Generation of two-mode Gaussian-type entangled states of light via a quantum beat laser

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The generation of two-mode Gaussian-type entangled states of the cavity field is studied in a quantum beat laser, where three-level atoms of the V-type level configuration are coupled to two quantized modes of the cavity field and two upper levels of the atom are driven by a strongly classical field. According to Simon's criterion that is a necessary and sufficient condition for the inseparability of two-mode Gaussian states with a general form of covariance matrices, we analytically and numerically investigate the influence of phase and Rabi frequency of the classical driving field, cavity loss, and the purity and nonclassicality of the initial state of the cavity field on the entanglement property of the resulting two-mode Gaussian state. In the limit of weak cavity loss and strong driving, we show that the cavity field can be periodically into an ideal two-mode entangled Gaussian-type state.

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I. INTRODUCTION

Two-mode entangled states of the radiation field possess nonlocal correlations between phase-quadrature components that are analogous to position and momentum operators of a massive particle, and become a fundamental resource for quantum information processes of continuous variables such as quantum teleportation and quantum cryptography [1]. Among various types of two-mode entangled states, Gaussian-type states including the two-mode squeezed state as a special case have attracted a lot of attention [2]. On the aspect of theory, a necessary and sufficient condition for the inseparability of bipartite Gaussian states has been established by Simon [3], and Duan and coworkers [4]. The evaluation of entanglement of formation as a measure of entanglement can also be performed for symmetric two-mode Gaussian states [5] and even for arbitrary two-mode Gaussian states [6]. From the point of view of experiment, bipartite Gaussian entangled states can be completely characterized and easily generated in experiment [7,8]. For example, the two-mode squeezed state that is one of typical two-mode Gaussian-type entangled states (TMGES) can be generated from an optical parametric oscillator (OPO) operating below threshold [9]. The TMGES can be also generated in a very simple way by mixing at a beam splitter two single-mode squeezing beams out of a degenerate OPO [10].

Besides the nonlinear interaction of light with crystals such as in an OPO and the linear optical device such as beam splitter, the interaction of two-mode cavity fields with atoms coherently driven by laser fields is often employed to generate the TMGES. It has been shown that correlated photons in a two-mode Gaussian state are produced via four-wave parametric interactions of strongly driven two-level atoms with cavity fields [11]. By means of the interaction of properly driven V-type three-level atoms with two cavity modes, Li *et al.* [12] showed that the TMGES can be generated via the four-wave mixing process with high efficiency. Pielawa *et al.* [13] showed that the TMGES with high purity such as the two-mode squeezed vacuum state can be obtained with a two-step preparation by randomly injecting two-level atoms into a high-Q microwave cavity. The correlated emission laser (CEL) [14] involves the interaction of three-level atoms of the cascade level configuration with two modes of the cavity field, which upper and lower levels are coherently driven by a strong classical field. Xiong, Scully, and Zubairy showed that the two-mode entangled light with a large number of photons in each mode can be generated via a CEL [15]. It has also been shown that the cavity field generated in a CEL is in a two-mode Gaussian state when each of two modes of the cavity field is initially in a Gaussian state [16]. A CEL-based scheme of four-level atoms with the Raman-driven coherence has also been proposed for generating the TMGES [17].

In a quantum beat laser (QBL), three-level atoms of the V-type level configuration interact with two modes of the radiation in a doubly resonant cavity [18]. In order to make the two modes beat and couple each other, two upper levels of the atom are coherently driven by a strong classical field. In a recent study, Qamar *et al.* [19] considered the generation of two-mode entangled states of the cavity field via a quantum beat laser. Assuming that each of the two modes is initially in the specific states such as the squeezed state, vacuum state, number state, and coherent state, they numerically studied the variation of entanglement of the field in time according to the two sufficient conditions for the existence of entanglement in bipartite systems, which were proposed by Duan, Giedke, Cirac, and Zoller (DGCZ) [4], and Hillery and Zubairy [20]. They found that the two criteria lead to different time intervals for the entanglement existence and the two-mode cavity field can be kept in an entangled state for a longer time as the Rabi frequency of the classical driving field becomes larger. A four-level Raman-driven QBL-based scheme has been also proposed for generating two-mode entangled states of the cavity field [21].

