Lorentz force on an electron in a strong plane-wave laser field and the low-frequency limit for ionization

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A motion of a classical free charge in an electromagnetic plane wave can be found exactly in a fully relativistic case. I have found an approximate non-parametric form of the suitable equations of motion. In a linearly polarized wave, in the simplest frame of reference, the charge moves along the well-known figure-eight path. I have numerically calculated the Lorentz force acting on the charge as a function of time. By virtue of this, for the low-frequency ionization (or detachment) rate, I discuss a manifestation of nondipole and relativistic effects. When intensity of the plane wave increases, these effects can first appear in angular distributions, then in spectra of outgoing electrons, but have quite little effect on total ionization rates. I try to give an explanation of the latter fact.

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I. INTRODUCTION

Let us consider a classical point charge interacting with an arbitrary intense electromagnetic plane-wave field. The charge can move with a relativistic velocity. As I shall demonstrate in this article, studying such motion is important from the point of view of theories describing ionization (or detachment) in strong laser fields. In Ref. [1] (Sec. 48, p. 134) there are exact solutions to suitable equations of motion in the simplest frame of reference (i.e., in which the charge is at rest on the average). The solutions for a linear polarization and for a circular polarization of the plane wave [1] have been generalized recently [2]. My result for an electron in the laser field of any elliptical polarization is the following (in the present work I use atomic units: $\hbar = e = m_e = 1$, and I substitute explicitly -1 for the electronic charge):

$$x = \frac{a^2}{8c\,\omega\varepsilon^2}\cos\delta\sin 2(\omega t - kx) \equiv x_0 \sin 2(\omega t - kx), \quad (1a)$$

$$y = \mp\frac{a}{\omega\varepsilon}\sin(\delta/2)\cos(\omega t - kx) \equiv y_0\cos(\omega t - kx), \quad (1b)$$

$$z = \frac{a}{\omega\varepsilon}\cos(\delta/2)\sin(\omega t - kx) \equiv z_0\sin(\omega t - kx). \quad (1c)$$

In Eqs. (1) I have assumed that the laser field propagates along the x axis, and its wave vector is $k = \omega/c$ (where ω is the laser frequency and c is the velocity of light). Also, a is the amplitude of the vector potential describing the field, δ is the ellipticity parameter ($\delta \in [0, \pi/2]$; for the linear polarization, $\delta = 0$, and for the circular polarization, $\delta = \pi/2$), $\varepsilon = \sqrt{c^2 + a^2/2c^2}$, and the signs $\mp$ correspond to two different helicities. The electric field vector ($\vec{E} = -c^{-1}\partial\vec{A}/\partial t$) has the amplitude $E_0 = (a\omega/c)\cos(\delta/2)$. (See Ref. [2] for more detail.) In Eqs. (1) I have also defined x_0, y_0, and z_0—the amplitudes of motion along the respective axes.

My work is organized as follows. In Sec. II I solve Eqs. (1) analytically in an approximate way, and I compute also a velocity and an acceleration of the electron for any ellipticity parameter of the laser field. In Sec. III I discuss a general form of "tunnelinglike" formula, and I numerically calculate the Lorentz force acting on the electron in the linearly polarized plane-wave laser field. Nonrelativistic dipole and nonrelativistic nondipole approximations to this force are compared with the fully relativistic result. In Sec. IV I derive simple nonrelativistic nondipole formulas (approximately valid for low-frequency fields) describing total ionization rates, for both linear and circular polarizations. Final remarks and conclusions are given in Sec. V.

II. SOLUTIONS TO CLASSICAL EQUATIONS OF MOTION

Equations (1) are nonlinear and in general require a numerical treatment to find x, y, and z as functions of t. However, when the condition

$$kx_0 \ll 1 \quad (2)$$

is satisfied, one can expand the right-hand sides of Eqs. (1) in a Taylor series. Later in this work I assume that Eq. (2) is valid. For any finite ω, E_0, one can easily show that $kx_0 < 1/4$ always [3]. The latter value is for $\omega = \text{const}$, $E_0 \to \infty$ or $E_0 = \text{const}$, $\omega \to 0$. Neglecting terms of the order of $(kx_0)^2$ and higher, one finds the following approximate solutions to Eqs. (1):

