

Entanglement is not a critical resource for quantum metrology

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(Received 20 May 2009; revised manuscript received 15 December 2009; published 11 February 2010)

We investigate high-precision measurements beyond the standard quantum limit, utilizing nonclassical states. Although entanglement was considered a resource for achieving the Heisenberg limit in measurements, we show that any advantage expected from using entanglement is dependent on the measurement in question. We investigate several measurement scenarios and illustrate the role of entanglement as a resource for quantum high-precision measurement. In particular, we demonstrate that there is a regime wherein entanglement not only does not help, but prevents the achievement of the fundamental limit.

DOI: [10.1103/PhysRevA.81.022108](https://doi.org/10.1103/PhysRevA.81.022108)

PACS number(s): 03.65.Ta, 03.65.Wj, 03.67.–a, 42.50.Dv

I. INTRODUCTION

Quantum metrology is a field that investigates characteristic fundamental properties of measurements under the laws of quantum mechanics [1]. Quantum high-precision measurements, in particular, focus on realizing more precise measurements, that is, measurements beyond their classical counterparts. They have the ultimate goal of achieving the information theoretical bounds allowed by the laws of quantum mechanics. The typical and classic examples are measurements utilizing squeezed states. It is well known that squeezed states exhibit sensitivities beyond the standard quantum limit [2] both theoretically and experimentally. Squeezed states can approach the Heisenberg limit with infinite squeezing, which indicates the infinite amount of energy necessary to achieve the fundamental quantum limit.

In recent years, quantum-information science has rapidly developed and quantum measurement schemes, as information gathering processes, were widely investigated from a quantum-information point of view. The development of experimental quantum-information processing (QIP) [3–9] ignited a strong motivation to realize quantum measurements beyond the squeezed-state regime and to test the ultimate limits for measurements. The fundamental technologies necessary for QIP led us to this new stage of quantum metrology research where entanglement plays a natural role.

Now that nonclassical states, typically entangled states, are available in laboratories, a number of protocols to implement quantum metrology were considered and several of them were actually implemented [3–9]. Given that entanglement has a central role in most QIP protocols, naturally, QIP developments invoked strong interest in whether the theory for nonclassicality including entanglement can contribute to investigations of quantum high sensitivities. Entangled photon number states were shown to help phase estimation and lithographic resolution beat the standard quantum limit [10,11]. Also, more general entangled light states were shown to greatly improve photodetector calibration [12], quantum imaging [13], and lidar [14]. In all of these successful protocols, entanglement has indeed played the central role

in achieving super-sensitivity below the shot-noise limit [1]. These results cultivated a belief that entanglement is essential to the mechanism to allow us to access the Heisenberg limit with a finite amount of energy.

The main aim of this article is to challenge this belief. Entanglement is a key resource for a large number of QIP applications and there are a number of indications that entanglement is the necessary element for achieving high-precision measurements [15,16]. However, there are somewhat different views about the role of entanglement in quantum mechanical and quantum computational processes [17,18]. For instance, the Gottesman-Knill theorem was the first to show that entanglement alone is not sufficient for universal quantum computation [19,20]. Although entanglement is necessary for scalable universal quantum computation [21], scalability might not necessarily be applicable to quantum metrology.

In this article, we demonstrate how the fundamental limits on sensitivity can be achieved under the principles of quantum mechanics. The scaling of the sensitivities approaching the Heisenberg limit will be investigated using several characteristic quantum states. We employ several states and Hamiltonians to illustrate the characteristics of the fundamental measurement limits, however, the discussion applies to a broad class of states and the main results hold, in general.

II. THE MODEL

To begin, we consider a physical parameter θ to be estimated, which is associated with the evolution represented by the unitary transformation

$$U(\theta) = e^{i\theta\hat{H}/\hbar}, \quad (1)$$

where \hat{H} is the effective Hamiltonian of the system. We assume $\hat{H} = \sum_{i=1}^N \hat{H}_i = \hbar \sum_{i=1}^N \hat{n}_i^k$, where N is the number of subsystems involved, $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ is the number operator for subsystem i , and k is the order parameter of the nonlinearity. $k = 1$ corresponds to a linear phase shift on the state of the system and $k = 2$ gives the well known nonlinear Kerr phase shift [22]. We can also consider nonintegers for k , such as $k = 1/2$, which represents a square-root-type phase shift useful for the preparation of Schrödinger-cat-like states [23].