In a quantum beat laser, no steady-state entangled states exist and the state of the cavity field at time t strongly depends on its initial state. It is also noticed that the interaction of a V-type three-level atom with two modes of the quantized cavity field can be approximated to be a beam splitter when the Rabi frequency of the classical field driving the two upper levels of the atom is much larger than the decay rate of the atom to other states. As was shown in Ref. [22], a beam splitter can

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have two output fields entangled. In this paper, considering that each of two modes of the cavity field is initially in a general single-mode Gaussian state that takes coherent state, vacuum state, and squeezed vacuum state as a special case, we study the generation of TMGESs of the cavity field in a quantum beat laser. For TMGESs, either Simon criterion [3] or Duan criterion [4] can be used to adjudge the entanglement existence. However, the later one becomes necessary and sufficient only when TMGESs have a standard form of covariance matrices, but the Simon criterion is necessary and sufficient for the entanglement existence of TMGESs with a general form of covariance matrices. Thus, by employing the Simon criterion in this paper, we analytically and numerically investigate the influence of phase and Rabi frequency of the driving field, cavity loss, and the purity and nonclassicality of the initial state of the cavity field on the entanglement property of the resulting two-mode Gaussian state. We find that the cavity field can be periodically into an ideal two-mode entangled Gaussian state under certain conditions.

This paper is organized as follows. In Sec. II, the master equation for the cavity field in a quantum beat laser is derived. In Sec. III, we analytically study the conditions for the entanglement existence of the cavity field in the limit of no photon leakage out of the cavity and strong driving. In Sec. IV, we numerically investigate the influence of cavity loss, Rabi frequency of the classical field, the purity and nonclassicality of the initial field state on the entanglement property of the resulting two-mode Gaussian state. In Sec. V, the main results of the present paper are summarized.

II. MODEL AND MASTER EQUATION

In Fig. 1, a three-level atom with the V configuration is shown. The transitions from level $|c\rangle$ to levels $|a\rangle$ and $|b\rangle$ are electrical dipole allowed and coupled to two modes of the quantized radiation field. The transition between levels $|a\rangle$ and $|b\rangle$ is electrical dipole forbidden and driven by a strong classical field. In the dipole and rotating-wave approximations, the Hamiltonian of the atom-field coupled system takes the form

$$H = H_0 + V, \tag{1}$$

where

$$\begin{aligned} H_0 &= \hbar(\omega_c + \nu_1 + \Delta)|a\rangle\langle a| + \hbar(\omega_c + \nu_2 + \Delta)|b\rangle\langle b| \\ &\quad + \hbar\omega_c|c\rangle\langle c| + \hbar\nu_1 a_1^\dagger a_1 + \hbar\nu_2 a_2^\dagger a_2, \end{aligned} \tag{2}$$

$$\begin{aligned} V &= \hbar g_1(a_1|a\rangle\langle c| + a_1^\dagger|c\rangle\langle a|) + \hbar g_2(a_2|b\rangle\langle c| + a_2^\dagger|c\rangle\langle b|) \\ &\quad - \frac{\hbar\Omega}{2}(e^{-i\Phi - i\nu_3 t}|a\rangle\langle b| + e^{i\Phi + i\nu_3 t}|b\rangle\langle a|). \end{aligned} \tag{3}$$

In the above, a_1 ($a_1^\dagger$) and a_2 ($a_2^\dagger$) are the annihilation (creation) operators of photons in two modes of frequencies ν_1 and ν_2, respectively, g_1 and g_2 are interaction constants of the atom with the quantized cavity modes. The classical field of frequency ν_3, Rabi frequency Ω, and phase Φ is resonant with the transition $|a\rangle \leftrightarrow |b\rangle$, that is, $\nu_3 = \omega_a - \omega_b$. The two modes of the cavity field is tuned from the atomic transitions $|a\rangle \leftrightarrow |c\rangle$ and $|b\rangle \leftrightarrow |c\rangle$ by an amount $\Delta = \omega_a - \omega_c - \nu_1 = \omega_b - \omega_c - \nu_2$.

