Geometry-induced potential on a two-dimensional section of a wormhole: Catenoid

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We show that a two-dimensional wormhole geometry is equivalent to a catenoid, a minimal surface. We then obtain the curvature-induced geometric potential and show that the ground state with zero energy corresponds to a reflectionless potential. By introducing an appropriate coordinate system we also obtain bound states for different angular momentum channels. Our findings can be realized in suitably bent bilayer graphene sheets with a neck, in a honeycomb lattice with an array of dislocations, or in nanoscale waveguides in the shape of a catenoid.

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I. INTRODUCTION

Quantum mechanics in flat two-dimensional space gives unusual results such as the quantum anticentrifugal force for waves with zero angular momentum [1,2]. Thus, it would be insightful to explore quantum mechanics in curved two-dimensional space. Of special interest are the minimal surfaces (i.e., with zero mean curvature) which play an important role in physics. Besides the plane, the two other examples include the helicoid and the catenoid. An interesting question in the context of the three-dimensional wormhole geometry [3,4] in cosmology is whether information can propagate across the wormhole. We study the analog of this problem in two dimensions and first show that the two-dimensional wormhole is a catenoid. We then obtain the curvature-induced quantum potential [5]. The latter is an attractive geometric potential, $V_G(q_1, q_2) = -(\hbar^2/8m_0)(\kappa_1 - \kappa_2)^2$, where κ_1, κ_2 denote the two position-dependent principal curvatures of the surface, (q_1, q_2) are the surface coordinates, and m_0 is the mass of the particle on the surface.

A two-dimensional wormhole geometry can conceivably be realized in a bilayer of honeycomb lattices with radially arranged dislocations or in bilayer graphene [6], where the curvature-induced suppression of local Fermi energy can lead to the control of local electronic properties. In the next section we demonstrate the equivalence of a two-dimensional wormhole and a catenoid. In Sec. III we obtain the effective curvature induced potential. In Sec. IV we introduce a suitable coordinate system to study the bound states of the resulting Schrödinger equation on the catenoid. We have previously studied bound states on the other minimal surface, namely, the helicoid [7]. Note that a genus one helicoid also exists [8]. Finally, in Sec. V we summarize our main conclusions and comment on the anticentrifugal force.

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II. CATENOID AS A TWO-DIMENSIONAL SECTION OF A WORMHOLE

For a catenoid $x = R\cosh(z/R)\cos\phi$, $y = R\cosh(z/R)\sin\phi$, and $z = z$ with $\phi \in [0, 2\pi]$ (Fig. 1). Thus the local radius $\rho = R\cosh(z/R)$ and the metric is given by the following elements:

$$g_{\rho\rho} = \frac{\rho^2}{\rho^2 - R^2}, \quad g_{\phi\phi} = \rho^2. \tag{1}$$

We now show that a two-dimensional wormhole geometry is equivalent to a catenoid (Fig. 1). In cylindrical coordinates (z, r, ϕ), a two-dimensional section of a wormhole is given by

$$z(r) = \pm b_0 \ln\left[\frac{r}{b_0} + \sqrt{\frac{r^2}{b_0^2} - 1}\right], \tag{2}$$

with $l = \pm\sqrt{r^2 - b_0^2}$. For the three-dimensional wormhole the line element is given by the following expression [4]:

$$ds^2 = dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

where the coordinates belong to the following intervals, $l \in [-\infty, +\infty]$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi]$, and b_0 is the shape function of the wormhole [in general $b = b(l)$ and for $l = 0$, $b = b(0) = b_0 = $ const represents the radius of the throat of the wormhole]. Here l is a radial coordinate measuring proper radial distance; θ and ϕ are spherical polar coordinates. In this article we consider the case of $\theta = \pi/2$ which represents an equatorial section of a three-dimensional wormhole (at constant time). For this section we thus get the following line element:

$$ds^2 = dl^2 + (b_0^2 + l^2)d\phi^2, \tag{4}$$

which is precisely equivalent to the line element of a catenoid (since $l^2 = r^2 - b_0^2$),

