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Finite-size behavior of quantum collective spin systems

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We discuss the finite size behavior of the adiabatic Dicke model, describing the collective coupling of a set of N two-level atoms (qubits) to a faster (electromagnetic) oscillator mode. The energy eigenstates of this system are shown to be directly related to those of another widely studied collective spin model, the uniaxial one. By employing an approximate continuum approach, we obtain a complete characterization of the properties of the latter, which we then use to evaluate the scaling properties of various observables for the original Dicke model near its quantum phase transition.

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I. INTRODUCTION

The interaction of N two-level systems (qubits) with a common single-mode quantum bosonic field is a paradigmatic example of collective quantum behavior. Dating back to the model put forward by Dicke, [1], this has become one of the most investigated problems in quantum optics and condensed matter physics, with proposed physical implementations ranging from superconducting nanodevices, [2] to ultracold atoms and Bose-Einstein condensates in cavity [3]. The Dicke model exhibits a second-order phase transition [4] and, due to its broad application range [5], it has been studied extensively in the past few years, [6–9]. It displays a rich dynamics, with many nonclassical features [10–13]; in particular, the ground-state entanglement [14,15] and the Berry phase [16,17] of the Dicke model have been diffusely analyzed and many aspects of its finite-size behavior have been obtained [18–20]. The continued interest in the Dicke model also stems from the fact that it pertains to the same universality class as other intensely studied many-body systems that possess infinite-range interactions and for which theoretical models typically allow for exact solutions in the thermodynamic limit.

The most general collective model of (effective) spin 1/2 systems, the biaxial model in arbitrary field, can be described by the Hamiltonian (see Ref. [21] for details)

$$\hat{H}_{XY}^{\perp,\parallel} = \sum_{k=x,y,z} \delta_k \hat{S}_k + g_x \hat{S}_x^2 + g_y \hat{S}_y^2, \tag{1}$$

where the $\hat{S}_k = \sum_{i=1}^{N} \hat{\sigma}_i^{(k)}$ are the collective Pauli operators that obey angular-momentum-like commutation relations $[\hat{S}_i, \hat{S}_j] = 2i\epsilon_{ijk}\hat{S}_k$. The energy eigenstates can be written in the angular-momentum basis (we employ the standard one, apart from a factor 2 in the definitions) $\{|s, s_z\rangle;\ s_z = -s, -s+2, \ldots, s-2, s\}$ constructed as the set of common eigenstates of both $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$ and $\hat{S}_z$. For a ferromagnetic interaction $g_{x,y} < 0$, the ground state of the Hamiltonian belongs to the symmetric subspace with $S^2 = N(N+2)$ and special and diffusely studied cases are the biaxial model in a transverse field ($\delta_x = \delta_y = 0$) $\hat{H}_{XY}^{\perp}$ (the well-known LGM model [22,23]) and the uniaxial model ($\delta_y = g_y = 0$) $\hat{H}_X^{\perp,\parallel}$.

Under the thermodynamic limit, the phase diagram of these collective spin models has been simply established by a mean-field approach [24]. For N large but finite, purely quantum effects become important and numerical analysis have been implemented using the continuous unitary transformation method [25] and a semiclassical approach [26]. A qualitative understanding of the LGM model is obtained in Ref. [27] by introducing a double well structure above the phase transition in a semiclassical treatment of the system.

In the present work, we establish an exact relationship between the Dicke model in the adiabatic regime (i.e., for the case of slow qubits coupled to a faster oscillator mode) and the uniaxial model, which is valid not only in the thermodynamic limit, but also for any finite number N of spins. We then present an alternative analytic method which relies on a continuum approach to solve the collective uniaxial model for large N. We show that this method is useful to determine the finite-size behavior and the entire $1/N$ expansion (i.e., critical exponents and prefactors) at the critical point for both the Dicke and the collective uniaxial spin models. These results corroborate several studies in which the exponents have already been derived.

The objective of the present study is thus threefold, and the article is organized accordingly: first, we consider the Dicke model in the regime in which the frequency of the quantum field is much larger than the energy spacing of the qubits; in this case, the field degree of freedom can be adiabatically separated from the qubit ones and an effective N-qubit interaction can be obtained by means of the Born-Oppenheimer approximation. This is done in Sec. II, where the relationship with the uniaxial model is established for any energy eigenstate. Afterward, we focus on the quantum phase transition of this collective model (Sec. III) for which we derive the $1/N$ expansion for some relevant physical observables and we also compute exactly various entanglement measures for the qubits. Finally, using these results together with those obtained in Sec. II, we obtain analogous $1/N$ expansions for the Dicke model (Sec. IV). A summary and some concluding remarks are finally given in Sec. V.

II. ADIABATIC DICKE MODEL

We consider a system of N qubits interacting with a single harmonic oscillator mode, described by the

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Hamiltonian ($\hbar = c = 1$)

$$\hat{H} = -\frac{\delta}{2}\hat{S}_x + \frac{\epsilon}{2}\hat{S}_z + \omega\hat{a}^\dagger\hat{a} + \frac{\lambda}{\sqrt{N}}(\hat{a}^\dagger + \hat{a})\hat{S}_z, \quad (2)$$

where a is the annihilation operator for the field mode of frequency ω, δ is the transition frequency of the qubit, ϵ is the level asymmetry, and λ is the strength of the coupling between the oscillator and the two-level systems.

