Matter-wave bistability in coupled atom-molecule quantum gases

Lei Jiang,[1] Han Pu,[1] Andrew Robertson,[2,3] and Hong Y. Ling[2]

[1]*Department of Physics and Astronomy, and Rice Quantum Institute, Rice University, Houston, Texas 77251, USA*

[2]*Department of Physics and Astronomy, Rowan University, Glassboro, New Jersey, 08028-1700, USA*

[3]*Joint Quantum Institute and Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA*

(Received 27 July 2009; published 20 January 2010)

We study the matter-wave bistability in coupled atom-molecule quantum gases, in which heteronuclear molecules are created via an interspecies Feshbach resonance involving either two-species Bose or two-species Fermi atoms at zero temperature. We show that the resonant two-channel Bose model is equivalent to the nondegenerate parametric down-conversion in quantum optics, while the corresponding Fermi model can be mapped to a quantum optics model that describes a single-mode laser field interacting with an ensemble of inhomogeneously broadened two-level atoms. Using these analogies and the fact that both models are subject to the Kerr nonlinearity due to the two-body s-wave collisions, we show that under proper conditions, the population in the molecular state in both models can be made to change with the Feshbach detuning in a bistable fashion.

DOI: 10.1103/PhysRevA.81.013619 PACS number(s): 03.75.Ss, 05.30.Fk, 05.30.Jp

I. INTRODUCTION

The ability to cool and trap neutral atoms down to quantum degenerate regime has created a host of new and exciting problems that are increasingly interdisciplinary, bridging in particular atomic, molecular, and optical physics and condensed-matter physics. The rich knowledge and experience accumulated over the past several decades in these fields have dramatically accelerated the progress of ultracold atomic physics. An example that serves to illustrate how the interdisciplinary fields learn and benefit from each other is the phenomenon of atomic pairing where a bosoinc molecule is coupled to two bosonic or fermionic constituent atoms via Feshbach resonance or photoassociation. So far this is the only viable approach to creating ultracold molecules. It is also an ideal testing ground for studying coupled atom-molecule condensates and the BCS–Bose-Einstein condensate (BEC) crossover [1–3]. The latter is thought to be underlying the mechanism of high-temperature superconductors and has been extensively studied in the realm of condensed-matter physics. In addition, the coupled atom-molecule systems have deep quantum optical analogies [4,5]: bosoinc molecules coupled to bosonic atoms (which we will refer to as the bosonic model in this article) is the matter-wave analog of parametric coupling of photons, which has important applications in generating nonclassical light fields and, more recently, in quantum information science; the system of bosonic molecules coupled to fermionic atoms (which we will refer to as the fermionic model) can be mapped to the Dicke model, where a light field interacts with an ensemble of two-level atoms, a model having fundamental importance in the field of quantum optics.

In this work, we will further explore these quantum optical analogies of the atom-molecule system and focus on the important effects of binary collisional interactions between atoms which are largely ignored in previous studies [4,5]. We show that the atom-atom interaction introduces extra nonlinear terms which, under certain conditions, give rise to matter-wave bistability in both bosonic and fermionic models. Hence, we may establish the connection between the coupled atom-molecule quantum gases and the nonlinear bistable systems [6] that have been extensively studied in the 1980s in the context of nonlinear optics, due both to its fundamental interest and to its many practical applications in fast optical switches, optical memory, laser pulse shaping, etc.

II. BOSONIC MODEL

In what we call the bosonic model, a molecule associated with annihilation operator $\hat{a}_m$ is coupled to two nonidentical atoms labeled as $|\uparrow\rangle$ and $|\downarrow\rangle$ with corresponding annihilation operators $\hat{a}_\uparrow$ and $\hat{a}_\downarrow$, respectively. Here we consider two types of atoms in order to make direct comparisons with the fermionic model to be treated in the next section, for which only unlike fermionic atoms can pair with each other and form a bosonic molecule. Futhermore, in this work we only consider zero-temperature homogeneous case so that all the bosons are condensed into zero center-of-mass momentum states.

The second quantized Hamiltonian reads

$$\hat{H} = \delta \hat{a}_m^\dagger \hat{a}_m + g(\hat{a}_m^\dagger \hat{a}_\uparrow \hat{a}_\downarrow + \text{H.c.}) + \sum_{i,j} \chi_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_i, \quad (1)$$

where the detuning δ represents the energy difference between the molecular and atomic levels which can be tuned by external field, g is the atom-molecule coupling strength, and $\chi_{ij} = \chi_{ji}$ is the s-wave collisional strength between modes i and j. This system has been studied in Ref. [7]. For completeness and better comparison with the fermionic model, we briefly state some of the main results relevant to the focus of this work—matter-wave bistability—and direct readers to Ref. [7] for more details.

