinct states for the values $\lambda < 1$ of the control parameter λ and at sufficiently low temperature, (ii) the code states of this memory have no instability except for the inevitable one, and (iii) the power of superposition of macroscopically distinct states in this memory is limited by space and time. We have also briefly discussed how the RVB memory improves the Z_2 quantum memory.

To conclude this article, let us briefly discuss the relations of our results to symmetry breaking in statistical physics.

Symmetry breaking is one of the most fundamental concepts in modern physics [24–26]. According to Landau-Ginzburg theory [27], the state of the phase is a local minimum of the effective potential as a function of the order parameter: if the temperature is high, the potential has the unique minimum at the origin and therefore the system is in the symmetric phase, whereas at sufficiently low temperature the effective potential becomes the double-well type whose local minima correspond to the symmetry-broken phases.

Although such an intuitive picture of the symmetry breaking has contributed to the progress of physics for a long time, it has been often pointed out by many researchers that such a picture is not always correct if the system is of a finite volume [28–33]. For example, if the many-body Hamiltonian $\hat{H}$ and the order operator $\hat{O}$ do not commute with each other $[\hat{H}, \hat{O}] \neq 0$ (typical examples in condensed matter physics are the transverse Ising model, the Heisenberg antiferromagnet, and the Hubbard model), the exact ground state of a finite volume is often nondegenerate and therefore symmetric. This symmetric exact ground state is completely different from the symmetry-broken "mean-field ground states," which are inherently separable since the mean-field approximation neglects the correlations among sites [34].

Such a symmetric exact ground state $|E_0\rangle$ often has the peculiar property that the relative fluctuation of a macroscopic observable $\hat{A}$, which is mostly the order operator $\hat{O}$, does not vanish even in the thermodynamic limit:

$$\lim_{N\to\infty} \frac{\sqrt{\langle E_0|\hat{A}^2|E_0\rangle - \langle E_0|\hat{A}|E_0\rangle^2}}{N} \neq 0,$$

where N is the number of total sites (alias the volume of the system). Since this means that a macroscopic observable does not have a definite value even in the thermodynamic limit, $|E_0\rangle$ is an anomalous state [5] from the viewpoint of thermodynamics where any macroscopic observable is supposed to have definite value [5,27]. In terms of index p, $|E_0\rangle$ has $p = 2$ and therefore contains superposition of macroscopically distinct states. In other words, the symmetry of the ground state is "obscured" by such large quantum fluctuation. This effect is often called "obscured symmetry breaking" [28].

However, when the exact ground state has such an anomalous property, it is often the case that the energy gap between the exact ground state and the low-lying eigenstates decays very fast as $N \to \infty$ [28,29]. Then, it is physically allowed to take a linear combination of the exact ground state and some of the low-lying eigenstates to form an approximate ground state $|E_0'\rangle$. Thus constructed $|E_0'\rangle$ are believed to break the symmetry and be "ergodic" in the sense that

$$\lim_{N\to\infty} \frac{\sqrt{\langle E_0'|\hat{A}^2|E_0'\rangle - \langle E_0'|\hat{A}|E_0'\rangle^2}}{N} = 0$$

for any macroscopic observable $\hat{A}$ [28,29]. In terms of index p, this means that $|E_0'\rangle$ has no superposition of macroscopically distinct states since $p < 2$.

Indeed, Horsch and von der Linden [29] introduced the trial state $\hat{O}|E_0\rangle$, which approximates the first excited state, and showed that the energy gap between the exact ground state and the trial state decays as fast as or faster than $1/N$. They also showed that a linear combination of the exact ground state and the trial state exhibits the desired Z_2 symmetry breaking.

Koma and Tasaki [28] rigorously showed that the linear combination $|\psi\rangle$ of the exact ground state and the trial state is also the ergodic state in the sense that

$$\frac{1}{N^2}[\langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2] = 0 \tag{9}$$

for any translationally invariant $\hat{A}$.