We assume a *slow* qubit and work in the regime $\omega \gg \delta$ by employing the Born-Oppenheimer approximation, Refs. [19,28]. The standard procedure is to separate the Hamiltonian of Eq. (2) in two parts, containing slow and fast variables, respectively [29]

$$\hat{H} = \hat{H}_s + \hat{H}_f, \quad (3)$$

where

$$\hat{H}_f = \omega\hat{a}^\dagger\hat{a} + \frac{\epsilon}{2}\hat{S}_z + \frac{\lambda}{\sqrt{N}}(\hat{a}^\dagger + \hat{a})\hat{S}_z, \quad \tilde{H}_s = -\frac{\delta}{2}\hat{S}_x. \quad (4)$$

The eigenstates of the composite system can be written as a coherent superposition of the eigenkets of H_f, having a parametric dependence on (i.e., conditioned by) the values of the slow variables:

$$|\psi_r\rangle = \sum_{\{n,s_z\}} \phi_{s_z}^{(n,r)}|n[s_z]\rangle, \quad (r = 0,\ldots,s), \quad (5)$$

where the displaced number states of the oscillator are given by

$$|n[s_z]\rangle = e^{-\frac{\lambda}{\omega\sqrt{N}}(a^\dagger - a)s_z}|n\rangle \otimes |s, s_z\rangle. \quad (6)$$

They are the eigenstates of the fast Hamiltonian

$$\hat{H}_f|n[s_z]\rangle = V_n(s_z)|n[s_z]\rangle, \quad (7)$$

with eigenvalues

$$V_n(s_z) = \omega n + \frac{\epsilon}{2}s_z - \frac{\lambda^2}{N\omega}s_z^2. \quad (8)$$

For different n, $V_n(s_z)$ contribute an effective adiabatic potential felt by the slow subsystem so that the wave function $\phi_{s_z}^{(n,r)}$ of the N qubit system is determined by

$$\hat{H}_{\rm eff}\phi_{s_z}^{(n,r)} = E_{(n,r)}(s_z)\phi_{s_z}^{(n,r)}, \quad (9)$$

where the effective Hamiltonian is reduced to the form

$$\hat{H}_{\rm eff} = \frac{\delta}{2}\hat{S}_x + V_n(\hat{S}_z) = \omega\hat{a}^\dagger\hat{a} + \hat{H}_Z^{\perp,\parallel}, \quad (10)$$

with $\hat{H}_Z^{\perp,\parallel}$ being the Hamiltonian of the uniaxial model introduced in the previous section, with coupling constant $g_z = -\frac{\lambda^2}{N\omega}$:

$$\hat{H}_Z^{\perp,\parallel} = -\frac{\delta}{2}\hat{S}_x + \frac{\epsilon}{2}\hat{S}_z + g_z\hat{S}_z^2. \quad (11)$$

The ground state of the coupled qubit-oscillator system is given by

$$|\psi_0\rangle = \sum_{m=-N}^{N} \varphi_m e^{-\frac{m\lambda}{\sqrt{N}\omega}(a^\dagger - a)}|0\rangle \otimes |N,m\rangle, \quad (12)$$

where $\varphi_m \equiv \phi_m^{(0,0)}$.

The uniaxial (as well as the LGM model) and the Dicke model are known to be equivalent in the thermodynamic limit. From the discussion of this section, we see that there is a strict relationship between the ground states of the two model systems as both can be expressed in the angular-momentum basis with the same amplitudes φ_m. However, this last equation shows that at finite size there can be differences between their behaviors since, in the case of the Dicke model, these coefficients gets effectively modified due to the presence of the displacement operator, whose argument depends explicitly on the number of qubits N. This implies that we will find small differences in the $1/N$ expansions for the two models.

In order to continue the discussion on the Dicke model, we need to evaluate the amplitudes φ_m. Therefore, we now turn our attention to the uniaxial model with N qubits. Once the coefficients φ_m are obtained, we will use them in Sec. IV to complete the description of the finite-size behavior in the Dicke model.

III. UNIAXIAL MODEL

A. Continuum approach

The Hamiltonian of a uniaxial model for a spin system with a collective coupling can be written as

$$\hat{H}_Z^{\perp,\parallel} = -\frac{\delta}{2}\hat{S}_x + \frac{\epsilon}{2}\hat{S}_z - \frac{g}{N}\hat{S}_z^2 \quad (13)$$

with $\delta \geqslant 0$ and where we have rescaled the ferromagnetic coupling constant by the number of spins, $g_z = -g/N$. This is equivalent to the more diffusely found $\hat{H}_X^{\perp,\parallel}$ Hamiltonian that can be obtained after the rotation $e^{i\pi S_y/4}$. To connect this model to the discussion of the previous section, one simply has to take $g = \lambda^2/\omega$.