For our purpose, we take the standard mean-field approximation and replace operators $\hat{a}_j$ with c-numbers $a_j = \sqrt{N_j} e^{i\varphi_j}$. The mean-field Hamiltonian takes the form

$$H = 2\Lambda(y^2 - y) + 2vy + (1 - 2y)\sqrt{2y}\cos\varphi, \quad (2)$$

where

$$y = 0.5[1 - (N_\uparrow + N_\downarrow)/N] = N_m/N, \quad \varphi = \varphi_\uparrow + \varphi_\downarrow - \varphi_m$$

are a pair of conjugate variables representing the molecular population and the relative phase, respectively. Other quantities are defined as

$$G = g\sqrt{2N},$$
$$\Lambda = N(\chi_{\uparrow\uparrow} + \chi_{\downarrow\downarrow} + \chi_{mm} + 2\chi_{\uparrow\downarrow} - 2\chi_{m\uparrow} - 2\chi_{m\downarrow})/G,$$
$$\nu = [\delta + \chi_{\uparrow\uparrow} + \chi_{\downarrow\downarrow} + (N-1)\chi_{mm} - N\chi_{m\uparrow} - N\chi_{m\downarrow}]/G,$$

with $N \equiv N_\uparrow + N_\downarrow + 2N_m$ a constant of motion representing the total number of atoms, and we have assumed that the number of atoms in states $|\uparrow\rangle$ and $|\downarrow\rangle$ are equal, that is, $N_\uparrow = N_\downarrow$.

The dynamical equations governing φ and y are

$$\dot{y} = \frac{\partial H}{\partial \varphi} = -(1-2y)\sqrt{2y}\sin\varphi,$$
$$\dot{\varphi} = -\frac{\partial H}{\partial y} = -2\Lambda(2y-1) - 2\nu - \sqrt{2y}\left(\frac{1}{2y} - 3\right)\cos\varphi.$$

At equilibrium, we have $\dot{y} = \dot{\varphi} = 0$. The first of these equations shows that $\varphi = \pi$ or 0. We will focus on the stationary states with $\varphi = \pi$, which has lower energies than the ones with $\varphi = 0$.

A. Quantum optical analogy

It is quite clear from the form of the second-quantized Hamiltonian in Eq. (1) that without the collisional terms our model will reduce to the trilinear Hamiltonian describing the nondegenerate parametric down-conversion in quantum optics [8,9]. In this analogy, the molecular mode plays the role of the pump photon, where the two atomic modes are the signal and idler photons, respectively. The collisional terms would correspond to the Kerr-type cubic nonlinearity which will be present in the optical system if the light fields propagate in some nonlinear medium [10].

B. Bistability

In the absence of the collisions or Kerr nonlinearity (i.e., $\Lambda = 0$), the system does not exhibit bistability. This can be seen by studying the properties of the mean-field Hamiltonian H in Eq. (2), which can be simplified as (taking $\varphi = \pi$)

$$H = 2\nu y - (1-2y)\sqrt{2y}. \tag{3}$$

The stationary solution of the equation

$$\frac{\partial H}{\partial y} = 2\nu + 3\sqrt{2y} - 1/\sqrt{2y} = 0, \tag{4}$$

with $0 < y < 0.5$, corresponds to a mixed atom-molecule state in which none of the (atomic or molecular) components is zero so that the relative phase φ is well defined. However, this system is also known to support a pure molecular (atomic) state with $y = 0.5$ (=0) [11], as one can easily verify directly from the Hamiltonian in Eq. (1). Thus, with these considerations, we find, for a given detuning ν, a unique stationary state in the form of

$$y_0(\nu) = \begin{cases} 0.5, & \nu < -1 \\ \frac{1}{18}(-\nu + \sqrt{\nu^2+3})^2, & \nu \geqslant -1. \end{cases} \tag{5}$$

Equation (5) shows that the system goes through a second-order phase transition at the critical detuing $\nu = -1$, which separates a pure molecular state for $\nu < -1$ from a mixture of atom-molecule state for $\nu > -1$.

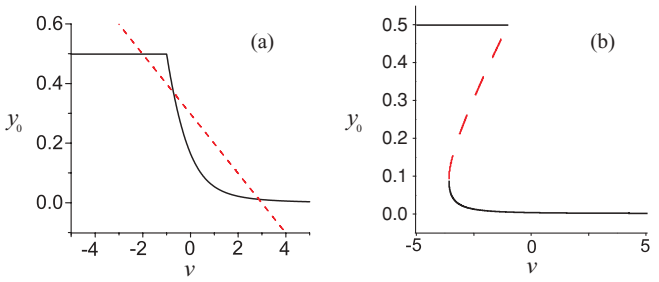

FIG. 1. (Color online) (a) For given Λ and ν, the thick solid line represents $y_0(\nu')$ in Eq. (8) and the thin dashed straight line represents Eq. (7). The intersections are the stationary solutions. Here we take $\Lambda = -5$ and $\nu = -0.4\Lambda$. For this particular value of ν, there are three stationary solutions. (b) Steady-state molecular population y_0 as a function of detuning ν for $\Lambda = -5$. The state represented by the red dashed line is dynamically unstable.