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Equation (1) can be written in the form

$$U(\theta) = U_1(\theta) \otimes U_2(\theta) \otimes \cdots \otimes U_N(\theta), \quad (2)$$

as each \hat{H}_i acts only on its own subspace. We assume this nonentangling operation and the no-cost resource on the evolution so as to first demonstrate the role of entanglement in high-precision measurements. However, more generally, the Hamiltonian can be an entangling interaction and the unitary transformation might be realized with a resource cost. We will later extend our discussion beyond these assumptions.

To evaluate the parameter θ , we first prepare a probe in a quantum state $|\psi\rangle$. Due to the assumption of no-cost resource on the unitary transformation, we do not need to consider strategies for efficient use of the physical resource to generate the unitary transformation, the state simply evolves via Eq. (1) to a state $|\psi'\rangle = U(\theta)|\psi\rangle$. Using $|\psi'\rangle$ we can estimate the smallest θ that can be detected or resolved with the following Cramer-Rao uncertainty relation [24,25]

$$\left\langle \left(\frac{\theta_{\text{est}}}{\partial \langle \theta_{\text{est}} \rangle_{\theta} / \partial \theta} - \theta \right)^2 \right\rangle \geq \frac{\partial \theta^2}{4M(1-d)}, \quad (3)$$

where $d = |\langle \psi | \psi' \rangle|^2$ is the distinguishability and M the number of classical measurements employed. If $d \sim 0$ (i.e., the states $|\psi\rangle$ and $|\psi'\rangle$ are effectively orthogonal) then the inequality is most easily satisfied and so $d \sim 0$ will be our metric here.

For illustrative purposes we employ several initial states $|\psi\rangle$ to highlight the effect of entanglement on the distinguishability d . The initial states undergo the unitary transformation given by $U(\theta)$, where θ is the parameter that indicates the sensitivity. The first choice of initial states are several, distinct, nonclassical states

$$|\psi\rangle_C = \frac{1}{\sqrt{2}}(|0\rangle + |\sqrt{N}\alpha\rangle), \quad (4)$$

$$|\psi\rangle_E = \frac{1}{\sqrt{2}}(|0\rangle_1 \cdots |0\rangle_N + |\alpha\rangle_1 \cdots |\alpha\rangle_N), \quad (5)$$

$$|\psi\rangle_S = \frac{1}{\sqrt{2^N}}(|0\rangle + |\alpha\rangle)^{\otimes N}, \quad (6)$$

where $\alpha^2 \gg 1$ for the state normalization. These three states typically exhibit different superposition and entanglement properties, nonetheless each of these states has the same mean photon number $\bar{n} = N|\alpha|^2/2$, which gives the same energy constraint on these states [26,27].

Now, let us see the effect of Eq. (1) on the initial states in Eqs. (4) through (6). This is best seen through the calculation of the distinguishability for each state

$$d_C \sim \frac{1}{2} \{1 + \cos[\theta(N|\alpha|^2)^k]\}, \quad (7)$$

$$d_E \sim \frac{1}{2} \{1 + \cos[\theta N|\alpha|^{2k}]\}, \quad (8)$$

$$d_S \sim \frac{1}{2^N} \{1 + \cos[\theta|\alpha|^{2k}]\}^N, \quad (9)$$

where we assumed that $\theta^2|\alpha|^{2k} \ll 1$ [28,29]. By setting $d = \delta \ll 1$ we may determine the sensitivity for θ in each case. For instance, the distinguishability between $|\psi'\rangle_C$ and $|\psi\rangle_C$ gives

$$\theta_C \sim \frac{\cos^{-1}[2\delta - 1]}{(N|\alpha|^2)^k} \sim \frac{[(2l+1)\pi - 2\sqrt{\delta}]}{(N|\alpha|^2)^k}, \quad (10)$$

TABLE I. The minimal detectable θ_{\min} for the unitary transformation given by Eq. (1) with the initial states (4) through (6). The general k case is given, as well as the special cases $k = 1/2$, 1, and 2. These show that the entangled coherent state superposition is better in the region $0 < k < 1$ while the single mode coherent state superposition is better for $k > 1$, indicating entanglement makes the precision worse. For $k = 1$ (the linear phase shift case) there is no real advantage from entanglement.