The ground state of $\hat{H}_Z^{\perp,\parallel}$ lies in the maximum spin sector $s \equiv N$. In this subspace, spanned by the states $\{|N,m\rangle;\, m = -N, -N+2, \ldots, N-2, N\}$, the ground state can be written as

$$|\phi_0\rangle = \sum_{m=-N}^{N} \varphi_m |N,m\rangle, \quad (14)$$

where φ_m are real coefficients.

We limit our present discussion to the case of the symmetric phase $\epsilon = 0$, that is, the one relevant for the description of the Dicke phase transition occurring at $\epsilon = 0$ with $\lambda^2 = \omega\delta/4$, corresponding to $g = \delta/4$.

In the angular-momentum basis, the Hamiltonian takes a $(N+1) \otimes (N+1)$ tridiagonal symmetric (Jacobi) matrix form with double symmetries along both the main and the second diagonal:

$$T_N = \begin{pmatrix} \lambda_{-N} & \delta_{-N} & 0 & \ldots & 0 & 0 & 0 \\ \delta_{-N} & \lambda_{-N+2} & \delta_{-N+2} & \ldots & 0 & 0 & 0 \\ 0 & \delta_{-N+2} & \lambda_{-N+4} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_{N-4} & \delta_{N-4} & 0 \\ 0 & 0 & 0 & \ldots & \delta_{N-4} & \lambda_{N-2} & \delta_{N-2} \\ 0 & 0 & 0 & \ldots & 0 & \delta_{N-2} & \lambda_N \end{pmatrix}, \quad (15)$$

where

$$\lambda_m = -\frac{\lambda^2}{N} m^2, \quad -N \leqslant m \leqslant N \tag{16}$$

and

$$\delta_m = -\frac{\delta}{4}\sqrt{N(N+2) - m(m+2)}. \tag{17}$$

These coefficients satisfy a confluence property $\lambda_m/N \to \lambda(z)$ and $\delta_m^2/N^2 \to \delta^2(z)$ where $z = m/N$ as $m, N \to \infty$ [30]. By application of theorems on the zeros of orthogonal polynomials [31] one finds that the ground-state energy density in the $m, N \to \infty$ limit is given in general by

$$\varepsilon_0(\infty) = \inf\{\lambda(z) - 2\delta(z)\}. \tag{18}$$

Introducing the dimensionless parameter $\alpha = 4g/\delta$ the minimum is found at

$$z_0 = \begin{cases} 0, & (\alpha \leqslant 1), \\ \pm\sqrt{1 - 1/\alpha^2}, & (\alpha > 1), \end{cases} \tag{19}$$

and the corresponding thermodynamic limit of the ground-state energy per spin is

$$\lim_{N\to\infty} \frac{\varepsilon_0(N)}{N} = \begin{cases} -\frac{\delta}{2}, & (\alpha \leqslant 1), \\ -\frac{\delta}{4}\left(\alpha + \frac{1}{\alpha}\right), & (\alpha > 1). \end{cases} \tag{20}$$

For finite N, the solution of the eigenvalues problem for the ground state reduces to the recurrence relation

$$\delta_{m-2}\varphi_{m-2} + \lambda_m\varphi_m + \delta_m\varphi_{m+2} = \varepsilon_0\varphi_m \tag{21}$$

that can be rewritten as a second-order linear difference equation

$$2(\delta_m + \delta_{m-2})\,\triangle_2\,\varphi_m + 2(\delta_m - \delta_{m-2})\,\triangle_1\,\varphi_m + (\delta_m + \delta_{m-2} + \lambda_m)\varphi_m = \varepsilon_0\varphi_m, \tag{22}$$

where $\triangle_2\varphi_m = (\varphi_{m+2} + \varphi_{m-2} - 2\varphi_m)/4$ and $\triangle_1\varphi_m = (\varphi_{m+2} - \varphi_{m-2})/4$ are finite differences of second and first orders, respectively.

A simple analytic behavior of the coefficients φ_m for $N \gg 1$ can be derived by considering m/N as a continuous variable and by expanding the recursion relation (22) in series around the minima of Eq. (18). For $\alpha \leqslant 1$, expanding in series (21) around $m = 0$ and neglecting corrections of order $1/N^2$, one obtains

$$\varphi_m'' + \left[\frac{\varepsilon_0(N)}{N\delta} + \frac{1}{2}\left(1 + \frac{1}{N}\right) - \frac{1-\alpha}{4N^2}m^2\right]\varphi_m \simeq 0 \tag{23}$$

whose solution is

$$\varphi_m \simeq \left(\frac{2k}{\pi N}\right)^{1/4} e^{-km^2/4N} \tag{24}$$

with $k = \sqrt{1-\alpha}$. The ground-state energy per spin is given by

$$\frac{\varepsilon_0(N)}{N} \simeq -\frac{\delta}{2}\left(1 + \frac{1 - \sqrt{1-\alpha}}{N}\right). \tag{25}$$