For $\Lambda \neq 0$, using Eq. (2), the stationary condition is given by

$$\frac{\partial H}{\partial y} = 2\nu' + 3\sqrt{2y} - 1/\sqrt{2y} = 0, \tag{6}$$

where we have defined

$$\nu' = \nu + \Lambda(2y-1). \tag{7}$$

Note that Eqs. (4) and (6) have the same form. In other words, we can express the effect of collisions as a nonlinear phase shift for molecules that modifies the detuning ν. Consequently, the stationary solution for $\Lambda \neq 0$ should have the same form as in Eq. (5), only with ν replaced by ν':

$$y_0(\nu') = \begin{cases} 0.5, & \nu' < -1 \\ \frac{1}{18}(-\nu' + \sqrt{\nu'^2+3})^2, & \nu' \geqslant -1. \end{cases} \tag{8}$$

In Eq. (8), y_0 is an implicit function of the detuning ν. To find the explicit dependence of y_0 on ν, we can use the graphic method as illustrated in Fig. 1(a). For the example given, we obtain three stationary states. The explicit dependence of y_0 on ν is shown in Fig. 1(b). In a certain region, there are three stationary solutions. Further analysis shows that the middle solution is dynamically unstable and the other two are stable solutions [7]. Such a behavior is typical in bistable systems [6].

Figure 1 also shows that, in order to have multiple stationary solutions, the slope of the straight line (given by $1/2\Lambda$) must be negative and cannot be too steep. More specifically, the slope of the straight line has to be larger than the slope of the curve at $\nu = -1$, and this leads to the condition

$$\Lambda < -1, \tag{9}$$

in order for the system to exhibit bistability.

III. FERMIONIC MODEL

In the fermionic model, we denote $\hat{a}_{\mathbf{k},\sigma}$ as the annihilation operator for an atom with spin $\sigma(=\uparrow, \downarrow)$, momentum $\hbar\mathbf{k}$, and energy $\epsilon_k = \hbar^2k^2/(2m)$, and as before denote $\hat{a}_m$ as the annihilation operator for a molecule in state $|m\rangle$ with zero

momentum. The second quantized Hamiltonian reads

$$\hat{H} = \sum_{\mathbf{k},\sigma} \epsilon_k \hat{a}^\dagger_{\mathbf{k}\sigma} \hat{a}_{\mathbf{k}\sigma} + U \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \hat{a}^\dagger_{\mathbf{k}\uparrow} \hat{a}^\dagger_{-\mathbf{k}+\mathbf{q}\downarrow} \hat{a}_{-\mathbf{k}'+\mathbf{q}\downarrow} \hat{a}_{\mathbf{k}'\uparrow} + \nu \hat{a}^\dagger_m \hat{a}_m + \frac{g}{\sqrt{V}} \sum_{\mathbf{k}} (\hat{a}^\dagger_m \hat{a}_{-\mathbf{k}\downarrow} \hat{a}_{\mathbf{k}\uparrow} + \text{H.c.}), \quad (10)$$

where V is the quantization volume. Hamiltonian (10) has the form of the two-channel model of BCS-BEC crossover where only the condensed molecule part is considered [3,12]. Following the Hartree-Fock-Bogoliubov mean-field approach [13] by dividing the two-body collision into a part related to the BCS gap potential $\Delta = Up$, where

$$p = \sum_{\mathbf{k}} \langle \hat{a}_{-\mathbf{k}\downarrow} \hat{a}_{\mathbf{k}\uparrow} \rangle / V,$$

and a part related to the Hartree potential

$$V_h = U \sum_{\mathbf{k}\sigma} \langle \hat{a}^\dagger_{\mathbf{k}\sigma} \hat{a}_{\mathbf{k}\sigma} \rangle / (2V), \quad (11)$$

where we again assume equal population in $|\uparrow\rangle$ and $|\downarrow\rangle$ atomic states, i.e., $\langle \hat{a}^\dagger_{\mathbf{k}\uparrow} \hat{a}_{\mathbf{k}\uparrow} \rangle = \langle \hat{a}^\dagger_{\mathbf{k}\downarrow} \hat{a}_{\mathbf{k}\downarrow} \rangle$, we may express the Hamiltonian as

$$\hat{H} = \sum_{\mathbf{k},\sigma} (\epsilon_k + V_h) \hat{a}^\dagger_{\mathbf{k}\sigma} \hat{a}_{\mathbf{k}\sigma} + \nu \hat{a}^\dagger_m \hat{a}_m + \sum_{\mathbf{k}} [(Up + g\hat{a}_m/\sqrt{V}) \hat{a}^\dagger_{\mathbf{k}\uparrow} \hat{a}^\dagger_{-\mathbf{k}\downarrow} + \text{H.c.}]. \quad (12)$$