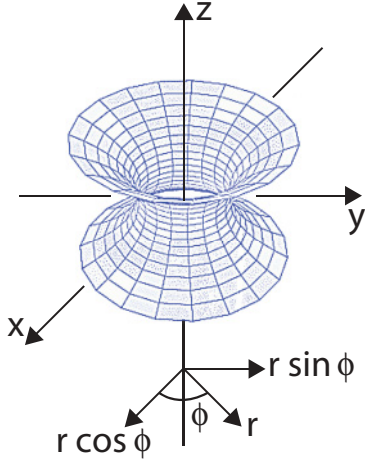$$ds^2 = \frac{r^2}{r^2 - b_0^2}dr^2 + r^2 d\phi^2. \tag{5}$$

FIG. 1. (Color online) A two-dimensional section (catenoid) of a three-dimensional wormhole geometry with its axis along z and the throat radius R.

Note that if we consider any other section of the three-dimensional wormhole, say for $\theta = \theta_0$, the line element will change to

$$ds^2 = \frac{r^2}{r^2 - b_0^2}dr^2 + a^2 r^2 d\phi^2, \tag{6}$$

where $a^2 = \sin^2 \theta_0$ and obviously $a^2 \in [0, 1]$. For the catenoid this will mean only a rescaling of the radius of the catenoid from R to aR. The line element Eq. (6) corresponds to a catenoid with $x = aR \cosh(z/aR) \cos \phi$, $y = aR \cosh(z/aR) \sin \phi$, and $z = z$. Thus all θ sections of the three-dimensional wormhole represent a catenoid with radius aR. The catenoid with the biggest radius corresponds to the equatorial section $\theta = \frac{\pi}{2}$ and the catenoid with zero radius corresponds to $\theta = \pi$.

III. EFFECTIVE POTENTIAL

Returning to the catenoid and focusing on the (z, ϕ) coordinates (instead of ρ, ϕ), the line element is given by

$$ds^2 = \cosh^2(z/R)dz^2 + R^2 \cosh^2(z/R)d\phi^2, \tag{7}$$

with the principal curvatures

$$\kappa_1 = \frac{1}{R}\text{sech}^2(z/R), \quad \kappa_2 = -\frac{1}{R}\text{sech}^2(z/R). \tag{8}$$

This implies that the mean curvature $H = (\kappa_1 + \kappa_2)/2 = 0$ (i.e., a minimal surface) and the Gaussian curvature $K = \kappa_1 \kappa_2 = -(1/R^2)\text{sech}^4(z/R)$. If a particle is confined to move on a curved surface (with finite thickness) and the thickness is allowed to go to zero, then an effective potential in the Schrödinger equation, known as the da Costa potential [5], will appear. (For a flat surface the potential is zero). The corresponding curvature-induced da Costa potential for a catenoid is

$$V(z) = -\frac{\hbar^2}{2m_0}(H^2 - K) = -\frac{\hbar^2}{2m_0 R^2}\text{sech}^4(z/R). \tag{9}$$

Note that for $a^2 \ll 1$ the geometric potential becomes very deep and localized at the origin.

The relevant Schrödinger equation is

$$-\frac{\hbar^2}{2m_0 R \cosh^2(z/R)}\left[R\frac{\partial^2 \psi}{\partial z^2} + \frac{1}{R}\frac{\partial^2 \psi}{\partial \phi^2}\right]$$
$$-\frac{\hbar^2}{2m_0 R^2}\text{sech}^4(z/R)\psi = E\psi, \tag{10}$$

or, simplifying,

$$-R\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{R}\frac{\partial^2 \psi}{\partial \phi^2} - \frac{\text{sech}^2(z/R)}{R}\psi = \frac{2m_0 R}{\hbar^2}\cosh^2(z/R)E\psi. \tag{11}$$

Using the cylindrical symmetry along the z axis, we set $\psi = e^{imz}\Phi$ and we get the following equation for Φ:

$$\Phi_{zz} - \frac{m^2}{R^2}\Phi + \frac{\text{sech}^2(z/R)}{R^2}\Phi + \frac{2m_0 E \cosh^2(z/R)}{\hbar^2}\Phi = 0. \tag{12}$$

