

Angular performance measure for tighter uncertainty relations

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The uncertainty principle places a fundamental limit on the accuracy with which we can measure conjugate quantities. However, the fluctuations of these variables can be assessed in terms of different estimators. We propose an angular performance that allows for tighter uncertainty relations for angle and angular momentum. The differences with previous bounds can be significant for particular states and indeed may be amenable to experimental measurement with the present technology.

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Apart from interpretational issues, the main goal of quantum mechanics is to make predictions on the outcomes of experiments. In fact, in many modern setups one is led to measurements that simultaneously estimate two noncommuting variables. The precision with which they are jointly estimated obeys a fundamental constraint dictated by the uncertainty principle [1].

The archetypal example is the case of continuous variables, such as position and linear momentum of a single particle. The standard formalization of the uncertainty principle is presented in terms of the associated variances [defined as $(\Delta A)^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$], and it reads [2] (with $\hbar = 1$ throughout)

$$(\Delta x)^2 (\Delta p)^2 \geq \frac{1}{4}. \quad (1)$$

These variances are a measure of the width of the corresponding probability distributions in the quantum state. However, it has long been argued that some experiments do not measure variances and encouraging reformulations of Eq. (1) have been proposed in terms of other resolution measures [3,4]. In other words, one can assign different measures of inaccuracy (each one with its own pros and cons) to a particular measurement and this proves crucial to properly set its ultimate resolution limits. The price one has to pay is that establishing an uncertainty principle in terms of these measures can turn out to be very intricate [5–9]. The situation is even more ambiguous for magnitudes that cannot be measured, but must be only inferred, as it happens with, e.g., entanglement [10].

Angular variables are also riddled with the same kind of problems, but aggravated by the peculiarities of their periodic character [11–14]. Though this is an old question, it experiences periodic revivals in connection with some hot topics. Nowadays, a renewed interest in these features has been triggered by the treatment of rotating Bose-Einstein condensates [15,16] and the quantum optics of vortex beams [17]. It is worth remarking that we have at hand very simple experimental schemes to test in practice ideal angle concepts.

There is agreement in using the variance $(\Delta L)^2$ to characterize fluctuations in angular momentum (although, since this variable is unbounded, the variance may fail in some instances to provide a satisfactory expression for the uncertainty principle [18]). In contrast, there is no wide consensus concerning the proper assessment of the conjugate angle fluctuations. Periodicity may lead to serious troubles

when using variance, since the powers of the angle are not periodic functions, so that their mean values depend on the origin chosen. There are several proposals that avoid these problems, such as the Süßmann measure [19–21], circular variance [22–26], entropies [27–31], reciprocal peak height [32–34], origin-optimized angle variance [35,36], and other nonstandard quantities [37,38]. In short, for periodic variables there are a lot of candidates for assessing fluctuations, each one surely with its virtues, but no undisputed champion.

As commented before, if we decide to choose, e.g., the circular variance (which is computed as the standard one, but using the moments of the complex exponential of the angle rather than the angle itself and is the simplest natural choice from a pure statistical viewpoint [39]), the resulting uncertainty relation is rather involved and cannot be saturated, except in very trivial cases [40,41].

All these difficulties motivate this Brief Report. We shall seek for an angular performance measure that, apart from properly quantifying angle fluctuations, provides simple and feasible bounds for the conjugate variable.

To be as self-contained as possible, we first introduce some basic notions for the problem at hand. We are concerned (assuming cylindrical symmetry) with the planar rotations by an angle ϕ generated by the angular momentum along the z axis, which for simplicity will be denoted henceforth as \hat{L} . Classically, a point particle is necessarily located at a single value of the periodic angular coordinate ϕ , defined within a chosen window. The corresponding quantum wave function, however, is an object extended around the unit circle and so can be directly affected by the nontrivial topology.

One may be tempted to think that angle should stand in the same relationship to angular momentum as ordinary position stands to linear momentum. This would prompt to interpret the angle operator as multiplication by ϕ while \hat{L} is the differential operator $\hat{L} = -i\partial_\phi$. However, the use of this operator may entail many pitfalls for the unwary: in particular, single-valuedness restricts the Hilbert space to the subspace of 2π -periodic functions, which, among other things, rules out the angle coordinate as a *bona fide* observable [42,43].