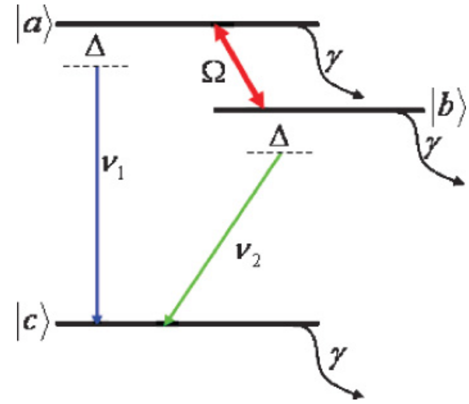

FIG. 1. (Color online) Atomic level configuration.

In the interaction picture, the Hamiltonian (1) is transformed into the form

$$\begin{aligned} V_I &= \hbar g_1(a_1|a\rangle\langle c| + a_1^\dagger|c\rangle\langle a|) + \hbar g_2(a_2|b\rangle\langle c| + a_2^\dagger|c\rangle\langle b|) \\ &\quad - \frac{\hbar\Omega}{2}(e^{-i\Phi}|a\rangle\langle b| + e^{i\Phi}|b\rangle\langle a|) + \hbar\Delta(|a\rangle\langle a| + |b\rangle\langle b|). \end{aligned} \tag{4}$$

From the equation of motion for the density matrix of the atom-field coupled system

$$\dot{\rho} = -\frac{i}{\hbar}[V_I, \rho], \tag{5}$$

tracing over the atomic variables, we have the equation of motion for the reduced density matrix of the cavity field

$$\begin{aligned} \dot{\rho}_f &= -\frac{i}{\hbar}Tr_{\text{atom}}[V_I, \rho] \\ &= -ig_1[a_1^\dagger, \rho_{\text{ac}}] - ig_2[a_2^\dagger, \rho_{\text{bc}}] + \text{H.c.} \end{aligned} \tag{6}$$

In the limit of $\Omega \gg g_1, g_2$, upon keeping the coupling constants of the cavity field up to the second order [23], the equations of motion for the matrix element operators ρ_{ac} and ρ_{bc} can be approximately written as

$$\begin{aligned} \dot{\rho}_{\text{ac}} &= -(\gamma + i\Delta)\rho_{\text{ac}} + \frac{i\Omega}{2}e^{-i\Phi}\rho_{\text{bc}} \\ &\quad + ig_1\rho_{aa}a_1 + ig_2\rho_{ab}a_2 - ig_1a_1\rho_{cc}, \end{aligned} \tag{7}$$

$$\begin{aligned} \dot{\rho}_{\text{bc}} &= -(\gamma + i\Delta)\rho_{\text{bc}} + \frac{i\Omega}{2}e^{i\Phi}\rho_{\text{ac}} \\ &\quad + ig_1\rho_{ba}a_1 + ig_2\rho_{bb}a_2 - ig_2a_2\rho_{cc}, \end{aligned} \tag{8}$$

where the decay of the atomic levels to other states at rate γ is phenomenologically included. Correspondingly, we keep the equations of motion for the matrix element operators ρ_{aa}, ρ_{bb}, ρ_{ab}, ρ_{ba}, and ρ_{cc} which have been multiplied by the coupling constant of the cavity field in Eqs. (7) and (8) up to the zero-order

$$\dot{\rho}_{aa} = -\gamma\rho_{aa} + \frac{i\Omega}{2}e^{-i\Phi}\rho_{ba} - \frac{i\Omega}{2}e^{i\Phi}\rho_{ab} + r_a\rho_f, \tag{9}$$

$$\dot{\rho}_{bb} = -\gamma\rho_{bb} + \frac{i\Omega}{2}e^{i\Phi}\rho_{ab} - \frac{i\Omega}{2}e^{-i\Phi}\rho_{ba}, \tag{10}$$