Defining dimensionless length $\zeta = z/R$ and energy $\epsilon = 2m_0 E R^2/\hbar^2$ we get the following effective Schrödinger equation:

$$-\Phi_{\zeta\zeta} + V(\zeta)\Phi(\zeta) = 0, \tag{13}$$

where the geometric potential now reads

$$V(\zeta) = [m^2 - \epsilon \cosh^2(\zeta)] - \text{sech}^2(\zeta). \tag{14}$$

This potential for $m \neq 0$ bears some similarity to the corresponding geometric potential for the three-dimensional wormhole [3]. Note that in the ground state ($\epsilon = 0$, called also a *critically bound state* [9]) the aforementioned potential becomes the reflectionless Bargmann's potential [10] and the Schrödinger equation becomes the hypergeometric equation with the ground-state wave function (or the Goldstone mode) given by $\Phi(\zeta) = \text{sech}(\zeta)$. This result is remarkable in that the minimal surface of the catenoid enables complete transmission across it for a quantum particle. This does not seem to be the case for a three-dimensional wormhole. For nonzero and positive ϵ the aforementioned potential is an inverted double-well potential (see Fig. 2).
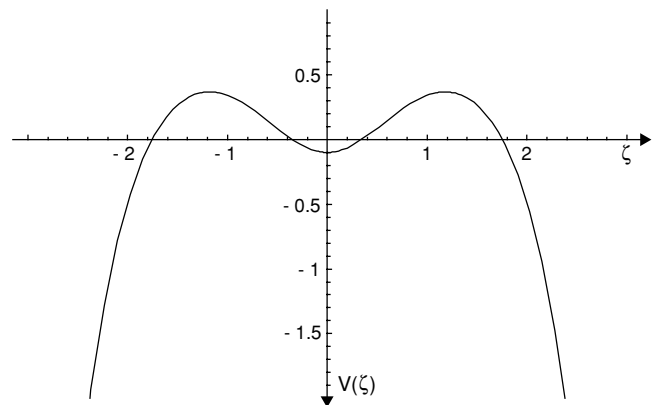

FIG. 2. The inverted double-well potential $V(\zeta)$ with $m = \pm 1$ and $\epsilon = 0.1$.

IV. BOUND STATES

Let us consider Eq. (13) in more detail. We see that

$$\lim_{\zeta \to \pm\infty} |V| \to \infty. \tag{15}$$

The behavior of the potential at infinity is strange since the physical geometry in the catenoid in these regions approaches the usual Euclidean one. This feature can be traced to the coordinates used since the proper length per unit in the ζ direction diverges when $\zeta \to \pm\infty$. This can be remedied by introducing another set of coordinates on the catenoid.

Quantum theory in curved spaces is generally a challenge since the theory is not generally covariant. Classical quantum theory is not even Lorentz invariant. This puts a severe constraint on the coordinate system in which one wishes to describe the physics in order to be able to extract the physical content of the theory. This challenge was even central in the early days of general relativity theory itself in connection with the physical interpretation of the Schwartzschild metric, for example, just like as in general relativity theory one is usually safe concerning the physical interpretation as well as the definitions of physical quantities when the manifold in question is asymptotically Minkowski (Euclidean). In such asymptotic regions we expect on physical grounds to re-derive the usual flat space physical results. The asymptotic properties thus in some sense anchor the curved region and its physics to reality as we know it. The catenoid is an asymptotic Euclidean object, thus making this manifold a space anchored to "reality."

Considering the two-dimesional Schrödinger equation in the plane in polar coordinates we get the Bessel equation. Clearly, the boundary condition at the origin is suspect here. However, in our case we can as a first approximation consider a deformation of the plane in a region around the origin. In the deformed region the Schrödinger equation will generally be very complicated but the solutions of it must nevertheless be matched to the Bessel functions which survive sufficiently far from the deformed region. This reasoning goes *ad verbum* through also on the catenoid even though we here, in addition to curvature corrections, have a topology change when compared to the plane. Hence, we should seek coordinates on the catenoid such that the Schrödinger equation gives rise to the Bessel equation in the asymptotic region on the catenoid. The coordinates should in particular result in a metric which is reminiscent of polar coordinates at infinity. It is possible to find such coordinates if one covers the entire manifold with two coordinate patches. One patch covers the region $\zeta > 0$ and the other one $\zeta < 0$. In the upper part we choose