$$x(t) = x_0\sin 2\omega t(1 - 2kx_0\cos 2\omega t), \quad (3a)$$

$$y(t) = y_0\cos\omega t(1 + kx_0 - kx_0\cos 2\omega t), \quad (3b)$$

$$z(t) = z_0\sin\omega t(1 - kx_0 - kx_0\cos 2\omega t). \quad (3c)$$

Then one can easily find components of the velocity vector of the electron,

$$\dot{x}(t) = 2\omega x_0(\cos 2\omega t - 2kx_0\cos 4\omega t), \quad (4a)$$

$$\dot{y}(t) = -\omega y_0\sin\omega t(1 - kx_0 - 3kx_0\cos 2\omega t), \quad (4b)$$

$$\dot{z}(t) = \omega z_0\cos\omega t(1 + kx_0 - 3kx_0\cos 2\omega t), \quad (4c)$$

and components of its acceleration vector,

$$\ddot{x}(t) = -4\omega^2 x_0\sin 2\omega t(1 - 8kx_0\cos 2\omega t), \quad (5a)$$

$$\ddot{y}(t) = -\omega^2 y_0\cos\omega t(1 + 5kx_0 - 9kx_0\cos 2\omega t), \quad (5b)$$

$$\ddot{z}(t) = -\omega^2 z_0\sin\omega t(1 - 5kx_0 - 9kx_0\cos 2\omega t). \quad (5c)$$

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Equations (5) determine the Lorentz force (coming from both the electric $\vec{E}$ and the magnetic $\vec{B}$ components of the laser field) acting on the electron as a function of time. There is an extensive discussion of a classical relativistic dynamics in a strong plane-wave field in the old article by Sarachik and Schappert [4]. Equations (3)–(5) are one of two main (analytical) results of this article. Let us denote a position of the electron (in the simplest frame of reference) as $\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$ (where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are real unit vectors). Then the Lorentz force (defined as a time derivative of a relativistic momentum) is given by

$$\vec{F}_{\rm rel}(t) = \frac{1}{\sqrt{1-\frac{\dot{\vec{r}}(t)^2}{c^2}}}\ddot{\vec{r}}(t) + \frac{\dot{\vec{r}}(t)\cdot\ddot{\vec{r}}(t)}{c^2\left[1-\frac{\dot{\vec{r}}(t)^2}{c^2}\right]^{3/2}}\dot{\vec{r}}(t). \tag{6}$$

[There is a scalar product of $\dot{\vec{r}}(t)$ and $\ddot{\vec{r}}(t)$ in a numerator of the second term in Eq. (6).] If $|\dot{\vec{r}}(t)| \ll c$, one obtains from Eq. (6) the nonrelativistic approximation to the Lorentz force:

$$\vec{F}_{\rm nonrel}(t) = \ddot{\vec{r}}(t). \tag{7}$$

Equation (7) contains Eqs. (5) and takes into account nondipole effects. If one also puts $x_0 = 0$ in Eqs. (3)–(5), one obtains from Eq. (7) the nonrelativistic Lorentz force in the dipole (or long-wavelength) approximation. One usually assumes that the dipole approximation is valid when $k = \omega/c \ll 1$ a.u. However, according to Reiss [5], nondipole effects should appear for the H(1s) atom, if

$$x_0 \geqslant\sim 1 \text{ a.u.} \tag{8}$$

(x_0 here is denoted as β_0 in Ref. [5]). I agree that nondipole effects can appear in angular distributions of photoelectrons from strong-field ionization, if criterion (8) is obeyed. Nevertheless, if one looks at a total ionization (or detachment) rate, Eq. (8) may be too restrictive, particularly in the low-frequency limit of strong-field ionization, as I shall demonstrate later in this article.

III. LORENTZ FORCE AND TUNNELINGLIKE FORMULA

Let us consider now the linear polarization ($\delta = 0$), which is of most experimental interest. From Eqs. (1) one obtains $y(t) = 0$, and the electron moves in the x-z plane. The motion takes place along the figure-eight path $ABCDAEFGA$ (shown schematically in Fig. 1), which is covered every laser cycle. It follows from Eqs. (1) that the ratio x_0/z_0 grows monotonically with increasing a laser field intensity I (for the linear polarization $I = E_0^2$) from 0 (for $I = 0$) to $\sqrt{2}/8 \approx 0.177$ (for $I \to \infty$). For strong laser fields, almost for all t, the distance $\sqrt{x(t)^2 + y(t)^2 + z(t)^2}$ (calculated from Eqs. (3) for any elliptical polarization) is much larger than a radius of an atom (or ion). Therefore, total forces acting on the ionized (or detached) electron during its motion in strong laser fields are nearly equal to those of a free motion, because binding forces (Coulomb or short-range) are much weaker. Moreover, according to the quasistatic limit of the ionization theory by Keldysh [6], the electron escapes when both the $\vec{E}$ and the $\vec{B}$ fields are close to their maximum values during the laser cycle. If the laser frequency ω is much lower than a characteristic atomic frequency, the Keldysh adiabaticity parameter γ [6] obeys the condition