As is pointed out in Ref. [10], the low-temperature coherence properties of quantum memory are closely related to the obscured symmetry breaking in statistical physics. Indeed, our results in this article are considered an improvement of the previous results, since Hamiltonian Eq. (2) is equivalent to that of the transverse Ising model [35] in condensed matter physics. (For example, this model was used to describe the order-disorder transition in some double-well ferroelectric systems, such as potassium dihydrogen phosphate (KH_2PO_4) crystals [36].)

First, by numerical calculations, we have explicitly shown how the macroscopic coherence properties of the exact ground state changes when the transverse magnetic field is changed (Fig. 1). We have also visualized the structure of the macroscopic superposition in the exact ground state (Fig. 2) and seen that the exact ground state is approximately an equal weight superposition of two symmetry-broken phases.

Second, we have shown that only $\hat{M}_z$ fluctuates macroscopically in the $\lambda < 1$ phase (Fig. 3). Since the ground state is symmetric, this means that the second moment of $\hat{M}_z$ is of $O(N^2)$ in that phase. In terms of statistical physics, this means that $\hat{M}_z$ is the unique order operator in this phase.

Third, we have shown that the equal-weight superposition of the exact ground state and the first excited state has $p = 1$ (Fig. 5). This means that the superposition of the exact ground state and the first excited state is ergodic. Although similar results have been obtained, the advantages of our results are (i) instead of the trial state, we have directly used the first excited state to show the ergodicity, (ii) our result that the fluctuation is of $O(N)$ is stronger than Eq. (9), and (iii) we have shown the ergodicity for any additive operator, whereas only translationally invariant additive operators are considered in the previous studies. (A disadvantage of our results is that they are less general since

we have used numerical calculations.) In summary, to consider index p for the ground states of many-body Hamiltonians in condensed matter physics is very useful for the study of the foundation of statistical physics.

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APPENDIX

A. Proof of Eq. (7)

Note that $|\Psi\rangle$ is a simultaneous eigenvector of $\hat{M}_x$, $\hat{M}_y$, and $\hat{M}_z$ corresponding to the eigenvalue 0, since a singlet is a simultaneous eigenvector of x, y, and z components of the total magnetization corresponding to the eigenvalue 0. Since each of the states

$$\hat{\sigma}_x(l)|\Psi\rangle,\quad \hat{\sigma}_y(l)|\Psi\rangle,\quad \hat{\sigma}_x(l)\hat{\sigma}_x(l')|\Psi\rangle,\quad \hat{\sigma}_x(l)\hat{\sigma}_y(l')|\Psi\rangle,$$
$$\hat{\sigma}_x(l)\hat{\sigma}_z(l')|\Psi\rangle,\quad \hat{\sigma}_y(l)\hat{\sigma}_y(l')|\Psi\rangle,\quad \hat{\sigma}_y(l)\hat{\sigma}_z(l')|\Psi\rangle,$$

for $|l-l'|\geqslant 2$, has no component in the eigenspace of $\hat{M}_z$ corresponding to the eigenvalue $M_z=0$, they are orthogonal to $\langle\Psi|$. In the same way,

$$\hat{\sigma}_z(l)|\Psi\rangle,\quad \hat{\sigma}_z(l)\hat{\sigma}_z(l')|\Psi\rangle,$$

for $|l-l'|\geqslant 2$, are orthogonal to $\langle\Psi|$ since each of them has no component in the eigenspace of $\hat{M}_x$ corresponding to the eigenvalue $M_x=0$. Hence we have shown Eq. (7). ■

B. Proof of Eq. (8)

Before showing the equation, let us note some useful relations. First, the projection operator $\hat{t}_{2,3}$ "swaps" the entanglement

$$\hat{t}_{2,3}|\circ\circ\bullet\bullet\rangle=-\tfrac{1}{2}|\circ\bullet\bullet\circ\rangle,$$

where singlet pairs are schematically represented: sites represented by the circle of the same color make a singlet pair (see Fig. 8).