Defining $\hat{N} = 2\hat{a}^\dagger_m \hat{a}_m + \sum_{\mathbf{k},\sigma} \hat{a}^\dagger_{\mathbf{k}\sigma} \hat{a}_{\mathbf{k}\sigma}$ as the number operator, we may rewrite the term proportional to V_h in (12) as

$$\sum_{\mathbf{k},\sigma} V_h \hat{a}^\dagger_{\mathbf{k}\sigma} \hat{a}_{\mathbf{k}\sigma} = V_h (\hat{N} - 2\hat{a}^\dagger_m \hat{a}_m) = V_h \hat{N} - (Un - 2U \langle \hat{a}^\dagger_m \hat{a}_m \rangle / V) \hat{a}^\dagger_m \hat{a}_m, \quad (13)$$

where $n = \langle \hat{N} \rangle / V$ is the constant total atom number density. In our derivation, V_h arises from the two-body atom-atom collision. In general, additional terms representing atom-molecule and molecule-molecule collisions are also present. These additional terms will modify the coefficient U in the definition of V_h [Eq. (11)], which is the counterpart of Λ in the bosonic model, but the general form of Eq. (13) will remain valid [14]. In the following, we will refer to this term as the collisional term. Through Eq. (13), we have expressed the effect of the two-body collisions as a nonlinear energy shift of the molecules (along with an energy bias $V_h N$, where we have replaced the number operator by its expectation value as in the stardard mean-field treatment), in complete analogy with the bosonic model. We remark that in the usual one-channel model of the mean-field BCS theory valid when the molecular population is negligible, the collisional term just represents an unimportant constant energy shift.

As usual, $\hat{a}_{\mathbf{k}\sigma}(t)$ and $\hat{a}_m(t)$ obey the Heisenberg equations of motion based on Hamiltonian (12). By replacing Bose operator $\hat{a}_m$ with the related c-number $c = \langle \hat{b} \rangle / \sqrt{V}$ and Fermi operators $\hat{a}_{\mathbf{k}\sigma}(t)$ with the familiar $u_k(t)$ and $v_k(t)$ parameters through the Bogoliubov transformation $\hat{a}_{\mathbf{k}\uparrow} = u^*_k \hat{\alpha}_{\mathbf{k}\uparrow} + v_k \hat{\alpha}^\dagger_{-\mathbf{k}\downarrow}$ and $\hat{a}^\dagger_{-\mathbf{k}\downarrow} = -v^*_k \hat{\alpha}_{\mathbf{k}\uparrow} + u_k \hat{\alpha}^\dagger_{-\mathbf{k}\downarrow}$, where $\hat{\alpha}_{\mathbf{k}\sigma}$ are the Fermi quasiparticle operators, we arrive at the following set of mean-field equations of motion:

$$i\hbar \dot{c} = \nu_e c + gp, \quad (14a)$$

$$i\hbar \dot{u}_k = -\epsilon_k u_k + \Delta_e v_k, \quad (14b)$$

$$i\hbar \dot{v}_k = \Delta_e u_k + \epsilon_k v_k, \quad (14c)$$

where $p = \sum_{\mathbf{k}} u^*_k v_k / V$, $\Delta_e = gc + Up$, and

$$\nu_e = \nu - Un + 2U |c|^2 \quad (15)$$

is the effective detuning which contains a Kerr nonlinear term $2U|c|^2$ whose origin can be traced to the two-body collisional shift. This set of equations describes the dynamics at zero temperature where the state of the system can be described as the quasiparticle vacuum.

A. Quantum optical analog

In several previous studies where the collisional term is neglected, it has been pointed out that the fermionic model can be mapped to the Dicke model in quantum optics [4,15], as schematically shown in Fig. 2 (see later in this article for details). In fact, this model was recently shown to display collective dynamics similar to photon echo and solitonlike oscillations in transient collective coherent optics [16]. Such a connection can be traced to Anderson's spin analogy [17] for the BCS problem.