$$\dot{\rho}_{ab} = -\gamma\rho_{ab} + \frac{i\Omega}{2}e^{-i\Phi}\rho_{bb} - \frac{i\Omega}{2}e^{-i\Phi}\rho_{aa}, \tag{11}$$

$$\dot{\rho}_{ba} = -\gamma\rho_{ba} + \frac{i\Omega}{2}e^{i\Phi}\rho_{aa} - \frac{i\Omega}{2}e^{i\Phi}\rho_{bb}, \tag{12}$$

$$\dot{\rho}_{cc} = 0, \tag{13}$$

where we have assumed that the atoms are pumped at a rate r_a into the level $|a\rangle$. By setting all the time derivatives on the left side of Eqs. (7)–(13) to be zero, we can find the steady-state solutions for $\rho_{\rm ac}$ and $\rho_{\rm bc}$. It follows on inserting the steady-state solutions into Eq. (6) and taking the damping of the cavity field to the vacuum into account that we obtain the master equation for the reduced density operator of the cavity field

$$\begin{aligned}\dot{\rho_f} = &-\tfrac{1}{2}[\alpha_{11}^* a_1 a_1^\dagger \rho_f + \alpha_{11}\rho_f a_1 a_1^\dagger - (\alpha_{11}+\alpha_{11}^*)a_1^\dagger\rho_f a_1]\\ &-\tfrac{1}{2}[\alpha_{22}^* a_2 a_2^\dagger \rho_f + \alpha_{22}\rho_f a_2 a_2^\dagger - (\alpha_{22}+\alpha_{22}^*)a_2^\dagger\rho_f a_2]\\ &-\tfrac{1}{2}[\alpha_{21}^* a_1^\dagger a_2 \rho_f + \alpha_{12}\rho_f a_1^\dagger a_2 - (\alpha_{12}+\alpha_{21}^*)a_1^\dagger\rho_f a_2]e^{-i\Phi}\\ &-\tfrac{1}{2}[\alpha_{12}^* a_1 a_2^\dagger \rho_f + \alpha_{21}\rho_f a_1 a_2^\dagger - (\alpha_{21}+\alpha_{12}^*)a_2^\dagger\rho_f a_1]e^{i\Phi}\\ &-\kappa_1(a_1^\dagger a_1\rho_f - 2a_1\rho_f a_1^\dagger + \rho_f a_1^\dagger a_1)\\ &-\kappa_2(a_2^\dagger a_2\rho_f - 2a_2\rho_f a_2^\dagger + \rho_f a_2^\dagger a_2),\end{aligned} \tag{14}$$

where $\kappa_j (j = 1, 2)$ is the relaxation rate of the cavity mode j and the other parameters are given by the following equations:

$$\begin{aligned}\alpha_{11} = \frac{g^2 r_a}{2\gamma(\gamma^2+\Omega^2)}\Bigg(&\frac{(2\gamma^2+\Omega^2+i\Omega\gamma)[\gamma - i(\Delta-\Omega/2)]}{[\gamma^2+(\Delta-\Omega/2)^2]}\\ &+\frac{(2\gamma^2+\Omega^2-i\Omega\gamma)[\gamma - i(\Delta+\Omega/2)]}{[\gamma^2+(\Delta+\Omega/2)^2]}\Bigg),\end{aligned} \tag{15}$$

$$\begin{aligned}\alpha_{12} = \frac{g^2 r_a\Omega}{2\gamma(\gamma^2+\Omega^2)}\Bigg(&\frac{[\gamma - i(\Delta-\Omega/2)]}{[\gamma^2+(\Delta-\Omega/2)^2]}(\Omega - i\gamma)\\ &-\frac{[\gamma - i(\Delta+\Omega/2)]}{[\gamma^2+(\Delta+\Omega/2)^2]}(\Omega + i\gamma)\Bigg),\end{aligned} \tag{16}$$