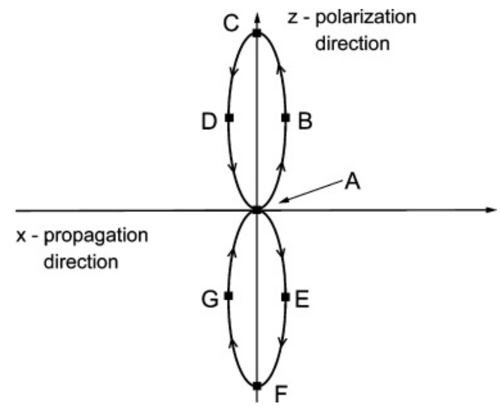

FIG. 1. Motion of the charge along the figure-eight path (shown schematically) in a linearly polarized plane-wave laser field (in the simplest frame of reference, in the fully relativistic case; see the text for more detail).

$$\gamma = \frac{\omega\sqrt{2E_B}}{E_0} \ll 1. \tag{9}$$

E_B denotes here a binding energy of the atom or ion. Later in this work I assume that the inequality (9) is satisfied. In the limit $\omega \to 0$ (then also $\gamma \to 0$, if $E_0 = \text{const}$) the ionization rate Γ is approximately given by an expression of the type

$$\Gamma \approx f(E_0)\exp\left(\frac{-C}{E_0}\right), \tag{10}$$

where $C > 0$ is a constant (or nearly a constant), $f(E_0)$ is a relatively slowly varying function of E_0, and $\exp(-C/E_0)$ grows rapidly with E_0. [Both $f(E_0)$ and C depend also on E_B and the initial-state wave function.] In Keldysh's theory (see Eq. (20) of Ref. [6]) the pre-exponential factor $f(E_0)$ is not the same as in the static-field theories [7–9] when the ionization rate is averaged over the cycle of the electric field [$E(t) = E_0 \sin\omega t$]. However, the exponential factor $\exp(-C/E_0)$ remains the same. One should stress that the well-known dependence (10) is typical not only for the early tunneling theories [6–15], where one usually assumes that $E_0 \ll E_{\rm BSI}$ (BSI denotes barrier-suppression ionization). Dörr *et al.* [16] investigated the static-field limit in multiphoton ionization with the help of the Floquet method. For the linear polarization they found that Eq. (10) describes the ionization rate, but does not account for intermediate resonances (which occur for some specific values of ω and E_0). Ilkov *et al.* [17] confirmed experimentally, that Eq. (10) is approximately valid for $\gamma < 0.5$ and $E_0 < E_{\rm BSI}$. Buerke and Meyerhofer [18] confirmed experimentally a high accuracy of the semiclassical approach [10–15] in the tunneling regime. Scrinzi *et al.* [19] calculated exactly the static-field ionization rate $\Gamma_{\rm stat}$ up to $E_0 = 1$ a.u. for the H(1s) atom. When this $\Gamma_{\rm stat}$ is averaged over the cycle of the electric field [$E(t) = E_0 \sin\omega t$], one numerically obtains the ionization rate $\Gamma^{\rm av}_{\rm stat}$, which is really of the type of Eq. (10). For 0.03 a.u. $\leqslant E_0 \leqslant 1$ a.u., $\Gamma^{\rm av}_{\rm stat}$ is equal to the averaged ionization rate of Landau [8] times a factor of the order of 0.1–1, as shown in Fig. 2 of Ref. [20]. At the same time, for 0.03 a.u. $\leqslant E_0 \leqslant 1$ a.u., $\Gamma^{\rm av}_{\rm stat}$ changes over about ten orders of magnitude. An approximate empirical formula