For $\alpha > 1$, by expanding in series Eq. (21) around $m \simeq \pm m_0 = \pm N\sqrt{1 - 1/\alpha^2}$, one gets

$$\varphi_m'' + \alpha\left[\frac{\varepsilon_0(N)}{N\delta} + \frac{1}{4}\left(\alpha + \frac{1}{\alpha}\right) + \frac{\alpha}{2N} - \frac{\alpha(\alpha^2-1)}{4N^2}(m \pm m_0)^2\right]\varphi_m \simeq 0 \tag{26}$$

whose approximate solution is the symmetric superposition

$$\varphi_m \simeq \frac{1}{\sqrt{2}}(\varphi_m^+ + \varphi_m^-) \tag{27}$$

with

$$\varphi_m^\pm = \left(\frac{2\bar{k}}{\pi N}\right)^{1/4} e^{-\bar{k}(m \mp m_0)^2/4N}, \tag{28}$$

where $\bar{k} = \alpha\sqrt{\alpha^2 - 1}$. In this regime one has

$$\frac{\varepsilon_0(N)}{N} \simeq -\frac{\delta}{2}\left[\frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) + \frac{\alpha - \sqrt{\alpha^2-1}}{N}\right]. \tag{29}$$

In this language, the transition is readily understood: above the coupling value corresponding to $\alpha = 1$ a drastic change in the form of the ground-state wave function takes place, with a breaking of the "inversion" symmetry around $m = 0$. For a finite-size system, the transition becomes smother and smother and the wave function φ_m gradually changes from a one peaked Gaussian to the superposition with two peaks that emerge progressively as the value of z_0 moves away from the origin (i.e., as α increases).

For large enough N, we can check the continuum approximation by comparing it to the behavior obtained by solving the tridiagonal matrix numerically. In Fig. 1 φ_m for $N = 200$ is shown with $\alpha = 0.3$ and $\alpha = 1.3$ compared with the analytic expressions of Eqs. (24), (25), (26), and (27).

B. Finite-size corrections

Having obtained the ground-state coefficients φ_m (together with the ground-state energy), we may evaluate the average

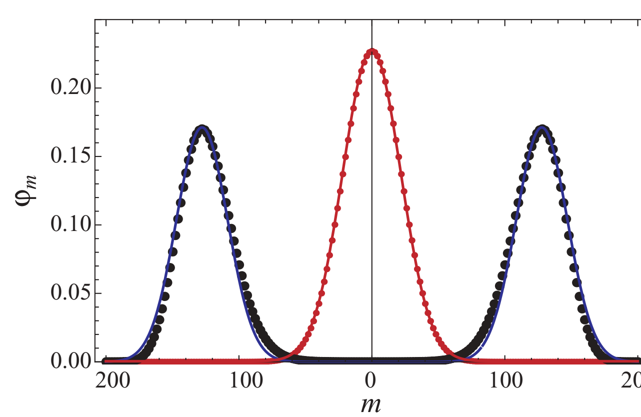

FIG. 1. (Color online) Normalized φ_m function for $\alpha = 0.3$ (small red circles) and $\alpha = 1.3$ (black circles) for a spin system of size $N = 200$ obtained by numerically solving the tridiagonal matrix (15). Comparison is made with the analytic expressions of Eq. (24) (dashed red line) and Eq. (27) (continuous blue line).

values of every physical observable; in particular, we concentrate on the total spin components. One immediately gets

$$\frac{\langle S_x\rangle}{N} = -\frac{2}{N}\frac{\partial\varepsilon_0(N)}{\partial\delta} = \begin{cases} 1+\frac{1}{N}\left(1+\frac{\alpha-2}{2\sqrt{1-\alpha}}\right), & (\alpha\leqslant 1);\\ \frac{1}{\alpha}+\frac{1}{N\sqrt{\alpha^2-1}}, & (\alpha>1)\end{cases} \quad (30)$$

and

$$\frac{\langle S_z^2\rangle}{N^2} = -\frac{1}{N}\frac{\partial\varepsilon_0(N)}{\partial g} = \begin{cases} \frac{1}{N\sqrt{1-\alpha}}, & (\alpha\leqslant 1);\\ 1-\frac{1}{\alpha^2}+\frac{2}{N}\left(1-\frac{\alpha}{\sqrt{\alpha^2-1}}\right), & (\alpha>1).\end{cases} \quad (31)$$

The expressions for $\langle S_{x,y}^2\rangle$ are, instead, a bit more complicated

$$\langle S_{x,y}^2\rangle = \frac{1}{2}\left[N(N+2)-\langle S_z^2\rangle\right] \pm 2\sum_{m=-N+2}^{N-2} a_m^+ a_m^- \varphi_{m-2}\varphi_{m+2}\,. \quad (32)$$