Many of these difficulties can be avoided by simply selecting angular coordinates that are both periodic and continuous instead. A single such quantity cannot uniquely specify a point on the circle because periodicity implies extrema, which excludes a one-to-one correspondence and

hence is incompatible with uniqueness. Perhaps the simplest choice [44,45] is to adopt two angular coordinates, such as, e.g., cosine and sine. In classical mechanics this is indeed a good definition, while in quantum mechanics one would have to show that these variables, we shall denote by \hat{C} and \hat{S} to make no further assumptions about the angle itself, form a complete set of commuting operators. One can concisely condense all this information using the complex exponential of the angle $\hat{E} = \hat{C} - i\hat{S}$, which satisfies the commutation relation

$$[\hat{E}, \hat{L}] = \hat{E}. \quad (2)$$

In mathematical terms, this defines the Lie algebra of the two-dimensional Euclidean group E(2). Interestingly enough, E(2) is the canonical symmetry of the cylinder, which is the phase space for our system.

The action of \hat{E} on the angular momentum basis $|\ell\rangle$ is $\hat{E}|\ell\rangle = |\ell - 1\rangle$, and it possesses then a simple implementation by means of phase mask removing a unit charge from a vortex state [40]. Since the integer ℓ runs from $-\infty$ to $+\infty$, \hat{E} is a unitary operator whose eigenvectors

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} e^{-i\ell\phi} |\ell\rangle \quad (3)$$

describe states with well-defined angle. Although the proposal that this operator represents the angle conflicts with the orthodox view of describing observables by Hermitian operators, the option for \hat{E} is actually very natural. Note that one could expect a Fourier relationship between angle and angular momentum. In this context, this can be expressed as

$$e^{-i\phi'\hat{L}}|\phi\rangle = |\phi + \phi'\rangle, \quad (4)$$

which can be easily verified by using the explicit form in Eq. (3).

Let us turn to the corresponding uncertainty relations. The Robertson inequality [46,47] (which remains valid for unitary operators) can be applied to obtain

$$(\Delta L)^2 \geq \frac{1}{4} \frac{[1 - (\Delta E)^2]}{(\Delta E)^2}, \quad (5)$$

where we have rearranged terms to facilitate comparison with the next steps in our analysis. Here we have used the natural extension of variance for unitary operators [22]

$$(\Delta E)^2 = \langle \hat{E}^\dagger \hat{E} \rangle - \langle \hat{E}^\dagger \rangle \langle \hat{E} \rangle = 1 - |\langle \hat{E} \rangle|^2, \quad (6)$$

with exactly agrees with the circular variance [39]. The form (5) has been advocated by many authors. However, although correct, it does not provide the tightest lower bound and equality cannot be attained except for some trivial states [38].

To face this disadvantage, let us first recast Eq. (2) in terms of the corresponding Hermitian components

$$[\hat{C}, \hat{L}] = -i\hat{S}, \quad [\hat{S}, \hat{L}] = i\hat{C}, \quad (7)$$

while $[\hat{C}, \hat{S}] = 0$. Moreover, for reasons that will be apparent soon, we look at their rotated versions

$$\hat{C}_\alpha = \hat{C} \cos \alpha - \hat{S} \sin \alpha, \quad \hat{S}_\alpha = \hat{S} \cos \alpha + \hat{C} \sin \alpha. \quad (8)$$

This means that we allow the reference frame in which we compute the trigonometric functions to be rotated by an angle α . One can check that they satisfy a commutation relation identical to Eq. (7). Therefore, the associated uncertainty relations are

$$(\Delta S_\alpha)^2 (\Delta L)^2 \geq \frac{1}{4} |\langle \hat{C}_\alpha \rangle|^2, \quad (\Delta C_\alpha)^2 (\Delta L)^2 \geq \frac{1}{4} |\langle \hat{S}_\alpha \rangle|^2. \quad (9)$$

Since Eqs. (9) are fully equivalent to Eq. (5), they cannot be saturated simultaneously. In fact, there are further unfavorable aspects of them that have been reviewed in Ref. [18].

A common way of going on is to look for intelligent states minimizing, e.g., the first one of these equations. Although this can be seen as dealing only with ‘‘half’’ the uncertainty principle, the resulting states are often referred to as circular squeezed states [38] and exhibit amazing properties. They are defined by

$$(\hat{L} - i\kappa \hat{C}_\alpha)|\Psi\rangle = \lambda|\Psi\rangle, \quad (10)$$

where κ and λ are real parameters. Using the angle representation, this extremal equation reads as

$$-i \frac{d}{d\phi} \Psi(\phi) = [\lambda + i\kappa \cos(\phi + \alpha)] \Psi(\phi), \quad (11)$$

whose integration yields the normalized solution

$$\Psi(\phi) = \frac{1}{\sqrt{2\pi I_0(2\kappa)}} \exp[i\lambda\phi - \kappa \sin(\phi + \alpha)], \quad (12)$$

I_0 being the modified Bessel function of order 0. These are called von Mises states, since the associated probability distribution is precisely the von Mises, a very close analog of the Gaussian distribution on the circle [41]. The meaning of the parameters is clear: λ is the mean value of the angular momentum, whereas κ determines the angular spread.