k	$\theta_{\min C}$	$\theta_{\min E}$	$\theta_{\min S}$
$\hat{H} = \hat{n}^{1/2}$	$\frac{\pi}{\sqrt{2\bar{n}}}$	$\frac{\pi}{\sqrt{2N\bar{n}}}$	$\frac{\pi}{\sqrt{2\bar{n}}}$
$\hat{H} = \hat{n}$	$\frac{\pi}{2\bar{n}}$	$\frac{\pi}{2\bar{n}}$	$\frac{\pi\sqrt{N}}{2\bar{n}}$
$\hat{H} = \hat{n}^2$	$\frac{\pi}{4\bar{n}^2}$	$\frac{\pi N}{4\bar{n}^2}$	$\frac{\pi N^{3/2}}{4\bar{n}^2}$
$\hat{H} = \hat{n}^k$	$\frac{\pi}{(2\bar{n})^k}$	$\frac{\pi N^{k-1}}{(2\bar{n})^k}$	$\frac{\pi N^{k-1/2}}{(2\bar{n})^k}$

where l is a nonnegative integer. Setting $l = 0$ gives the first minimum, which corresponds to the highest precision the scheme can provide. Applying this line of reasoning to the other cases and recalling $\bar{n} = N|\alpha|^2/2$ we have

$$\theta_{\min C} \sim \frac{\pi - 2\sqrt{\delta}}{(2\bar{n})^k}, \quad (11)$$

$$\theta_{\min E} \sim \frac{(\pi - 2\sqrt{\delta})N^{k-1}}{(2\bar{n})^k}, \quad (12)$$

$$\theta_{\min S} \sim \frac{(\pi - 2\sqrt{\delta})N^{k-1/2}}{(2\bar{n})^k}. \quad (13)$$

III. RESOLUTION

We can now investigate the scaling for these three states with respect to the nonlinear phase shifts. Equations (11) through (13) are scaling as $1/\bar{n}^k$, as expected, but the precision limit for each scheme is critically dependent on the factors N and k in the coefficient. The precision limits comparing Eqs. (12) and (13) definitely indicate that entanglement always helps. However, it depends on k as to whether Eq. (11) is better or worse than Eq. (12), that is, whether entanglement helps or not. For the case $0 < k < 1$, a large N is optimal. We obtain an improvement by dividing our given mean energy resource over distributed modes using entanglement. The precision given by $\theta_{\min E}$ is the optimal choice and achieves the Cramer-Rao bound. This agrees with the previous results [30,31].

In contrast to the scaling for $0 < k < 1$, the regime for $k > 1$ typically shows that large N actually reduces the precision. In this regime, entanglement does not help the precision and is a major disadvantage. The disadvantage from the energy distribution over multiple modes exceeds the precision gain provided by entanglement and so it is better to consider a single mode superposition of coherent states (4). Such a strategy allows one to also reach the Cramer-Rao bound. The $k = 1$ case, in between these two regimes, is where the gain and loss are balanced and there is no difference in $\theta_{\min C}$ and $\theta_{\min E}$. We summarize the scaling properties in Table I for the $\delta = 0$ situation.

As shown in Table I, the scaling properties are defined by the energy distribution and the entanglement in the states,

so our results can be easily extended to different types or superpositions of coherent states. We can, for instance, consider the more general state

$$|\phi\rangle = \sum_{i_1, i_2, \dots, i_N} c_{i_1, i_2, \dots, i_N} |\alpha_{i_1}\rangle |\alpha_{i_2}\rangle \dots |\alpha_{i_N}\rangle. \quad (14)$$

This state does not require the even distribution of energy over the modes and there is a large class of states that can be categorized by the same scaling. In fact it is straightforward to show, once the total mean energy is fixed, that the optimal $|\phi\rangle$ corresponds to Eq. (5) for $0 < k < 1$ and Eq. (4) for $k > 1$. We also do not need to restrict our attention to superpositions of coherent states. We can employ superpositions of number states instead, typically giving the equivalent states $(1/\sqrt{2})(|0\rangle + |N\rangle)$ and $(1/\sqrt{2})(|0\rangle_1 \dots |0\rangle_N + |1\rangle_1 \dots |1\rangle_N)$, where the same scaling properties can be obtained. We note that the same generalization holds with the number state as well as the coherent states (see the Appendix).

The $k > 1$ situation is interesting because it places entangled resources at a disadvantage. We can consider possibilities where entanglement regains the capability to achieve the sensitivity limit $\theta_{\min C}$ by using an entangling unitary transformation. Changing the Hamiltonian in Eq. (1) from $\hat{H} = \hbar \sum_{i=1}^N \hat{n}_i^k$ to $\hat{H} = \hbar (\sum_{i=1}^N \hat{n}_i)^k$, an N -body interaction [15,16], the entangled state case (5) gives the sensitivity $\theta_{\min} = \pi/(2\bar{n})^k$, the same as the one given in Eq. (4). The entangled state merely regains the same sensitivity as the nonentangling state.