However, they can be simplified by making use of the simple results $\varphi_{m-2}\varphi_{m+2} = e^{-2k/N}\varphi_m^2$ for $\alpha\leqslant 1$ and $\varphi_{m-2}\varphi_{m+2} = e^{-2\bar{k}/N}\varphi_m^2$ for $\alpha>1$ that are easily derived from our analytic expressions for φ_m. Thus, one obtains

$$\frac{\langle S_x^2\rangle}{N^2} \simeq \begin{cases} 1+\frac{2}{N}\left(1-\frac{1}{\sqrt{1-\alpha}}\right), & \alpha\leqslant 1;\\ \frac{1}{\alpha^2}+\frac{1}{N}\frac{\alpha^2+1}{\alpha\sqrt{\alpha^2-1}}, & \alpha>1\end{cases} \quad (33)$$

$$\frac{\langle S_y^2\rangle}{N} \simeq \begin{cases} \sqrt{1-\alpha}, & \alpha\leqslant 1;\\ \sqrt{1-\frac{1}{\alpha^2}}, & \alpha>1.\end{cases} \quad (34)$$

In the region $\alpha\sim 1$ we must take into account also the next to leading order in the expansion of the recursion relation (21) that gives a non-negligible contribution near the phase transition point. We thus need to consider the quartic-oscillator-like equation

$$\varphi_m'' + \left[\frac{\varepsilon_0(N)}{N\delta}+\frac{1}{2}\left(1+\frac{1}{N}\right)-\frac{1-\alpha}{4N^2}m^2-\frac{m^4}{16N^4}\right]\varphi_m \simeq 0. \quad (35)$$

Using the approach presented in a previous work [19], Eq. (35) can be reduced to a single-parametric problem with the help of Symanzik scaling procedure [32]. This is done, by recasting Eq. (35) into the equivalent form

$$\varphi_n'' + [e_0(\zeta)-\zeta n^2 - n^4]\varphi_n \simeq 0, \quad (36)$$

where $n = m(2N)^{-2/3}$ is a scaled variable. The only remaining scale parameter is then $\zeta = (2N)^{2/3}(1-\alpha)$, while the ground-state energy is rewritten as

$$\frac{\varepsilon_0(N)}{N} = -\frac{\delta}{2}\left(1+\frac{1}{N}\right)+\delta\frac{e_0(\zeta)}{(2N)^{4/3}}. \quad (37)$$

For $\zeta\sim 0$ (that is, very close to the transition point), we can resort to perturbation theory and obtain the ground-state energy as an expansion in powers of ζ,

$$e_0(\zeta) = \sum_{n=0}^{\infty}\beta_n\zeta^n. \quad (38)$$

It is easy to show that $\beta_0 = e_0(0)\simeq 1.06036$ is the lowest eigenvalue of the pure quartic oscillator and $\beta_1 = e_0'(0)\simeq 0.36203$.

Using these results to obtain an approximate expression for the ground-state energy and for the coefficients φ_m, it is easy to derive the following leading nontrivial finite-size corrections for one- and two-spin correlation functions

$$\frac{\langle S_x\rangle}{N} \simeq 1-\frac{2\beta_1}{(2N)^{2/3}} \quad (39)$$

$$\frac{\langle S_z^2\rangle}{N^2} \simeq \frac{4\beta_1}{(2N)^{2/3}} \quad (40)$$

$$\frac{\langle S_x^2\rangle}{N^2} \simeq 1-\frac{4\beta_1}{(2N)^{2/3}} \quad (41)$$

$$\frac{\langle S_y^2\rangle}{N^2} \simeq \frac{8\beta_0}{3(2N)^{4/3}}. \quad (42)$$

The critical exponents in these expressions are in full agreement with those reported in Ref. [25]. The present method not only corroborates the results for the exponents reported in the Literature but also allowes us to obtain the prefactors of the finite-size expansion that cannot be determined with typical scaling arguments and that are important to transfer these results to the case of the Dicke model. In particular, we relied on a continuum approximation to solve the eigenvalue problem for the matrix T_N of Eq. (15). This kind of approach is not limited to the present problem but can be applied whenever the confluence property holds (namely, whenever the matrix element of a spin Hamiltonian depends only on the ratio m/N).

In Fig. 2 we make a comparison of the analytical results for the leading nontrivial finite-size corrections with those obtained from a direct numerical solution at the critical point.

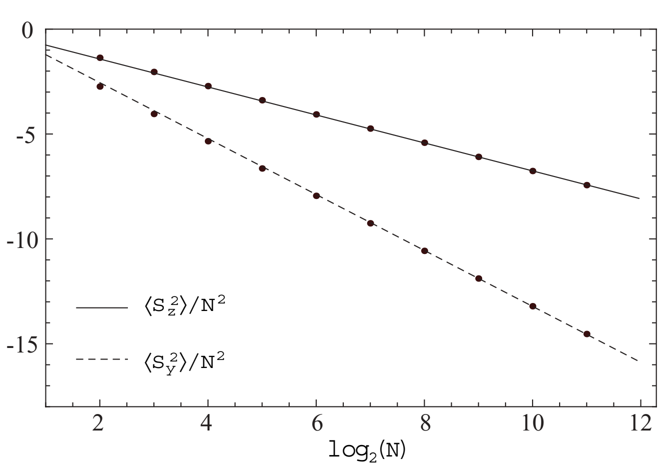

FIG. 2. Comparison in $\log_2$-$\log_2$ scale between numerical (points) and analytical (lines) results for the scaling of two-spins correlation functions $\langle S_z^2\rangle/N^2$ and $\langle S_y^2\rangle/N^2$ [analytical results refer to the Eq. (40) and Eq. (42), respectively] as a function of N at the critical point $\alpha = 1$.