To show what is the quantum optical analogy of the collisional term, let us rewrite Eqs. (14) in a form more familiar in cavity optics. To this end, we first introduce a set of new variables,

$$P_k = 2u^*_k v_k, \quad D_k = |u_k|^2 - |v_k|^2, \quad \mathcal{E}_L = 2i\Delta_e,$$

and recast Eqs. (14b) and (14c) as

$$\hbar \dot{P}_k = -i2\epsilon_k P_k - \mathcal{E}_L D_k, \quad (16a)$$

$$\hbar \dot{D}_k = (\mathcal{E}^*_L P_k + \mathcal{E}_L P^*_k)/2. \quad (16b)$$

Interpreting P_k and D_k as the microscopic polarization and population inversion, respectively, Eqs. (16) then become the optical Bloch equation that describes the interaction between a local electromagnetic field $\mathcal{E}_L$ and a fictitious two-level atom, characterized by a transition energy $2\epsilon_k$ [18]. This analogy is

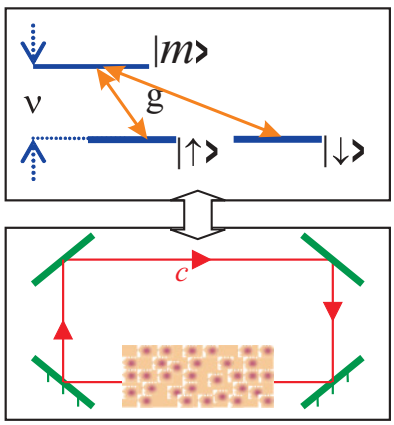

FIG. 2. (Color online) Mapping of the two-channel resonant Fermi superfluid model to the Dicke model. The bosonic molecules and the fermionic atoms in the former are mapped to the cavity laser field and an ensemble of two-level atoms in the latter, respectively. See text for details.

consistent with the fact that there exists a one-to-one mapping between pairs of fermion operators and Pauli matrices when the BCS pairing mechanism is taken into account [17].

In this optical analogy, the local electric field $\mathcal{E}_L = \mathcal{E} + \mathcal{E}_i$ contains two contributions because of $\Delta_e = gc + Up$. The first of these ($\mathcal{E} = i2gc$) is equivalent to an average macroscopic field, whose dynamics is described by Eq. (14a), which can now be interpreted as the Maxwell's equation for the cavity field $\mathcal{E}$ with cavity detuning ν_e, driven by a macroscopic polarization density $p = \sum_{\mathbf{k}} P_k/(2V)$ of an inhomogeneously broadened medium (see Fig. 2). The second part $\mathcal{E}_i = i2Up$ may be regarded as the internal field at the atom due to the collective dipole polarization of the nearby two-level atoms in the ensemble. As such, $\mathcal{E}_L = \mathcal{E} + \mathcal{E}_i$ here bears a direct analogy to the Lorentz-Lorenz relation in optics [19]. Note that had the collisional term been neglected (i.e., $U = 0$), there would have been no internal field contribution, nor would there have been the Kerr nonlinearity in the equation for the bosonic mode. For $U \neq 0$, both of these terms will be present. Under such a circumstance, Eqs. (14a) and (16) represent the generalized optical-Bloch equations in which the Lorentz-Lorenz relation is explicitly incorporated [20] and hence can lead to interesting nonlinear phenomena just as they do in optical systems.

B. Bistability

Having established this analogy, we now look for the steady-state solution from Eqs. (14a) and (16). As is well known, the operation frequency of a laser field is not known *a priori* but is established through the so-called mode pulling—the dynamical competition between atomic and cavity resonances. A similar argument holds for the molecular field c. For this reason, we adopt the following steady-state ansatz:

$$c \to ce^{-2i\mu t/\hbar}, \quad P_k \to P_k e^{-2i\mu t/\hbar}, \quad D_k \to D_k,$$

where the same symbols are used for both dynamical and steady-state variables for notational simplicity. The molecular chemical potential, 2μ, is just the corresponding lasing frequency in the cavity optics model. From the steady-state equations obtained by inserting this stationary ansatz into Eqs. (14a) and (16), we can easily find that there always exists (a) a trivial solution or a "nonlasing" state with $\Delta_e = 0$ or equivalently $c = 0$, which corresponds to the nonsuperfluid normal Fermi sea; and (b) a nontrivial solution with its μ, Δ_e, and c determinedself-consistently from the gap equation

$$\frac{1}{U - g^2/(\nu_e - 2\mu)} = -\frac{1}{2V}\sum_{\mathbf{k}} \frac{1}{E_k}, \tag{17}$$

with $E_k = \sqrt{(\epsilon_k - \mu)^2 + \Delta_e^2}$, the number equation

$$2|c|^2 + \frac{1}{V}\sum_{\mathbf{k}} \left(1 - \frac{\epsilon_k - \mu}{E_k}\right) = n, \tag{18}$$

and an auxiliary relation

$$|g\Delta_e| = |c(\nu_e - 2\mu)[U - g^2/(\nu_e - 2\mu)]|. \tag{19}$$

The integral in the gap equation (17), under the assumption of contact interaction, is known to be ultraviolet divergent. To eliminate this problem, we renormalize the interaction strength U and g, as well as the detuning ν in (17), while U in the collisional term is replaced by the background interaction strength U_0 [21,22].