$$\begin{aligned}\alpha_{21} = \frac{g^2 r_a}{2\gamma(\gamma^2+\Omega^2)}\Bigg(&\frac{(2\gamma^2+\Omega^2+i\Omega\gamma)[\gamma - i(\Delta-\Omega/2)]}{[\gamma^2+(\Delta-\Omega/2)^2]}\\ &-\frac{(2\gamma^2+\Omega^2-i\Omega\gamma)[\gamma - i(\Delta+\Omega/2)]}{[\gamma^2+(\Delta+\Omega/2)^2]}\Bigg),\end{aligned} \tag{17}$$

$$\begin{aligned}\alpha_{22} = \frac{g^2 r_a\Omega}{2\gamma(\gamma^2+\Omega^2)}\Bigg(&\frac{[\gamma - i(\Delta-\Omega/2)]}{[\gamma^2+(\Delta-\Omega/2)^2]}(\Omega - i\gamma)\\ &+\frac{[\gamma - i(\Delta+\Omega/2)]}{[\gamma^2+(\Delta+\Omega/2)^2]}(\Omega + i\gamma)\Bigg).\end{aligned} \tag{18}$$

In the limit of $\Omega \gg \gamma$, the master equation (14) can be approximated to

$$\begin{aligned}\dot{\rho_f} = &-\frac{i}{2}[\alpha_1 a_1 a_1^\dagger\rho_f - \alpha_1\rho_f a_1 a_1^\dagger] - \frac{i}{2}[\alpha_1 a_2 a_2^\dagger\rho_f\\ &-\alpha_1\rho_f a_2 a_2^\dagger] - \frac{i}{2}[\alpha_2 a_1^\dagger a_2\rho_f - \alpha_2\rho_f a_1^\dagger a_2]e^{-i\Phi}\\ &-\frac{i}{2}[\alpha_2 a_1 a_2^\dagger\rho_f - \alpha_2\rho_f a_1 a_2^\dagger]e^{i\Phi} - \kappa_1(a_1^\dagger a_1\rho_f - 2a_1\rho_f a_1^\dagger\\ &+\rho_f a_1^\dagger a_1) - \kappa_2(a_2^\dagger a_2\rho_f - 2a_2\rho_f a_2^\dagger + \rho_f a_2^\dagger a_2),\end{aligned} \tag{19}$$

where

$$\alpha_1 = \frac{K\gamma}{2}\left(\frac{1}{\Delta-\Omega/2} + \frac{1}{\Delta+\Omega/2}\right), \tag{20}$$

$$\alpha_2 = \frac{K\gamma}{2}\left(\frac{1}{\Delta-\Omega/2} - \frac{1}{\Delta+\Omega/2}\right), \tag{21}$$

with $K = g^2 r_a/\gamma^2$. The master equation (19) can be rewritten in the form

$$\begin{aligned}\dot{\rho_f} = &-i[H_{\rm eff}, \rho_f] - \kappa_1(a_1^\dagger a_1\rho_f - 2a_1\rho_f a_1^\dagger + \rho_f a_1^\dagger a_1)\\ &-\kappa_2(a_2^\dagger a_2\rho_f - 2a_2\rho_f a_2^\dagger + \rho_f a_2^\dagger a_2),\end{aligned} \tag{22}$$

with the effective Hamiltonian

$$H_{\rm eff} = \tfrac{1}{2}\alpha_1(a_1 a_1^\dagger + a_2 a_2^\dagger) + \tfrac{1}{2}\alpha_2(e^{-i\Phi}a_1^\dagger a_2 + e^{i\Phi}a_1 a_2^\dagger). \tag{23}$$

Thus, in the strongly driving limit, the two-mode interaction induced via coherence between the upper levels of the atom is equivalent to an optical beam splitter. It has been shown that two output fields may become entangled via a beam splitter [22].