$$\eta^+ = e^\zeta - 1; \quad \zeta > 0. \tag{16}$$

In the lower part we correspondingly choose

$$\eta^- = -(e^{-\zeta} - 1); \quad \zeta < 0. \tag{17}$$

Clearly $\eta^+ = \eta^-$ at $\zeta = 0$. The invariant line-element can then be written as

$$ds^2 = \frac{[(\eta^\pm \pm 1)^2 + 1]^2}{4(\eta^\pm \pm 1)^4}(d\eta^\pm)^2 + \frac{1}{4}\left[\frac{(\eta^\pm \pm 1)^2 + 1}{\eta^\pm \pm 1}\right]^2 d\phi^2, \tag{18}$$

In the limit $\eta^\pm \to \pm\infty$ the metric reduces to

$$ds^2 = \frac{1}{4}(d\eta^\pm)^2 + \frac{1}{2}[(\eta^\pm \pm 1)^2 + 1]d\phi^2$$
$$\simeq \frac{1}{4}(d\eta^\pm)^2 + \frac{1}{2}(\eta^\pm)^2 d\phi^2. \tag{19}$$

Hence, the asymptotic form of this metric is very similar to the usual polar coordinates. Clearly, these new coordinates should be well suited to explore the physical states of a quantum particle on the catenoid.

Let us now consider the Schrödinger equation. In terms of the new coordinates we have in particular that

$$\partial_u^2 \Phi = (\eta^\pm \pm 1)\partial_\pm[(\eta^\pm \pm 1)\partial_\pm \Phi], \tag{20}$$

$$\cosh u = \pm\frac{1}{2}\left[\frac{(\eta^\pm \pm 1)^2 + 1}{\eta^\pm \pm 1}\right]. \tag{21}$$

This gives rise to identical expressions for the Schrödinger equation in the two patches. In the upper patch the equation is explicitly given by

$$\partial_+^2 \Phi + \frac{1}{(\eta^+ + 1)}\partial_+ \Phi + \left(\frac{\epsilon}{4} - \frac{(m^2 - \epsilon/2)}{(\eta^+ + 1)^2}\right.$$
$$\left. + \frac{1}{4}\left\{\frac{\epsilon}{(\eta^+ + 1)^4} + \frac{16}{[(\eta^+ + 1)^2 + 1]^2}\right\}\right)\Phi = 0. \tag{22}$$

Clearly, letting $\eta^+ \to \infty$ we get the Bessel equation, which is well behaved at infinity.

The stationary Schrödinger equation, assuming a well-defined energy E eigenvalue problem, is formally given by

$$(-\nabla^2 + V)\Psi = E\Psi. \tag{23}$$

Hence, we have that

$$V - E = -\left(\frac{\epsilon}{4} - \frac{(m^2 - \epsilon/2)}{(\eta^+ + 1)^2}\right.$$
$$\left. + \frac{1}{4}\left\{\frac{\epsilon}{(\eta^+ + 1)^4} + \frac{16}{[(\eta^+ + 1)^2 + 1]^2}\right\}\right). \tag{24}$$

In the asymptotic region we find that

$$\lim_{\eta^+ \to \infty} V = E - \frac{1}{4}\epsilon > 0. \tag{25}$$

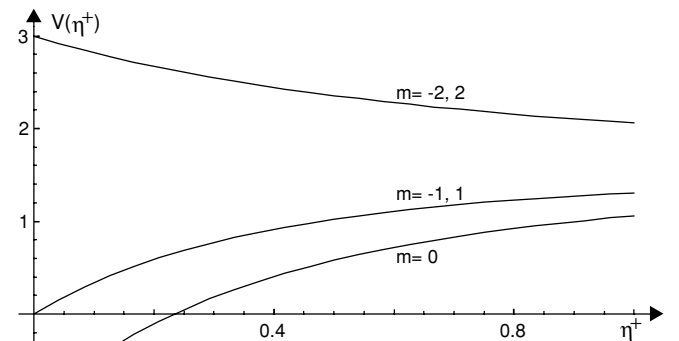We have plotted the potential for $m = 0, \pm 1, \pm 2$ and $\epsilon = 2$ in Fig. 3.

FIG. 3. The potential $V(\eta^+)$ with $m = 0, \pm 1, \pm 2$ and $\epsilon = 2$.