Note that there exists, in the single-mode inhomogeneously broadened laser theory [23], a similar set of steady-state integral equations, which, due to lasers being open systems, are obtained under different considerations. For example, the requirement that the cavity loss balance the saturated gain leads to the "gap" equation, whose primary role is to limit the laser intensity; while the phase-matching condition translates into the "number" equation, whose main responsibility is to assign the amount of mode pulling of the laser field relative to the cavity resonance.

An alternative way to derive Eqs. (17)–(19) is from the energy density. The zero-temperature energy density $f(\Delta_e, c, \mu) \equiv \langle \hat{H} \rangle / V$ can be calculated using Hamiltonian (12) and the Bogoliubov transformation as [22]

$$f = \sum_{\mathbf{k}} \frac{\epsilon_k - \mu - E_k}{V} - \frac{(\Delta_e - gc)^2}{U} + (\nu_e - 2\mu)|c|^2 + \mu n. \tag{20}$$

The extremum conditions $\partial f/\partial \Delta_e = \partial f/\partial c = 0$ lead to Eqs. (17)–(19), respectively, while the condition $\partial f/\partial \mu = 0$ results in the number equation (18).

Figure 3 illustrates the energy density in the $|c|^2$-Δ_e plane for different detuning ν. For any given pair of (c, Δ_e), μ is

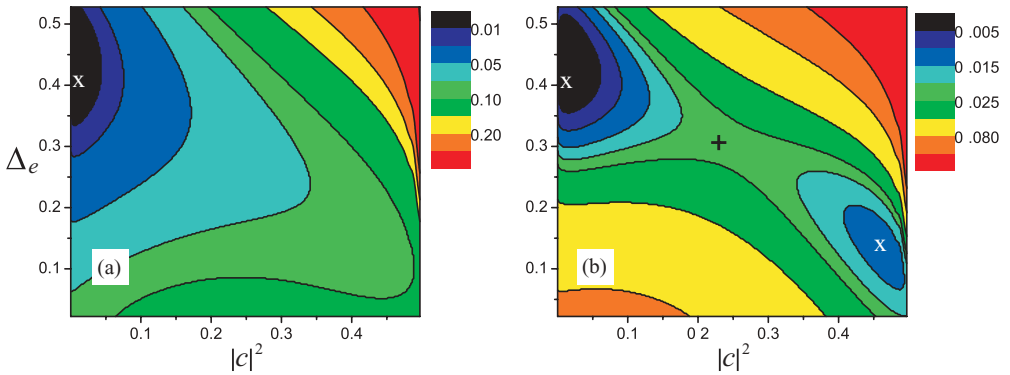

FIG. 3. (Color online) Free energy density f as a function of Δ_e and $|c|^2$ at $\nu = 0.2$ (a) and $\nu = 0.02$ (b). Extremum points are indicated with "x" (minimum) and "+" (saddle point). f, Δ_e, and ν are all in units of $E_F = (3\pi^2 n)^{2/3}/(2m)$, the Fermi energy of the noninteraction system. In all the examples shown in this article, the physical parameters corresponding to g_0 and U_0 are 1.2 $E_F/k_F^{3/2}$ and $-60.7E_F/k_F^3$, respectively.

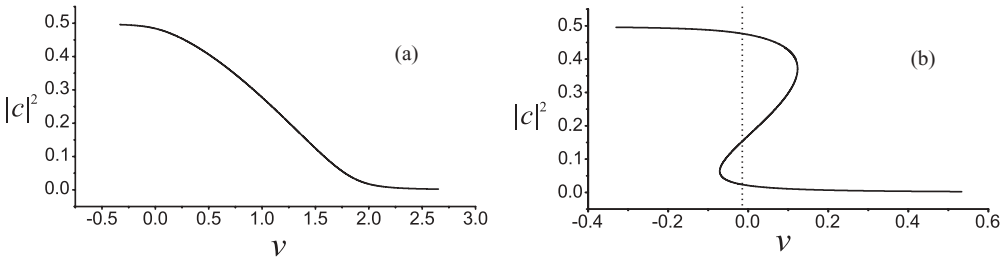

FIG. 4. Molecular population $|c|^2$ as a function of detuning. The vertical line in (b) indicates the critical point of a first-order phase transition. In (b), the collisional term is included, while it is neglected in (a).

calculated self-consistently using the number equation (18). Typically, f has only one extremum, which is a minimum point, as shown in Fig. 3(a). However, in the regime $\nu \in (-0.08, 0.13)E_F$, f possesses three extrema: two of them are local minima and the third a saddle point. An example with $\nu = 0.02$ is shown in Fig. 3(b).