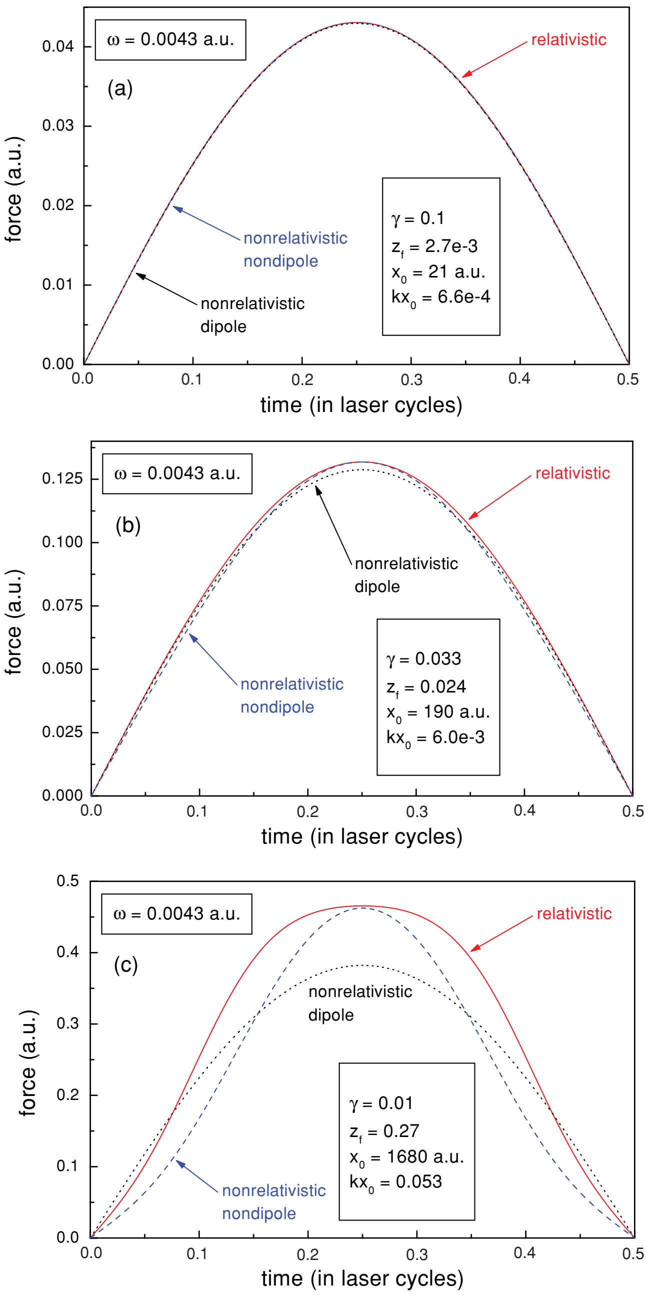

FIG. 2. (Color online) The Lorentz force acting on the electron as a function of time during the first half of the laser cycle, for $\omega = 0.0043$ a.u. Three values of the Keldysh parameter are fixed here [$\gamma = 0.1$, 0.033, and 0.01 in panels (a), (b), and (c), respectively]. The laser intensity increases from panel (a) to panel (c). $2.7e-3$ denotes 2.7×10^{-3} (see text for more detail).

(the so-called tunneling or above barrier ionization (TBI) formula) for the static-field ionization rates for atoms and molecules and fields up to $E_0 \geqslant\sim E_{\rm BSI}$ was found by Tong and Lin [21]. The TBI formula can also be treated as of the type of Eq. (10), with a slowly varying function $f(E_0)$. In Refs. [6–21] the nonrelativistic and dipole approximation was applied to a description of the ionization, but in other works [22–26] the magnetic-field component or relativistic effects of the laser field were taken into account. In the limit $\omega \to 0$, in all the cases [6–26] Eq. (10) is approximately valid.

In Eq. (10), the ionization rate Γ depends strongly on E_0, which is the laser field parameter. The Coulomb (or short-range) force acting on the electron is present in Eq. (10) only through constants included in $f(E_0)$ and C. Therefore, for a given initial state of the atom (or ion), the ionization rate Γ is determined by the amplitude of the electric field vector E_0. In the nonrelativistic and dipole approximations, E_0 is equal (in atomic units) to the maximal Lorentz force exerted on the electron during its motion along the figure-eight path (which simply becomes a line segment in this case). The force (in the simplest frame of reference) can be calculated from Eqs. (5) and (7). In the intermediate range of the laser field parameters one can keep the nonrelativistic theory, but one has to take into account the magnetic-field component of the laser [27]. Then both the electric field $\vec{E}$ and the magnetic field $\vec{B}$ depend only on time, and $\vec{B} = \hat{n} \times \vec{E}$ ($\hat{n}$ is a unit vector in the propagation direction). In the fully relativistic case, one should replace Eq. (7) with Eq. (6) to calculate the Lorentz force acting on the electron.