Clearly, the constant part of the potential can be renormalized to zero without any physical consequences. Hence, the renormalized potential V_r can be taken to be

$$V_r - E = -\left(-\frac{(m^2 - \epsilon/2)}{(\eta^+ + 1)^2} + \frac{1}{4}\left\{ \frac{\epsilon}{(\eta^+ + 1)^4} + \frac{16}{[(\eta^+ + 1)^2 + 1]^2} \right\} \right). \quad (26)$$

Consider V and the case when the energy is set to unity. Then the physical quantum states fall into four different categories. In the s channel ($m = 0$) the potential becomes negative sufficiently close to the origin. When $m = \pm 1$, the same patterns emerge but with a much faster fall off of the potential with increasing coordinate distance from the origin than in the s channel. When $m = \pm 2$ the potential is everywhere positive definite. Higher angular momentum modes will give rise to bound states at a distance from the origin and at a distance in the direction of increasing radial coordinate.

In general, if it is possible to redefine $\cosh^2(\zeta)\Phi_{\zeta\zeta} = \Phi_{\eta\eta}$ in Eqs. (13) and (14), then the equation for $\Phi(\eta)$ would correspond to the double sinh-Gordon potential which is quasi-exactly solvable [11,12], that is, for specific values of ϵ and m exact solutions can be obtained.

V. CONCLUSION

We demonstrated that a two-dimensional equatorial ($\theta = \frac{\pi}{2}$) or any other section of a wormhole is equivalent to the minimal surface of a catenoid. We then showed that the curvature-induced da Costa quantum potential [5] allows for a critically bound state ($\epsilon = 0$). This leads to a reflectionless transmission of a quantum particle across the catenoid (or a two-dimensional wormhole). By introducing an appropriate coordinate system we were able to obtain bound states for different angular momentum channels. It is interesting to note that the potential is attractive at the origin $\eta^{\pm} = 0$ for $m = 0$ and $m = \pm 1$ (anticentrifugal potential with bound states). In contrast, in the plane the anticentrifugal potential is present only for $m = 0$ but an additional δ-function potential is needed at the origin in order to introduce the missing length scale in the plane [1]. We note that a radial array of dislocations in a bilayer of honeycomb lattices or a suitably bent bilayer graphene sheet with a neck [6] may provide a physical realization for our findings. Another experimental means of measuring the potential [Eq. (14)] is to construct nanoscale waveguides in a catenoid shape.

[1] M. A. Cirone, K. Rzazewski, W. P. Schleich, F. Straub, and J. A. Wheeler, Phys. Rev. A **65**, 022101 (2001).

[2] I. Bialynicki-Birula, M. A. Cirone, J. P. Dahl, M. Fedorov, and W. P. Schleich, Phys. Rev. Lett. **89**, 060404 (2002).

[3] R. Dandoloff, Phys. Lett. **A373**, 2667 (2009).

[4] M. Morris and K. Thorne, Am. J. Phys. **56**, 395 (1988).

[5] R. C. T. da Costa, Phys. Rev. A **23**, 1982 (1981).

[6] Y. N. Joglekar and A. Saxena, Phys. Rev. B **80**, 153405 (2009).

[7] V. A. Atanasov, R. Dandoloff, and A. Saxena, Phys. Rev. B **79**, 033404 (2009); B. Jensen, Phys. Rev. A **80**, 022101 (2009).

[8] H. Karcher, F. S. Wei, and D. Hoffman, in *Global Analysis in Modern Mathematics*, edited by K. Uhlenbeck (Publish or Perish Press, Houston, TX, 1993).

[9] J. Lekner, Am. J. Phys. **75**, 1151 (2007).

[10] V. Bargmann, Rev. Mod. Phys. **21**, 488 (1949); R. E. Crandall, J. Phys. A **16**, 3005 (1983).

[11] M. Razavy, Phys. Lett. **A72**, 89 (1979); Am. J. Phys. **48**, 285 (1980).

[12] A. Khare, S. Habib, and A. Saxena, Phys. Rev. Lett. **79**, 3797 (1997); S. Habib, A. Khare, and A. Saxena, Physica D (Amsterdam) **123**, 341 (1998).