For the general case, according to Eqs. (14), we can derive out the time evolution equations for expectation values of the photon number and the two-photon correlation

$$\begin{aligned}\frac{d}{dt}\langle a_1^\dagger a_1\rangle = &\left[\frac{1}{2}(\alpha_{11}+\alpha_{11}^*) - 2\kappa_1\right]\langle a_1^\dagger a_1\rangle + \frac{1}{2}(\alpha_{12}e^{-i\Phi}\langle a_1^\dagger a_2\rangle\\ &+\alpha_{12}^* e^{i\Phi}\langle a_1 a_2^\dagger\rangle) + \frac{1}{2}(\alpha_{11}+\alpha_{11}^*),\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{d}{dt}\langle a_2^\dagger a_2\rangle = &\left[\frac{1}{2}(\alpha_{22}+\alpha_{22}^*) - 2\kappa_2\right]\langle a_2^\dagger a_2\rangle + \frac{1}{2}(\alpha_{21}^* e^{-i\Phi}\\ &\times\langle a_1^\dagger a_2\rangle + \alpha_{21}e^{i\Phi}\langle a_1 a_2^\dagger\rangle) + \frac{1}{2}(\alpha_{22}+\alpha_{22}^*),\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{d}{dt}\langle a_1 a_2^\dagger\rangle = &\left[\frac{1}{2}(\alpha_{11}+\alpha_{22}^*) - (\kappa_1+\kappa_2)\right]\langle a_1 a_2^\dagger\rangle + \frac{1}{2}(\alpha_{21}^* e^{-i\Phi}\\ &\times\langle a_1^\dagger a_1\rangle + \alpha_{12}e^{-i\Phi}\langle a_2^\dagger a_2\rangle) + \frac{1}{2}(\alpha_{21}^* + \alpha_{12})e^{-i\Phi},\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{d}{dt}\langle a_1^\dagger a_2\rangle = &\left[\frac{1}{2}(\alpha_{11}^*+\alpha_{22}) - (\kappa_1+\kappa_2)\right]\langle a_1^\dagger a_2\rangle + \frac{1}{2}(\alpha_{21}e^{i\Phi}\\ &\times\langle a_1^\dagger a_1\rangle + \alpha_{12}^* e^{i\Phi}\langle a_2^\dagger a_2\rangle) + \frac{1}{2}(\alpha_{21}+\alpha_{12}^*)e^{i\Phi},\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{d}{dt}\langle a_1 a_2\rangle = &\left[\frac{1}{2}(\alpha_{11}+\alpha_{22}) - (\kappa_1+\kappa_2)\right]\langle a_1 a_2\rangle\\ &+\frac{1}{2}\alpha_{12}e^{-i\Phi}\langle a_2 a_2\rangle + \frac{1}{2}\alpha_{21}e^{i\Phi}\langle a_1 a_1\rangle,\end{aligned} \tag{28}$$

$$\frac{d}{dt}\langle a_1 a_1\rangle = (\alpha_{11} - 2\kappa_1)\langle a_1 a_1\rangle + \alpha_{12}e^{-i\Phi}\langle a_1 a_2\rangle, \tag{29}$$

$$\frac{d}{dt}\langle a_2 a_2\rangle = (\alpha_{22} - 2\kappa_2)\langle a_2 a_2\rangle + \alpha_{21}e^{i\Phi}\langle a_1 a_2\rangle. \tag{30}$$

By numerically solving these equations, one can investigate the entanglement property of the cavity field.

III. CONDITIONS FOR INSEPARABILITY OF THE CAVITY FIELD

We assume that two modes of the cavity field are initially in a product of two single-mode Gaussian states $\rho_1 \otimes \rho_2$, which covariance matrix is given by

$$\Gamma_0 = \Gamma_1 \oplus \Gamma_2 = \begin{pmatrix} c_1 & d_1 & 0 & 0 \\ d_1^* & c_1 & 0 & 0 \\ 0 & 0 & c_2 & d_2 \\ 0 & 0 & d_2^* & c_2 \end{pmatrix}, \tag{31}$$

where c_j is a real parameter and $d_j = |d_j|e^{i\phi_j}(j = 1, 2)$. Thus, we have the initial expectation values $\langle a_1^\dagger a_1\rangle_0 = c_1 - 1/2$, $\langle a_2^\dagger a_2\rangle_0 = c_2 - 1/2$, $\langle a_1 a_2^\dagger\rangle_0 = \langle a_1^\dagger a_2\rangle_0^* = 0$, $\langle a_1 a_2\rangle_0 = 0$, $\langle a_1 a_1\rangle_0 = -d_1$, $\langle a_2 a_2\rangle_0 = -d_2$. The parameters of the covariance matrix can be expressed in terms of the purity u_j and the nonclassical depth τ_j of the initial Gaussian state as follows [24]:

$$c_j = \frac{1}{2} + \frac{\tau_j^2 + 1/(2u_j)^2 - 1/4}{1 - 2\tau_j}, \tag{32}$$

$$|d_j| = \frac{\tau_j - \tau_j^2 + 1/(2u_j)^2 - 1/4}{1 - 2\tau_j}, \tag{33}$$

where $u_j = \mathrm{Tr}(\rho_j^2)$ and

$$\tau_j \equiv \max\{0, \tfrac{1}{2} - \eta_j\}, \quad j = 1, 2. \tag{34}$$

In Eq. (34), η_j is the smallest eigenvalue of the covariance matrix Γ_j.

When $\kappa_1 = \kappa_2 = \kappa$ and $\Omega \gg \gamma$, Eqs. (24)–(30) can be analytically solved and the solutions are as follows:

$$\langle a_1^\dagger a_1\rangle = \frac{1}{2}e^{-2\kappa t}[-1 + c_1 + c_2 + (c_1 - c_2)\cos(2\theta)], \tag{35}$$

$$\langle a_2^\dagger a_2\rangle = \frac{1}{2}e^{-2\kappa t}[-1 + c_1 + c_2 + (c_2 - c_1)\cos(2\theta)], \tag{36}$$

$$\langle a_1 a_2^\dagger\rangle = \frac{i}{2}(c_1 - c_2)e^{-2\kappa t - i\Phi}\sin(2\theta), \tag{37}$$

$$\langle a_1^\dagger a_2\rangle = -\frac{i}{2}(c_1 - c_2)e^{-2\kappa t + i\Phi}\sin(2\theta), \tag{38}$$

$$\langle a_1 a_2\rangle = -\frac{1}{4}e^{-2\kappa t - i\Phi - \frac{2iK\gamma t}{2\Delta+\Omega}}(d_1 e^{i(2\Phi+\phi_1)} + d_2 e^{i\phi_2})(e^{4i\theta} - 1), \tag{39}$$

$$\begin{aligned}\langle a_1 a_1\rangle = &-\frac{1}{4}e^{-2\kappa t - 2i\Phi - \frac{2iK\gamma t}{2\Delta+\Omega}}[d_2 e^{i\phi_2}(e^{2i\theta} - 1)^2 \\ &+ d_1 e^{i(2\Phi+\phi_1)}(e^{2i\theta} + 1)^2],\end{aligned} \tag{40}$$

$$\begin{aligned}\langle a_2 a_2\rangle = &-\frac{1}{4}e^{-2\kappa t - \frac{2iK\gamma t}{2\Delta+\Omega}}[d_2 e^{i\phi_2}(e^{2i\theta} + 1)^2 \\ &+ d_1 e^{i(2\Phi+\phi_1)}(e^{2i\theta} - 1)^2],\end{aligned} \tag{41}$$

where $\theta = K\gamma\Omega t/(\Omega^2 - 4\Delta^2)$. At time t, the cavity field is in a two-mode Gaussian state with the covariance matrix

$$\Gamma_t = \begin{pmatrix} \langle a_1^\dagger a_1\rangle + \frac{1}{2} & -\langle a_1 a_1\rangle & \langle a_1 a_2^\dagger\rangle & -\langle a_1 a_2\rangle \\ -\langle a_1 a_1\rangle^* & \langle a_1^\dagger a_1\rangle + \frac{1}{2} & -\langle a_1 a_2\rangle^* & \langle a_1 a_2^\dagger\rangle^* \\ \langle a_1 a_2^\dagger\rangle^* & -\langle a_1 a_2\rangle & \langle a_2^\dagger a_2\rangle + \frac{1}{2} & -\langle a_2 a_2\rangle \\ -\langle a_1 a_2\rangle^* & \langle a_1 a_2^\dagger\rangle & -\langle a_2 a_2\rangle^* & \langle a_2^\dagger a_2\rangle + \frac{1}{2} \end{pmatrix}, \tag{42}$$