Next we need to consider the scenario where there is no *a priori* knowledge about the magnitude of the parameter θ to estimate. We can evaluate the scalability with respect to the cost associated to achieve the sensitivity and compare it to the scaling on the sensitivity limit given by Eqs. (11) through (13). In that calculation, θ_{\min} requires θ to be restricted to the range $0 < \theta \leq \pi/\bar{n}$. This requires *a priori* knowledge that we may not have. However, a simple solution exists. We can start with $0 < \theta \leq 2\pi$ and use some of our resources to iteratively refine our θ region. This requires, at most, doubling our total mean photon number and $\log_2(\bar{n}) + 1$ steps. For instance, with our superposition of coherent states, the total mean photon number required becomes $\bar{n}_{\text{tot}} = 2\bar{n} - 1$ where \bar{n} is the mean photon number required for the measurement in the range $0 < \theta \leq \pi/\bar{n}$. This means we can easily write θ_{\min} in Eqs. (11) through (13) in terms of \bar{n}_{tot} and thus observe the scaling in terms of the total mean photon number.

So far we focused our attention on the sensitivity when the unitary transformation acts equally on the modes. In some cases, such as gravitational wave detection, this is the case. However, there are applications where the total resource to generate the unitary transformation $U(\theta)$ can be limited. In these applications, we can potentially be using N times the unitary resources of the single mode case. Here we extend our analysis to estimate the precision limit in such a scenario. Consider the situation where the total amount of unitary resource is fixed to $U(\theta)$, but is allowed to be split into smaller pieces, with each piece used only once. The minimum θ_{\min} for the three states of interest becomes

$$\theta_{\min C} = \frac{\pi}{(2\bar{n})^k}, \quad (15)$$

$$\theta_{\min E} = \frac{\pi N^k}{(2\bar{n})^k}, \quad (16)$$

$$\theta_{\min S} = \frac{\pi N^{k+\frac{1}{2}}}{(2\bar{n})^k}. \quad (17)$$

This result clearly shows that the single mode coherent state superposition given by Eq. (4) always achieves the best sensitivity limit, independent of the value of k . Naturally, entanglement is always a disadvantage. With constrained resources, $\theta_{\min E}$ and $\theta_{\min S}$ are worse than before by a factor of N .

As we have mentioned, these arguments can be readily generalized. In particular, for application purposes, various superpositions of number states may be of interest (see the Appendix). For the use of number states of the type $(1/\sqrt{2})(|0\rangle + |N\rangle)$ and $(1/\sqrt{2})(|0\rangle_1 \dots |0\rangle_N + |1\rangle_1 \dots |1\rangle_N)$ in the case where the unitary operation acts evenly for all modes, we have $\theta_{\min} = \pi/(2\bar{n})^k$ for the qudit superposition and $\theta_{\min} = \pi/(2\bar{n})$ for the entangled state (here $\bar{n} = N/2$). In the constrained resource case, we have $\theta_{\min} = \pi/(2\bar{n})^{k-1}$ and $\theta_{\min} = \pi$, respectively, showing that the previous argument exactly follows in the same way. In fact, these states were generated in optics from microwave to optical frequencies. In the case where the direct creation of $|0\rangle + |N\rangle$ is difficult, we alternatively can use a “ $N00N$ ” state of the form $|0\rangle_1 |N\rangle_2 + |N\rangle_1 |0\rangle_2$ [32]. Although the “ $N00N$ ” state is entangled, we are not making use of this property during the metrology part of the operation. Only one of the two modes interacts with the unitary operation. The other mode is passive and does not take an active part. This second mode can be measured in a basis that gives no information about the photon number components of that mode, in effect transforming the “ $N00N$ ” state to $|0\rangle_1 + |N\rangle_1$. Alternatively, it can be left there.

IV. CONCLUSION

In conclusion, we analyze the fundamental limitations on high-precision measurements utilizing quantum nature. It is clear that nonclassical states must be used to achieve the Heisenberg limit, even with finite energy. However, entanglement is not a necessary resource for quantum high-precision measurements. Our analysis shows that there are scenarios where entanglement can even be a disadvantage, preventing us from achieving the fundamental sensitivity limit (that is the Heisenberg limit). This result is rather counterintuitive. Nevertheless, when we carefully consider the distribution of resources for sensitivity, we observe that entanglement can be used to recover the disadvantage caused from the energy distribution over multiple modes to some degree. Yet the loss in sensitivity cannot always be recovered. This is not to say entanglement is not useful, it may just not be necessary during the quantum metrology process, but can be used in generating the initial quantum state resource. Finally, our considerations so far have been highly ideal in the sense that we have not considered loss and decoherence effects. Such effects might prevent our schemes from achieving the ultimate sensitivity shown in this article and additional mechanisms may be required to compensate for it.