To gain more insights into the bistable behavior, we may carry out an analogous analysis as in Sec. II B. In the absence of the collisional term, steady-state molecular population $|c|^2$ is a smooth monotonically decreasing function of ν and the system does not exhibit bistability: As ν increases, molecules decompose into atoms. This is shown in Fig. 4(a). When collisional term is included, the relevant equations of motion maintain the same forms if we substitute ν with

$$\nu' = \nu + 2U_0|c|^2. \tag{21}$$

Hence, the solution $|c|^2$ as a function of ν' is represented by the same curve as in Fig. 4(a). To find $|c|^2$ as a function of ν, we need to find the intersections between this curve and the straight line representing Eq. (21). In direct analogy to the graphic method in Fig. 1, for U_0 sufficiently large and negative, these two curves have three intersections and the system exhibits bistability. One example is shown in Fig. 4(b). The vertical line in Fig. 4(b) indicates the critical point of a first-order phase transition: across this line, the ground state jumps from the upper branch to the lower one. For the parameters used, this occurs at $\nu_c = -0.01E_F$.

To check the stability of these steady states, we have solved the dynamical equations (14) using the slightly perturbed steady-state solution as the initial condition. From the dynamical evolution of the system, one can see that, just like in the bosonic model, the states in the upper and lower branches are dynamically stable: when slightly perturbed, they exhibit damped oscillations around their equilibrium values. These oscillations can be further understood from the excitation spectrum of the corresponding steady state. This can be done using a linear stability analysis, which is also the standard tool for studying laser instabilities [23,24]. The spectrum is found to contain a discrete part which determines the oscillation frequencies, and a continuous part which contributes to the damping of these oscillations at a much longer time scale [25]. By contrast, the states in the middle branch are unstable as small perturbations will lead to large departures.

C. Dynamics

The bistability has important ramifications in atom-molecule conversion dynamics. When the collisional term is unimportant and negligible, one can easily create bosonic molecules from fermionic atoms by adiabatically sweeping the Feshbach detuning across the resonance. As long as the sweeping speed is sufficiently slow, the molecular population will follow the steady-state curve as shown in Fig. 4(a). By contrast, when bistability induced by the collisional term occurs, the adiabaticity condition will necessarily break down. Figure 5 displays the dynamical evolution of the bosonic population when the detuning is swept starting either from a large positive or a large negative value. We can see that the steady-state curve can be followed up to the point where the stable states of the upper and lower branches and the unstable states of the middle branch join each other (indicated by ν_1 and ν_2 in Fig. 5), where the population suddenly jumps between the two stable branches. The subsequent population dynamics depends not only on the steady-state structure but also the whole energy landscape as well. In the absence of dissipation as in our model, the system may not completely reach the steady state. Note that the critical detuning ν_c for the first-order phase transition as indicated by the vertical line in Fig. 4(b) lies between ν_1 and ν_2. The dynamical population curve thus exhibits hysteresis in the vicinity of the first-order phase transition. In this way, by tuning the detuning in the vicinity of ν_1 or ν_2, an atom-molecule switch can be realized. Similar behavior is also found in the bosonic model.

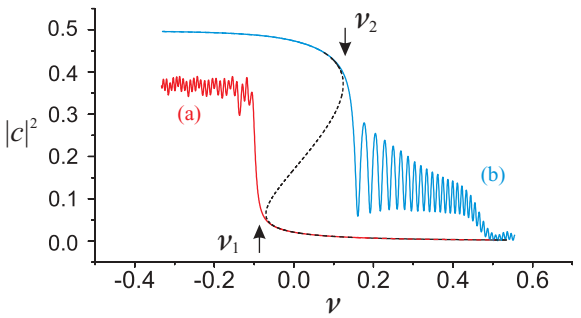

FIG. 5. (Color online) Dynamics of atom-molecule conversion as illustrated by the molecular population when the detuning ν is slowly swept. Curve (a) is obtained by sweeping ν from positive to negative values, while curve (b) is obtained by sweeping ν in the opposite direction. The dotted line is the steady-state molecular population, the same as in Fig. 4(b).

IV. CONCLUSION

In conclusion, we have studied the matter-wave bistability in coupled atom-molecule quantum gases in both the bosonic and the fermionic models. These two cases can be mapped to two very different quantum optical models: parametric down-conversion in the former and the generalized Dicke model in the latter. Nevertheless, one important common feature for both cases is that bistability can be induced by collisional interactions which give rise to Kerr nonlinearity. We hope that our work will motivate experimental efforts in demonstrating the matter-wave bistability we predicted here.