Taking into account my discussion related to Eq. (10), one can suppose that in the limit $\omega \to 0$, during the motion shown in Fig. 1, the electron is most probably emitted near the points C and F (i.e., when both the $\vec{E}$ and the $\vec{B}$ fields are close to their maxima). The tunneling picture of ionization in static electric fields [7–9] suggests that the electron is emitted mostly in the direction of the electric field vector. Figure 1 indicates that the electron may be emitted not only in the polarization direction (in the simplest frame of reference), but also at some little angle in relation to this direction. This happens when the electron is not emitted exactly from points C or F (note that for the circular polarization the electron always escapes in the polarization plane in the simplest frame of reference). Condition (8) is important, if one looks at angular distributions of photoelectrons. Indeed, such effects were theoretically predicted (usually within an exponential accuracy) for different polarizations in relativistic (or at least nondipole) strong-field photoionization (see, for example, Refs. [28–32]). As is generally known, the classical free point charge in the monochromatic plane-wave electromagnetic field moves with the so-called drift velocity, which is constant and parallel to the wave vector $\vec{k} = k\hat{n}$ [33,34]. As a result, the ionized electron has a greater momentum in the forward $\hat{n}$ direction than it would have in the dipole approximation, in which the drift velocity is zero. In the nonrelativistic approximation, the average drift per cycle in the propagation direction is of the order of $E_0^2/c\omega^3$. According to Joachain *et al.* [33], nondipole effects could appear if the aforementioned drift would be at least equal to 1 a.u. [for the H($1s$) atom]. The latter condition is equivalent to Eq. (8) (up to a constant factor of the order of unity). However, binding forces (Coulomb or short-range) make the electronic trajectory more complicated [35], and the magnetically induced drift may be overcome by the attraction of the nucleus [36].

In my opinion, Eq. (10) suggests that nondipole or relativistic effects could appear in the ionization rate if they would appear in the Lorentz force, that is, when Eq. (6) would differ significantly from Eq. (7). In Figs. 2 and 3 I have investigated the Lorentz force as a function of time during the motion along the first half of the figure-eight path. The solid

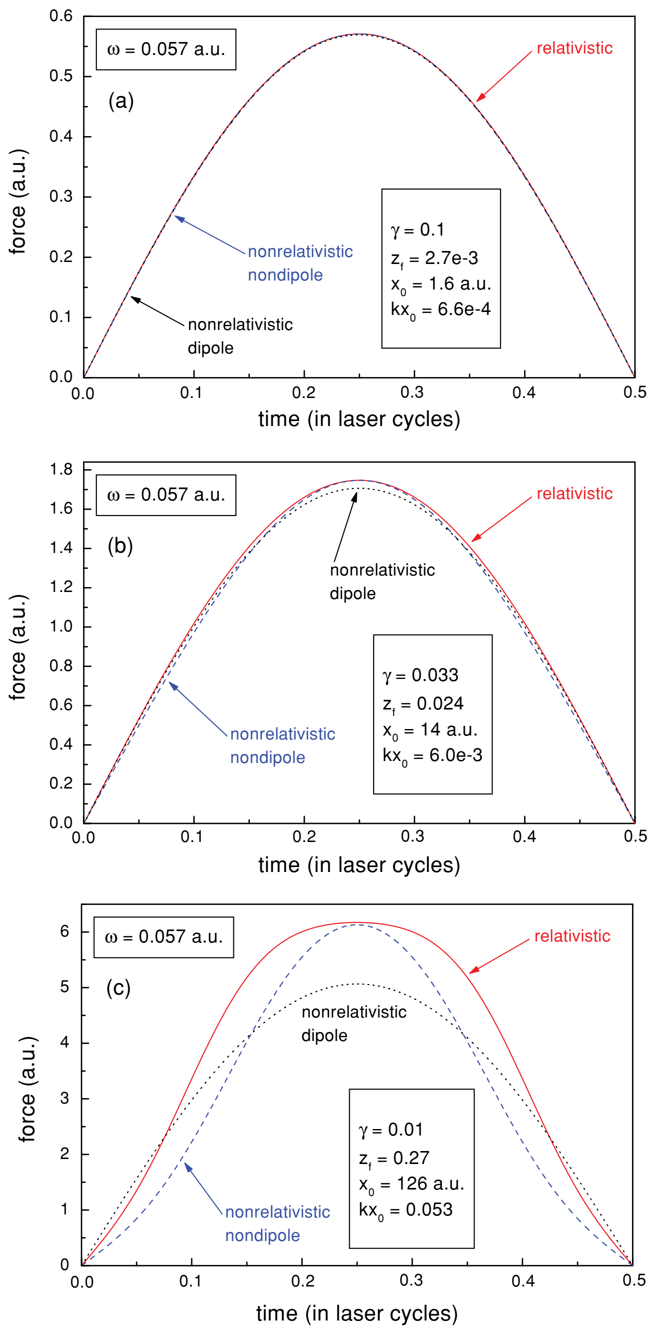

FIG. 3. (Color online) The same as for Fig. 2, but for $\omega = 0.057$ a.u.