ACKNOWLEDGMENTS

We thank Gerard Milburn, Samuel Braunstein, and E. C. G. Sudarshan for valuable discussions. This work was supported in part by MEXT, HP, and the EU project QAP.

APPENDIX: NUMBER STATES

Let us consider our superpositions of number states in slightly more detail. For

$$|\psi\rangle_C = \frac{1}{\sqrt{2}}(|0\rangle + |N\rangle), \quad (\text{A1})$$

$$|\psi\rangle_E = \frac{1}{\sqrt{2}}(|0\rangle_1 \cdots |0\rangle_N + |1\rangle_1 \cdots |1\rangle_N), \quad (\text{A2})$$

$$|\psi\rangle_S = \frac{1}{\sqrt{2^N}}(|0\rangle + |1\rangle)^{\otimes N}, \quad (\text{A3})$$

we can apply our transformation (1) and determine the distinguishability d . This gives

$$d_C = \frac{1}{2}(1 + \cos[\theta N^k]), \quad (\text{A4})$$

$$d_E = \frac{1}{2}(1 + \cos[\theta N]), \quad (\text{A5})$$

$$d_S = \frac{1}{2^N}(1 + \cos[\theta])^N, \quad (\text{A6})$$

and it is straightforward to show

$$\theta_C \sim \frac{\pi}{(2\bar{n})^k}, \quad (\text{A7})$$

$$\theta_E \sim \frac{\pi}{2\bar{n}}, \quad (\text{A8})$$

$$\theta_S \sim \frac{\pi}{\sqrt{2\bar{n}}}, \quad (\text{A9})$$

where for Eq. (A9) there was an additional $1/\sqrt{N}$ improvement from classical statistics. These results are summarized in Table II for various k .

From Table II we can already see that, for the phase shift case ($k = 1$), both N -mode entangled number states given by

TABLE II. The minimal detectable θ_{\min} for the unitary transformation given by Eq. (1) with the initial states (A1) through (A3). The general k case is given, as well as the special cases $k = 1/2, 1$, and 2. These show that the N -mode entangled number state is better in the region $0 < k < 1$ while the single mode number state is better for $k > 1$, indicating entanglement makes the precision worse. For $k = 1$ (the linear phase shift case) there is no real advantage from entanglement.

k	$\theta_{\min C}$	$\theta_{\min E}$	$\theta_{\min S}$
$\hat{H} = \hat{n}^{\frac{1}{2}}$	$\frac{\pi}{\sqrt{2\bar{n}}}$	$\frac{\pi}{2\bar{n}}$	$\frac{\pi}{\sqrt{2\bar{n}}}$
$\hat{H} = \hat{n}$	$\frac{\pi}{2\bar{n}}$	$\frac{\pi}{2\bar{n}}$	$\frac{\pi}{\sqrt{2\bar{n}}}$
$\hat{H} = \hat{n}^2$	$\frac{\pi}{(2\bar{n})^2}$	$\frac{\pi}{2\bar{n}}$	$\frac{\pi}{\sqrt{2\bar{n}}}$
$\hat{H} = \hat{n}^k$	$\frac{\pi}{(2\bar{n})^k}$	$\frac{\pi}{2\bar{n}}$	$\frac{\pi}{\sqrt{2\bar{n}}}$

Eq. (A2) and single mode number states given by Eq. (A1) are good to use. However, when we look at k values greater than 1, states of the form (A1) offer better resolution, at higher mean photon numbers than states of the form (A2). More interestingly, when $k < 1$, the situation reverses itself. States of the form (A2) become the better choice for high-precision measurements.

Moving to the situation where we constrain both the mean photon number of the system as well as the total resources $U(\theta)$ we find

$$\theta_{\min C}' = \frac{\pi}{(2\bar{n})^{k-1}}, \quad (\text{A10})$$

$$\theta_{\min E}' = \pi, \quad (\text{A11})$$

$$\theta_{\min S}' = \pi\sqrt{(2\bar{n})}. \quad (\text{A12})$$

This clearly shows that Eq. (A1) offers the best resolution for $k > 0$. Furthermore, both the unconstrained and constrained cases show the same scaling as we saw previously for coherent states.

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