One can see that the agreement is good even for small values of N.

C. Ground-state entanglement

Before going back to the Dicke model, we use the results we have obtained in order to discuss the critical behavior of the ground-state entanglement for the uniaxial model. In this respect, it is useful to make a partition of the N spins in two blocks of size L and $(N-L)$, respectively. Using the decomposition

$$|N,m\rangle = \sum_{l=-L}^{L} p_{lm}^{1/2} |N-L, m-l\rangle \otimes |L,l\rangle, \tag{43}$$

where

$$p_{lm} = \frac{\binom{L}{\frac{L+l}{2}}\binom{N-L}{\frac{N-L+m-l}{2}}}{\binom{N}{\frac{N+m}{2}}} \tag{44}$$

we obtain the ground-state reduced density matrix of the block of size L out of the total N spins in the form

$$\rho_{L,N} = \sum_{l_1=-L}^{L}\sum_{l_2=-L}^{L} |L,l_1\rangle\langle L,l_2| \times \sum_{m=-N}^{N} p_{l_1m}^{1/2} p_{l_2m-l_1+l_2}^{1/2} \varphi_m \varphi_{m-l_1+l_2}. \tag{45}$$

We then compute the linear entropy as a measure of the entanglement of the block of size L with the rest of the system

$$\tau_L = \eta_L\left[1 - \mathrm{Tr}\left(\rho_{L,N}^2\right)\right], \tag{46}$$

where the prefactor is chosen to be $\eta_L = \frac{2^L}{2^L-1}$ in order to bound τ_L to 1.

In particular, for $L=1$, the state of every single qubit is found to be

$$\rho_{1,N} = \frac{1}{2}\left(I + \frac{\langle S_x\rangle}{N}\sigma_x\right), \tag{47}$$

where I is the identity. We can then evaluate the one-tangle as

$$\tau_1 = 2\left[1 - \mathrm{Tr}\left(\rho_{1,N}^2\right)\right] \equiv 1 - \frac{\langle S_x\rangle^2}{N^2}. \tag{48}$$

One has

$$\tau_1 \simeq \begin{cases} \frac{1}{N}\left(2 + \frac{\alpha-2}{\sqrt{1-\alpha}}\right), & \alpha \leqslant 1; \\ 1 - \frac{1}{\alpha^2} + \frac{2}{N\alpha\sqrt{\alpha^2-1}}, & \alpha > 1 \end{cases} \tag{49}$$

and

$$\tau_1 \simeq \frac{4\beta_1}{(2N)^{2/3}}, \quad \alpha = 1. \tag{50}$$

The reduced density matrix of two qubits ($L=2$), can be written in the angular-momentum basis $\{|2,m\rangle\}$, with $m = 2, 0, -2$. In general, one should also consider the state $|0,0\rangle$; but its population is zero in our case, so we can erase the corresponding line and row and write $\rho_{2,N}$ in the form:

$$\rho_{2,N} = \begin{pmatrix} v_+ & \sqrt{2}x_+ & u \\ \sqrt{2}x_+ & 2w & \sqrt{2}x_- \\ u & \sqrt{2}x_- & v_- \end{pmatrix}, \tag{51}$$

where the matrix elements may be expressed in terms of the expectation values of the collective operators as [33]

$$v_\pm = \frac{N(N-2) + \langle S_z^2\rangle}{4N(N-1)} \pm \frac{\langle S_z\rangle}{2N} \tag{52}$$

$$w = \frac{N^2 - \langle S_z^2\rangle}{4N(N-1)} \tag{53}$$

$$u = \frac{\langle S_+^2\rangle}{N(N-1)} \tag{54}$$

$$x_\pm = \frac{\langle S_+\rangle}{2N} \pm \frac{\langle [S_+, S_z]_+\rangle}{4N(N-1)}. \tag{55}$$

The entanglement between two qubits can be expressed in terms of the concurrence [34]. Since the ground state lies in the maximum spin sector and has real coefficients in the basis $\{|N,m\rangle\}$, one has

$$C = \max\{0, C_y\}, \tag{56}$$

where

$$(N-1)C_y = 1 - \frac{\langle S_y^2\rangle}{N}. \tag{57}$$

Thus, the concurrence needs to be rescaled by the factor $N-1$ (that is, C vanishes $\sim 1/N$ in the thermodynamic limit), with $C_r = (N-1)C$. In the thermodynamic limit, only C_r remains finite:

$$C_r \simeq \begin{cases} 1 - \sqrt{1-\alpha}, & \alpha \leqslant 1; \\ 1 - \sqrt{1 - \frac{1}{\alpha^2}}, & \alpha > 1. \end{cases} \tag{58}$$

This result is consistent with the general formula for the concurrence of a class of symmetric states derived in Ref. [21].