Next, we observe that the associated uncertainty relation in Eq. (9) can be cast in the form

$$(\Delta L)^2 \geq U^2 \equiv \frac{1}{4} \max_\alpha \frac{|\langle \hat{C}_\alpha \rangle|^2}{(\Delta S_\alpha)^2}. \quad (13)$$

Let us introduce the following vectors:

$$\mathbf{x} = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \langle \hat{C} \rangle \\ \langle \hat{S} \rangle \end{pmatrix}, \quad (14)$$

and the covariance matrix

$$\mathbf{\Gamma} = \begin{pmatrix} (\Delta S)^2 & \Delta(SC) \\ \Delta(CS) & (\Delta C)^2 \end{pmatrix}, \quad (15)$$

where $\Delta(CS) = \langle \hat{C}\hat{S} \rangle - \langle \hat{C} \rangle \langle \hat{S} \rangle$. Then, U^2 can be written as

$$U^2 = \frac{1}{4} \max_{|\mathbf{x}|=1} \frac{(\mathbf{c}' \mathbf{x})^2}{\mathbf{x}' \mathbf{\Gamma} \mathbf{x}}, \quad (16)$$

and the superscript t denotes the transpose. The optimization over \mathbf{x} can be easily performed, getting

$$\frac{\mathbf{c}' \mathbf{x}}{\mathbf{x}' \mathbf{\Gamma} \mathbf{x}} \mathbf{c} - \left(\frac{\mathbf{c}' \mathbf{x}}{\mathbf{x}' \mathbf{\Gamma} \mathbf{x}} \right)^2 \mathbf{\Gamma} \mathbf{x} = 0, \quad (17)$$

whose solution gives the optimal value

$$U^2 = \frac{1}{4} \mathbf{c}^t \mathbf{\Gamma}^{-1} \mathbf{c}. \quad (18)$$

We stress that while the variances $(\Delta C)^2$ and $(\Delta S)^2$ are not invariant under rotations of the state around the z axis, this is not the case with U^2 , which constitutes a major advantage. In addition, U^2 combines the moments of \hat{C} and \hat{S} in a rather nontrivial way, since

$$\begin{aligned} |\langle \hat{E} \rangle|^2 &= 1 - \text{tr } \mathbf{\Gamma}, \\ |(\langle \Delta E \rangle)^2|^2 &= (\text{tr } \mathbf{\Gamma})^2 - 4 \det \mathbf{\Gamma}. \end{aligned} \quad (19)$$

The performance measure U^2 can be interpreted as a projection of the noise into the direction of the preferred angle, analogously to what was done for the ellipse representing a squeezed state in phase space [48].

Denoting by γ_- and γ_+ the smaller and larger eigenvalues of $\mathbf{\Gamma}$, a simple calculation allows us to estimate

$$U^2 \geq \frac{1}{4\gamma_+} |\mathbf{c}|^2 \equiv V^2 \geq \frac{1}{4} \frac{[1 - (\Delta E)^2]}{(\Delta E)^2}, \quad (20)$$

where we have introduced a resolution performance

$$V^2 = \frac{1}{4} \frac{2(1 - \text{tr } \mathbf{\Gamma})}{\text{tr } \mathbf{\Gamma} + \sqrt{(\text{tr } \mathbf{\Gamma})^2 - 4 \det \mathbf{\Gamma}}}, \quad (21)$$

that combines the two basic invariants of $\mathbf{\Gamma}$. Notice that V^2 is related to the covariance matrix (15) pretty much in the same way as the degree of polarization is linked to the polarization matrix. As we can see, it gives intermediate values between the bound in Eq. (5) (which cannot be attained for nontrivial states) and the one in Eq. (13) (which is saturated by all the von Mises states).

The second inequality in Eq. (20) is saturated only in trivial instances, such as, e.g., the eigenstates of \hat{L} [41]. A condition for the first inequality to be saturated is

$$\Delta(CS) = 0. \quad (22)$$

This holds if the associated probability distribution is symmetrical about some reference angle ϕ_0 , that is, $P(\phi_0 + \phi) = P(\phi_0 - \phi)$. In addition, $U^2 = V^2$ also implies the additional constraint

$$(\Delta S)^2 \geq (\Delta C)^2. \quad (23)$$

The Von Mises states are among those satisfying conditions (22) and (23). For the other cases, one has $U^2 > V^2$.