ACKNOWLEDGMENTS

We thank Dr. Satyan Bhongale for useful discussions. H.P. acknowledges support from NSF, the Robert A. Welch Foundation (Grant No. C-1669), and the W. M. Keck Foundation, and H.Y.L. acknowledges support from NSF and ARO.

[1] E. Timmermans, K. Furuya, P. W. Milonni, and A. K. Kerman, Phys. Lett. **A285**, 228 (2001); M. Holland, S. J. J. M. F. Kokkelmans, M. L. Chiofalo, and R. Walser, Phys. Rev. Lett. **87**, 120406 (2001).

[2] Q. Chen, J. Stajic, S. Tan, and K. Levin, Phys. Rep. **412**, 1 (2005).

[3] R. B. Diener, and T.-L. Ho, e-print arXiv:cond-mat/0405174.

[4] M. W. Jack and H. Pu, Phys. Rev. A **72**, 063625 (2005).

[5] I. Tikhonenkov, E. Pazy, Y. B. Band, and A. Vardi, Phys. Rev. A **77**, 063624 (2008); M. Ögren, C. M. Savage, and K. V. Kheruntsyan, *ibid*. **79**, 043624 (2009).

[6] H. M. Gibbs, *Controlling Light with Light* (Academic Press, Orlando, FL, 1985).

[7] L. Zhou, W. Zhang, H. Y. Ling, L. Jiang, and H. Pu, Phys. Rev. A **75**, 043603 (2007).

[8] G. Drobný, I. Jex, and V. Bužek, Phys. Rev. A **48**, 569 (1993).

[9] K.-P. Marzlin and J. Audretsch, Phys. Rev. A **57**, 1333 (1998).

[10] Y. R. Shen, *The Principle of Nonlinear Optics* (Wiley Interscience, New York, 1984).

[11] L. Zhou, J. Qian, H. Pu, W. Zhang, and H. Y. Ling, Phys. Rev. A **78**, 053612 (2008).

[12] M. L. Chiofalo, S. J. J. M. F. Kokkelmans, J. N. Milstein, and M. J. Holland, Phys. Rev. Lett. **88**, 090402 (2002).

[13] P.-G. de Gennes, *Superconductivity of Metals and Alloys* (Addison-Wesley, New York, 1989).

[14] Hamiltonian (10) has been widely used in the study of the two-channel model. The atom-molecule and molecule-molecule collision can be taken into account by including the following terms in the Hamiltonian: $U_{am}\hat{a}_m^\dagger\hat{a}_m\sum_{\mathbf{k},\sigma}\hat{a}_{\mathbf{k},\sigma}^\dagger\hat{a}_{\mathbf{k},\sigma}+U_{mm}\hat{a}_m^\dagger\hat{a}_m^\dagger\hat{a}_m\hat{a}_m$. With the addition of these terms, $\hat{N}$ remains a constant of motion. It is straightforward to show that the effect of these terms is to modify the coefficients of the Kerr nonlinear term in Eqs. (13) and (15) from $2U$ to $2(U-U_{am}+U_{mm})$. The essential physics therefore remains valid.

[15] C. P. Search, W. Zhang, and P. Meystre, Phys. Rev. Lett. **91**, 190401 (2003).

[16] A. V. Andreev, V. Gurarie, and L. Radzihovsky, Phys. Rev. Lett. **93**, 130402 (2004); R. A. Barankov and L. S. Levitov, *ibid*. **93**, 130403 (2004).

[17] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).

[18] L. Allen, and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975).

[19] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd edition.

[20] C. M. Bowden and J. P. Dowling, Phys. Rev. A **47**, 1247 (1993).

[21] S. J. J. M. F. Kokkelmans, J. N. Milstein, M. L. Chiofalo, R. Walser, and M. J. Holland, Phys. Rev. A **65**, 053617 (2002).

[22] M. Houbiers, R. Ferwerda, H. T. C. Stoof, W. I. McAlexander, C. A. Sackett, and R. G. Hulet, Phys. Rev. A **56**, 4864 (1997).

[23] N. B. Abraham, L. A. Lugiato, P. Mandel, L. M. Narducci, and D. K. Bandy, J. Opt. Soc. Am. B **2**, 35 (1985).

[24] L. M. Narducci and N. B. Abraham, *Laser Physics and Laser Instabilities* (World Scientific, Singapore, 1988).

[25] A. Robertson, L. Jiang, H. Pu, W. Zhang, and H. Y. Ling, Phys. Rev. Lett. **99**, 250404 (2007).