Second, by using this equation, we obtain

$$\begin{aligned}\langle\circ\bullet\bullet\circ|\circ\circ\bullet\bullet\rangle&=\langle\circ\bullet\bullet\circ|\hat{t}_{2,3}|\circ\circ\bullet\bullet\rangle\\&=-\tfrac{1}{2}\langle\circ\bullet\bullet\circ|\circ\bullet\bullet\circ\rangle\\&=-\tfrac{1}{2}\end{aligned}$$

and

$$\begin{aligned}\langle\circ\circ\bullet\bullet|\hat{t}_{2,3}|\circ\circ\bullet\bullet\rangle&=-\tfrac{1}{2}\langle\circ\circ\bullet\bullet|\circ\bullet\bullet\circ\rangle\\&=\tfrac{1}{4}.\end{aligned}$$

Third, by iterating the swap process,

$$|\mathrm{VB}_2\rangle=(-2)^{N/2-1}\prod_{l=1}^{N/2-1}\hat{t}_{2l,2l+1}|\mathrm{VB}_1\rangle,$$

and therefore

$$\langle\mathrm{VB}_2|\mathrm{VB}_2\rangle=(-2)^{N/2-1}\langle\mathrm{VB}_2|\prod_{l=1}^{N/2-1}\hat{t}_{2l,2l+1}|\mathrm{VB}_1\rangle,$$

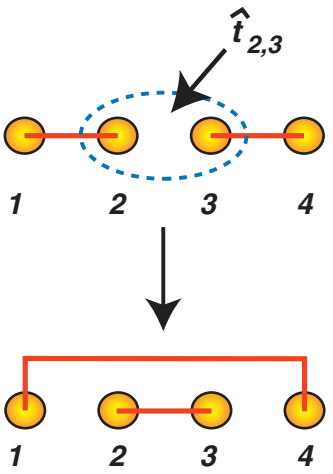

FIG. 8. (Color online) The projection operator $\hat{t}_{2,3}$ swaps the entanglement. Red (solid) lines represent singlet pairs.

which gives

$$\langle\mathrm{VB}_2|\mathrm{VB}_1\rangle=\left(-\tfrac{1}{2}\right)^{N/2-1}\simeq 0.$$

Let us define

$$|\phi_1\rangle\equiv\hat{T}|\mathrm{VB}_1\rangle+\frac{N}{2}|\mathrm{VB}_1\rangle=\sum_{l=\mathrm{even}}\hat{t}_{l,l+1}|\mathrm{VB}_1\rangle$$
$$|\phi_2\rangle\equiv\hat{T}|\mathrm{VB}_2\rangle-\frac{N}{2}|\mathrm{VB}_2\rangle=-\sum_{l=\mathrm{odd}}\hat{t}_{l,l+1}|\mathrm{VB}_2\rangle.$$

Then,

$$\langle\mathrm{VB}_i|\phi_j\rangle\simeq\delta_{i,j}(-1)^{i+1}\frac{N}{8},$$

which gives

$$\langle\mathrm{VB}_i|\hat{T}|\mathrm{VB}_j\rangle\simeq\delta_{i,j}(-1)^{i}\frac{3N}{8}.$$

Therefore,

$$\langle\Psi|\hat{T}|\Psi\rangle\simeq 0.$$

Also, we can show

$$\langle\phi_i|\phi_j\rangle\simeq\delta_{i,j}\left(\frac{N^2}{64}+\frac{3N}{32}\right),$$

which gives

$$\langle\mathrm{VB}_i|\hat{T}^2|\mathrm{VB}_j\rangle\simeq\delta_{i,j}\left(\frac{9N^2}{64}+\frac{3N}{32}\right).$$

Therefore,

$$\langle\Psi|\hat{T}^2|\Psi\rangle=O(N^2).$$

Hence we have shown Eq. (8). ■

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