lines show fully relativistic results, based on Eqs. (4)–(6). The dashed lines show the nonrelativistic and nondipole results, based on Eqs. (4), (5), and (7), and the dotted lines show the nonrelativistic results in the dipole approximation [Eqs. (4), (5), and (7) with $x_0 = 0$]. Each figure contains numerical values of some essential parameters, namely γ [with $E_B = 0.5$ a.u., for the H(1s) atom], $z_f = 2U_P/c^2$ (where U_P is the ponderomotive potential; see also Refs. [2–5]), x_0, and kx_0 [the latter value is shown to confirm validity of Eq. (2) in each case]. In Figs. 2(a)–2(c) $\omega = 0.0043$ a.u., which corresponds to CO_2 laser radiation ($\lambda = 10.6$ μm), and in Figs. 3(a)–3(c) $\omega = 0.057$ a.u., which corresponds to Ti:sapphire laser radiation ($\lambda = 800$ nm). The Keldysh parameter $\gamma = 0.1$, 0.033, and 0.01, respectively, in panels (a), (b), and (c) in Figs. 2 and 3. In Figs. 2(a) and 3(a) the three aforementioned Lorentz forces are nearly indistinguishable from each other, in spite of quite large values of x_0. In panels (b) and (c) of Figs. 2 and 3, the amplitude of the electronic motion in the propagation direction (x_0) grows significantly. In Figs. 2(b) and 3(b) I show that the nonrelativistic dipole approximation becomes insufficient for $x_0 = 190$ a.u. (if $\omega = 0.0043$ a.u.) and for $x_0 = 14$ a.u. (if $\omega = 0.057$ a.u.). As one should expect, for extremely intense fields, the relativistic description is necessary [see Figs. 2(c) and 3(c)]. However, it follows from Figs. 2(b) and 2(c) and 3(b) and 3(c) that the nonrelativistic nondipole approximation for the Lorentz force works much better than the nonrelativistic dipole approximation. Furthermore, the former one becomes the most accurate for $t = 0.25$ (in laser cycles), when the ionization is the fastest. There is a simple explanation of this fact, namely, near $t = 0.25$ (and $t = 0.75$), the velocity of the electron v achieves a local minimum during its figure-eight motion. (This corresponds to points C and F in Fig. 1). For example, in Figs. 2 and 3, for $t = 0.25$, one has $v/c = 0.0013$, 0.012, and 0.12 for panels (a), (b), and (c), respectively. It appears that even for extremely strong fields, the electron mostly moves with the nonrelativistic ($v \ll c$) velocity when it is ionized.

IV. NONDIPOLE TUNNELINGLIKE FORMULA

In this section, within the nonrelativistic approach, I derive a simple correction to tunnelinglike formula (10). The well-known Landau's result [8] (exact in the limit $E_0 \to 0$) for the static-field ionization of the H(1s) atom is

$$\Gamma_{\mathrm{stat}} = \frac{N}{E_0} \exp\left(\frac{-C}{E_0}\right), \tag{11}$$

where $N = 4$ and $C = 2/3$. To derive Eq. (11), one assumes that the electric field E_0 is constant in time and space, and the ionized electron is treated within the nonrelativistic quantum mechanics [7–9]. When the field changes harmonically only in time [$E(t) = E_0 \sin \omega t$], slowly enough, and $C/E_0 \gg 1$, averaging Eq. (11) over a cycle of the field, one obtains (see, for example, Sec. II of Ref. [37])

$$\Gamma_{\mathrm{stat}}^{\mathrm{av}} \approx N\sqrt{\frac{3}{\pi E_0}} \exp\left(\frac{-C}{E_0}\right). \tag{12}$$

If $\omega \ll E_B$, Eqs. (11) and (12) approximately describe ionization rates for the circular and the linear polarization, respectively [16,37,38]. For the linear polarization of the plane wave, its Lorentz force exerted on the ionized electron is given by

$$\vec{F}_{\mathrm{nonrel}}^{\mathrm{dip}}(t) = \ddot{\vec{r}}(t) = -\vec{E}(t) = -E_0\hat{z}\sin\omega t, \tag{13}$$

and Eqs. (11)–(13) are in the dipole approximation. To generalize Eq. (12) for nondipole effects, I replace Eq. (13) with

$$\vec{F}_{\mathrm{nonrel}}^{\mathrm{nondip}}(t) = \ddot{\vec{r}}(t) = \ddot{x}(t)\hat{x} + \ddot{z}(t)\hat{z}, \tag{14}$$

where $\ddot{x}(t)$ and $\ddot{z}(t)$ are given by Eqs. (5) and $\delta = 0$ [hence $y(t) = 0$]. Of course, Eq. (14) reduces to Eq. (13) for not-too-strong fields (when one can put $x_0 = 0$ and $z_0 = E_0/\omega^2$).