For finite N and at the critical point, Eq. (42) gives

$$C_r \simeq 1 - \frac{4\beta_0}{3\,(2N)^{1/3}}. \tag{59}$$

This shows that, at the critical point, the behavior of the concurrence is modified and C_r scales with N with a critical exponent of 1/3. This result is in agreement with Ref. [18], where the scaling exponents are obtained by arguing that there can be no singularity in any physical quantity at finite size.

IV. FINITE-SIZE SCALING OF THE DICKE MODEL

We can now make use of the expressions of the amplitudes φ_m obtained for uniaxial model in order to discuss the finite-size behavior of the Dicke model.

Whenever we are interested in a qubit observable, that is, whenever the result can be obtained by tracing out the oscillator, the only difference with the uniaxial model is the appearance of exponentials of the form $\exp\{-\lambda^2(m-m')^2/N\omega^2\}$, due to the overlap of different coherent states. This kind of terms modifies the behavior of the spin observables for small N, but for very large N one can expect to obtain very similar behaviors for the Dicke and the uniaxial models. One finds

$$\langle S_x\rangle \rightarrow e^{-\frac{\alpha D}{2N}}\langle S_x\rangle \tag{60}$$

and

$$\langle S_x^2 - S_y^2\rangle \rightarrow e^{-\frac{2\alpha D}{N}}\langle S_x^2 - S_y^2\rangle \tag{61}$$

which coincides, respectively, with Eq. (30) and Eq. (32) in the $D = \delta/\omega \rightarrow 0$ limit. Using these results, one can show, for example, that the ground-state energy at the critical point reads

$$\frac{\varepsilon_0(N)}{N} \simeq -\frac{\delta}{2}\left(1 + \frac{2-D}{2N} - \frac{2\beta_0}{(2N)^{4/3}}\right). \tag{62}$$

Once all of the average values of the spin observables are obtained, it is easy to get expressions for the various entanglement measures. In particular, the rescaled concurrence in the thermodynamic limit reads

$$C_r \simeq \begin{cases} 1 - D\alpha - \sqrt{1-\alpha}, & \alpha \leqslant 1; \\ 1 - \frac{D}{\alpha} - \sqrt{1 - \frac{1}{\alpha^2}}, & 1 < \alpha < \alpha_0, \end{cases} \tag{63}$$

where $\alpha_0 = (1 + D^2)/2D = (\delta^2 + \omega^2)/2\omega\delta$. For finite size, at the critical point one gets

$$C_r \simeq 1 - D - \frac{4\beta_0}{3\,(2N)^{1/3}}. \tag{64}$$

Thus, the concurrence scales with N exactly as in the uniaxial model. Figure 3 shows the C_r both for finite N and for $N \rightarrow \infty$.

The difference between the adiabatic Dicke model and the uniaxial one lies in the presence of the oscillator, which is far detuned from the spins but that can still be excited (because of the presence of the counter-rotating terms in the Hamiltonian) and becomes correlated with the qubits. In particular, the entanglement between the oscillator and the N qubits can be evaluated by the linear entropy which is of the form

$$\tau_N = \eta_N\left(1 - \mathrm{Tr}\{\rho_N^2\}\right), \tag{65}$$

where ρ_N is the reduced density matrix for the N-qubits subsystem, obtained from the ground-state density operator (12)

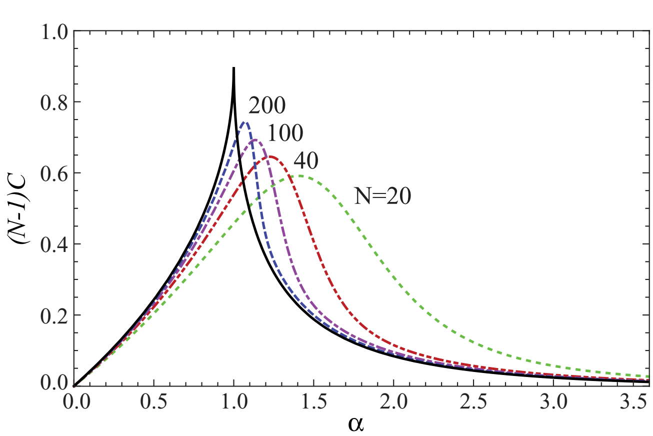

FIG. 3. (Color online) Scaled concurrence for the Dicke model as a function of α, for $D = 0.1$ and for system sizes $N = 10, 20, 40, 100$ and ∞ (bottom to top).