In consequence, the inequality

$$(\Delta L)^2 \geq V^2 \quad (24)$$

is always true, significantly improves the standard bound in Eq. (5), and the right-hand side is saturable. Given that a useful performance should be a simple expression of measurable quantities, we opt for using V^2 , which depends on the two basic invariants of the covariance matrix as one might expect, instead of U^2 . This latter quantity, in general, provides a slightly tighter bound. However, as we have shown above, for the majority of states of interest the two bounds (13) and (24) coincide, and both are saturated by the von Mises states. The difference between U^2 and V^2 is in most cases unimportant and more than compensated by the utility and feasibility of the proposed uncertainty relation Eq. (24).

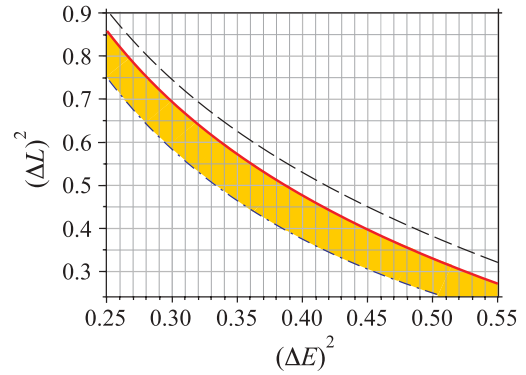


FIG. 1. (Color online) Plot of the different bounds for $(\Delta L)^2$ in terms of the dispersion ΔE for the state (25). From bottom to top, we show the equality in Eq. (5) (blue/dashed-dotted line), in Eq. (24) (red/thick line), and in Eq. (13) (black/dashed line).

In Fig. 1 we have condensed all this information for the state

$$\Psi(\phi) = \frac{1}{\sqrt{4\pi} I_0(2\kappa)} [\exp(\kappa \cos \phi) - i \exp(i\phi + \kappa \cos \phi)], \quad (25)$$

which corresponds to the superposition of two von Mises states with $\langle \hat{L} \rangle = 0$ and $\langle \hat{L} \rangle = 1$. This can be seen as an angular counterpart of a cat state, with a probability distribution

$$P(\phi) = (1 + \sin \phi) \exp(-2\kappa \cos \phi), \quad (26)$$

displaying a lack of symmetry. The proposed bound (24) constitutes a good improvement over the standard one (5), as we can see in the figure: all the area shaded corresponds to the values of $(\Delta L)^2$ that, for a given angular fluctuation $(\Delta E)^2$, are permitted by the standard uncertainty relation but not allowed according to our proposal. Obviously, the strongest bound in Fig. 1 is provided by relation (13). The family of states (26) was deliberately chosen so as to make the difference between V^2 and U^2 large; but even in that case the improvement of both Eqs. (24) and (13) over the standard bound is seen to be much larger than the difference between them.

Our arguments support the role of von Mises distribution on the circle as an analog of the Gaussian distribution on the line, at least as far as the uncertainty product is concerned. However, things may be not that simple with other aspects of quantum behavior. Indeed, our latest research indicates that the Wigner function of von Mises states is not positive (as it happens for Gaussian states in the line), since this property is reserved exclusively to the angular momentum eigenstates [49,50].

Finally, we observe that one could introduce a ladder operator [51]

$$\hat{X} = e^{-\hat{L}/2} \hat{E}. \quad (27)$$

Since this can be expressed also in terms of the nonunitary transformation $\hat{X} = e^{\hat{L}/2} \hat{E} e^{-\hat{L}/2}$ [52], the commutator $[\hat{X}, \hat{L}] = \hat{X}$ still remains valid. The construction of the accessible lower bound in this Brief Report can be thus repeated, provided the role of \hat{C} and \hat{S} is now taken by the

quadrature-like operators

$$\hat{Q} = \frac{1}{2}(\hat{X} + \hat{X}^\dagger), \quad \hat{P} = \frac{1}{2i}(\hat{X} - \hat{X}^\dagger). \quad (28)$$

Obviously, the (unnormalized) extremal states for these operators are given by the von Mises states, but transformed by $e^{-\hat{L}^2/2}$.

In summary, what we expect to have accomplished here is to present convincing arguments for the use of an angular

resolution measure that involves only invariant and measurable quantities, has no problem with periodicity, and gives an improved feasible criterion to assess minimal angle fluctuations.

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