Replacing $|E_0 \sin\omega t|$ with $g(t) \equiv \sqrt{\ddot{x}(t)^2 + \ddot{z}(t)^2}$ during the averaging of Eq. (11) over a cycle of the field ($T = 2\pi/\omega$), I obtain the following generalization of Eq. (12):

$$\Gamma_{\rm stat}^{\rm av,\,nondip} = \frac{1}{T/2}\int_0^{T/2} \frac{N}{g(t)} \exp\left[\frac{-C}{g(t)}\right] dt, \qquad (15a)$$

with

$$g(t) = \omega^2\big[16x_0^2 \sin^2 2\omega t(1 - 16kx_0 \cos 2\omega t) + z_0^2 \sin^2 \omega t(1 - 10kx_0 - 18kx_0 \cos 2\omega t)\big]^{1/2}, \qquad (15b)$$

where I have dropped terms of the order of $(kx_0)^2$ and higher. If $C/g(t) \gg 1$ for all t, the integral in Eq. (15a) may be calculated analytically in an approximate way [similarly one obtains Eq. (12) from Eq. (11)]. I put $\omega t = x + \pi/2$ (x is a new integration variable) in Eqs. (15), and I expand the function g in a Taylor series around $x = 0$ (for which the integrand has a local, strongly peaked maximum), keeping only terms with up to x^2 in the exponent. In the pre-exponential factor, it is sufficient to keep only the first term with x^0. Then I extend the limits of integration upon x to $\pm\infty$, obtaining a well-known Gaussian integral. Finally, I get the following expression:

$$\Gamma_{\rm stat}^{\rm av,\,nondip} \approx \frac{NA^{1/4}}{\omega}\sqrt{\frac{2}{\pi BC}} \exp\left(\frac{-C}{\omega^2\sqrt{A}}\right), \qquad (16)$$

with $A = z_0^2(1 + 8kx_0) > 0$ and $B = z_0^2(1 + 44kx_0) - x_0^2(64 + 1024kx_0) > 0$. [In the dipole approximation, when $x_0 = 0$ and $z_0 = E_0/\omega^2$, Eq. (16) reduces to Eq. (12), as it should.] I have numerically verified that making the step from Eqs. (15) to Eq. (16), one introduces some error, which is very small, if ω is low enough. For example, for the laser field parameters from Figs. 2(a) and 2(b) the error is roughly equal to 0.7%, and for Figs. 3(a) and 3(b) it is roughly equal to 8% and 12%, respectively.

However, let us note that Eq. (16) cannot be applied strictly in the limit: $E_0 = \text{const}$, $\omega \to 0$ [then expression (16) goes to zero]. In this limit one enters the fully relativistic region of the laser field parameters, so Eq. (7) should be replaced by Eq. (6), and Eq. (11) by its relativistic counterpart. This is beyond the scope of the present work. It follows from the approximations used here, that one can use Eqs. (15) or (16) for such field parameters that $kx_0 < 0.1$ and $z_f < 0.1$. From Eqs. (5) in Ref. [2] it follows that for $\delta = 0$ one has

$$kx_0 = \frac{\omega}{c}\,\frac{z}{2c(1+z_f)} < \frac{z\omega}{2c^2} = \frac{U_P}{2c^2} = \frac{1}{4}z_f. \qquad (17)$$

Therefore, to guarantee validity of my result (15,16), it is sufficient to obey the condition $z_f < 0.1$. Since the latter condition is much milder than the condition $x_0 \lesssim 1$ a.u., there is indeed quite large range of the laser field parameters, where nonrelativistic nondipole effects in the total ionization rate for the linear polarization might appear. To explore this range, I have numerically compared ionization rates given by Eqs. (12) and (15). The difference between dipole and nondipole ionization rates grows with increasing intensity if $\omega = \text{const}$. Nondipole rates are usually smaller than dipole ones, unless they are very close each other (within 1%). The largest differences, which occur for $z_f = 0.1$, are 14% (for $\omega = 0.0043$ a.u.) and 30% (for $\omega = 0.057$ a.u.). For the laser fields given in Figs. 2(a), 2(b), 3(a), and 3(b), these differences are, respectively, 0.4%, 2.8%, 8.6%, and 20%.