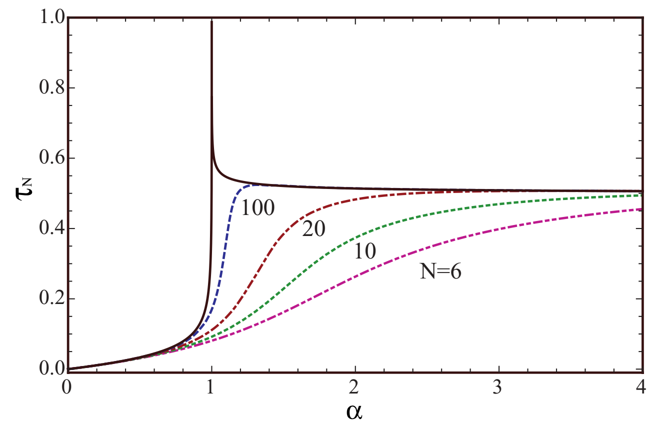

FIG. 4. (Color online) The tangle τ_N between the oscillator and the N qubits as a function of α, for $D = 0.1$ and for system sizes $N = 6, 10, 20, 100$ and ∞ (bottom to top).

by tracing out the field variables

$$\rho_N = \mathrm{Tr}_F\{|\psi_0\rangle\langle\psi_0|\} = \sum_{m_1,m_2=-N}^{N} e^{-\frac{\alpha D}{8N}(m_1-m_2)^2}\varphi_{m_1}\varphi_{m_2}|N,m_1\rangle\langle N,m_2|. \tag{66}$$

Evaluating the trace of ρ_N squared, one has

$$\tau_N = \eta\left(1 - \sum_{m_1,m_2=-N}^{N} e^{-\frac{\alpha D}{4N}(m_1-m_2)^2}\varphi_{m_1}^2\varphi_{m_2}^2\right). \tag{67}$$

In the thermodynamic limit the sum can be computed exactly to get

$$\tau_\infty = \begin{cases} 1 - \left(1 + \frac{D\alpha}{\sqrt{1-\alpha}}\right)^{-\frac{1}{2}} & (\alpha \leqslant 1) \\ 1 - \frac{1}{2}\left(1 + \frac{D}{\sqrt{\alpha^2-1}}\right)^{-\frac{1}{2}} & (\alpha > 1) \end{cases}, \tag{68}$$

which shows a cusp at the critical point, where $\tau_\infty = 1$. Figure 4 shows τ_N both for finite N and for $N \rightarrow \infty$.

When N is very large ($N \gg 4/D^3$), the entanglement scales as

$$\tau_N(\alpha = 1) \sim 1 - K\left(\frac{\pi}{D}\right)^{1/2}\left(\frac{4}{N}\right)^{1/6}, \tag{69}$$

where $K = \frac{1}{4}\int dn\varphi_n^4 \simeq 0.46$, and φ_n is the normalized solution of Eq. (36) for $\alpha = 1$. The fact that the leading term in the $1/N$ expansion of τ_N has exponent $1/6$ implies that the convergence of the series is slower (with respect to those found for other physical quantities) and that for small values of N, subsequent terms should be taken into account.

V. CONCLUDING REMARKS

We have discussed the finite-size critical behavior of the Dicke model for the case of a fast oscillator coupled to many slower qubits. We have derived a direct relationship between this system and the uniaxial model, describing the collective interaction among qubits residing on a fully connected graph. In particular, we have obtained a precise one-to-one correspondence between the energy eigenstates of the two models, showing that their critical behaviors are closely related both in the thermodynamic limit and at finite size. We have then adopted a continuum approximation in order to describe analytically the ground state of the uniaxial model which we used to reobtain all the known features of the model, such as its critical exponents. We have also obtained a full characterization of the $1/N$ expansion (including non universal features such as the prefactors) for many physical observables, among which we dedicated a particular emphasis to the description of the entanglement content of the ground state and to its behavior near the critical point.

Using the solution obtained for the uniaxial model, we have then been able to go back to the original Dicke model and to describe its critical behavior and its scaling properties, again obtaining not only the scaling exponents for various physical quantities, but their entire $1/N$ expansions (of which the first terms are shown and discussed explicitly).

The two models we have discussed obviously differ because of the presence of the bosonic mode in the Dicke case. From a physical point of view this implies that an entanglement is built up not only among the qubits (as in the uniaxial model) but also between qubits and oscillator. Formally, this manifests itself in the fact that the oscillator state is a displaced vacuum (i.e., coherent) state conditioned on the qubit magnetization in the direction of the coupling (i.e., on the value of S_z, in our notation). The presence of this quantum correlation with the oscillator also modifies the entanglement among qubits (formally, because of the presence of some exponential prefactors that essentially suppress entanglement), and this can be interpreted in terms of the monogamy of entanglement.

Apart from this aspect, the two models have many features in common; in particular, their critical behaviors are closely related and their quantum phase transitions are essentially the same, occurring at the same point in parameter space (once the proper relationship between the physical parameters is taken into account).

Our analysis can be of interest for a broad range of applications, ranging from quantum optics [5,35] to the description of solid state nanodevices. In particular, the Hamiltonian of Eq. (2) is written in the form usually employed in the rapidly evolving field of "circuit-QED" to describe superconducting nanocircuits comprising Josephson qubits interacting with an electromagnetic resonator in the dispersive (i.e., off-resonant) regime [36,37]. Due to strong coupling required to reach the critical point, solid-state devices are indeed the only effective candidates to show signatures of the Dicke collective critical behavior, and our results indicate that this goal can be achieved even with a relatively small number of qubits.

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