Equations (15) and (16) may be generalized for any elliptical polarization ($0 < \delta < \pi/2$) and other pre-exponential factors, which can appear in equations analogous to Eq. (11), describing ionization rates for other initial states of an atom or ion.

When the laser field is circularly polarized ($\delta = \pi/2$), it follows from Eqs. (1), (5), and (7) that $x_0 = 0$, $y_0 = z_0$, and one should replace E_0 with $E_0/\sqrt{1 + z_f}$ in Eq. (11), to take into account nonrelativistic nondipole effects. Therefore, the ionization rate formula, including these effects, is given by

$$\Gamma_{\rm stat}^{\rm nondip} = \frac{N\sqrt{1+z_f}}{E_0} \exp\left(\frac{-C\sqrt{1+z_f}}{E_0}\right), \qquad (18)$$

where $z_f = E_0^2/(\omega c)^2$, in this case. For $\omega = 0.0043$ a.u. and $z_f = 0.1$, one obtains a 13.5% greater result from Eq. (11) than from Eq. (18), and when E_0 decreases, both formulas become identical in the limit $E_0 \to 0$. For $\omega = 0.057$ a.u. and $z_f = 0.1$, one obtains 3.4% smaller result from Eq. (11) than from Eq. (18), and (of course) the same behavior in the limit $E_0 \to 0$. (For $\omega = 0.057$ a.u., the ratio of these two rates is slightly nonmonotonic when E_0 is close to zero, but then the rates differ by less than 1%.)

V. REMARKS AND CONCLUSIONS

In my opinion, the present results throw some light on a recent experiment by Chowdhury *et al.* [39] with argon and pulsed-laser 800-nm radiation of the linear polarization at intensities of up to 10^{19} W/cm^2. According to the authors of Ref. [39], total ionization rates, evoked by "relativistic" laser fields in the L-shell states of argon, can be quite well described by "nonrelativistic" strong-field theories. In Ref. [39] validity of "a widely used Ammosov-Delone-Krainov/WKB tunneling ionization model" [10–15] (which is the dipole approximation theory) was confirmed for the average Keldysh parameter $\gamma \approx 0.03$. One can also find similar conclusions in a later experimental work of Yamakawa *et al.* [40] with xenon, krypton, and argon. According to Chowdhury *et al.*, the results of their experiment may be interpreted within a two-step model, where the initial tunneling ionization process is dominated by nonrelativistic effects, while the photoelectron continuum dynamics are strongly relativistic. Although the authors of Ref. [39] discuss this phenomenon in Sec. IV of their work, they do not give any explanation of an absence of relativistic effects during the first step of the process in their experiment.

The Keldysh parameter γ, which is the most important one among the laser field parameters, depends on E_B, ω, and E_0 [see Eq. (9)]. For a few different initial ionic states of argon, used in the experiment [39], "ionization potentials extend from 422 V to 918 V," so their binding energies E_B extend from 15.5 to 33.7 a.u. Although the binding energies of the L-shell states of argon are much higher than $E_B = 0.5$ a.u. used in my present work, my Keldysh parameter from Figs. 2(b) and 3(b) ($\gamma = 0.033$) is close to the values from Ref. [39]. Therefore, from the point of view of an adiabaticity of the ionization process [crucial for my simple theory given by Eqs. (15), (16), and (18)], my laser field conditions for the $H(1s)$ atom resemble those from Ref. [39] for positively charged argon ions. In Sec. IV I have shown that for the linear polarization and nonrelativistic conditions ($z_f \leqslant 0.1$), the highest difference between dipole and nondipole ionization rates does not exceed 30%, but typically (for $\gamma \approx 0.03$) the difference is less than 20%.

In conclusion, I have derived a non-parametric form of the classical relativistic equations of motion for the electron in the monochromatic plane-wave laser field (of an arbitrary intensity and ellipticity). My approximate solution (in the simplest frame of reference) is based on Eq. (2), which is valid even for superstrong laser fields. I have numerically calculated the Lorentz force acting on the ionized electron during its figure-eight motion (as a function of time) in the linearly polarized field for two frequencies of an experimental interest. In the low-frequency limit, a manifestation of nonrelativistic and relativistic effects in the Lorentz force may be a quite good indication of such effects in the ionization rate. I have derived simple nonrelativistic expressions for $\omega \ll E_B$, and I have evaluated nondipole effects in total ionization rates. For higher frequencies, the simple tunnelinglike formula (10) is not valid. However, it is quite likely that my argument, based on the aforementioned effects in the Lorentz force, is of some importance.

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