According to Peres's partial transposition criterion [25], Simon [3] derived the necessary and sufficient condition for separability of two-bipartite Gaussian states

$$\Gamma_c = \Gamma_t + \tfrac{1}{2}T_1 E T_1 \geqslant 0, \tag{43}$$

where $E = \mathrm{diag}(1, -1, 1, -1)$ and

$$T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{44}$$

Substituting Eqs. (35)–(41) and (42) into the left side of the inequality (43), we work out the necessary and sufficient condition for separability of the cavity field at time t

$$\begin{aligned}&\{(2(|d_1|^2 + |d_2|^2 - (c_1 - c_2)^2)\cos 4\theta - 4|d_1||d_2| \\ &\quad\times(1 - \cos 4\theta)\cos(\phi_1 - \phi_2 + 2\Phi)) \\ &\quad + 2[2 + c_1^2 + c_2^2 - |d_1|^2 - |d_2|^2 - 4c_2 - 4c_1 + 6c_1c_2]\} \\ &\quad\times e^{-4\kappa t} + 4[2(c_1^2 - |d_1|^2)(2c_2 - 1) + 2(c_2^2 - |d_2|^2) \\ &\quad\times(2c_1 - 1) - 8c_1c_2 - 1 + 3(c_1 + c_2)]e^{-6\kappa t} \\ &\quad + [(2c_1 - 1)^2 - 4|d_1|^2][(2c_2 - 1)^2 - 4|d_2|^2]e^{-8\kappa t} \geqslant 0.\end{aligned} \tag{45}$$

It is obvious that the cavity field must finally evolute into a separable state since the exponential decay terms resulting from the cavity loss on the left side of the inequality (45) exists.

Now let us consider the following two cases.

1. When $\kappa \to 0$, and $\phi_1 - \phi_2 + 2\Phi = 0$, inequality (45) can be rewritten in terms of the parameters u_j and τ_j into the form

$$\lambda_1 + \lambda_2\cos 4\theta \geqslant 0, \tag{46}$$

where

$$\begin{aligned}\lambda_1 = &1 - 4(\tau_1 + \tau_2 - 2\tau_1\tau_2) - (1 - 2\tau_1)(1 - 2\tau_2)[u_1^2 + u_2^2 \\ &- u_1^2u_2^2(1 + \tau_1 + \tau_2 - 2\tau_1\tau_2)],\end{aligned} \tag{47}$$

$$\lambda_2 = [1 - u_1^2(1 - 2\tau_1)(1 - 2\tau_2)][1 - u_2^2(1 - 2\tau_1)(1 - 2\tau_2)]. \tag{48}$$

Obviously, $\lambda_2 \geqslant 0$. When $\theta = (2n + 1)\pi/4$ with $n = 0, 1, 2\ldots$, the inequality (46) becomes

$$-4[\tau_1(1 - \tau_2) + \tau_2(1 - \tau_1)][1 - u_1^2u_2^2(1-2\tau_1)(1-2\tau_2)] \geqslant 0. \tag{49}$$

Therefore, inequality (49) can be violated as long as either of the two modes is initially nonclassical, that is, $\tau_1 \neq 0$ or $\tau_2 \neq 0$. When $\theta = n\pi/2$ with $n = 1, 2, \ldots,$ inequality (46) becomes

$$(1 - u_1^2)(1 - u_2^2) \geqslant 0. \tag{50}$$

This inequality cannot be violated. Thus, in this case, the cavity field never be entangled no matter whether the cavity field is initially nonclassical or classical.

Therefore, we see that the cavity field is periodically entangled and separable in time since $\theta[= K\gamma\Omega t/(\Omega^2 - 4\Delta^2)]$ is proportional to time.

2. When $\kappa \to 0$ and $\phi_1 - \phi_2 + 2\Phi = \pi$, the left side of inequality (45) takes its maximum value. In terms of the parameters u_j and τ_j, the condition (45) is rewritten as

$$\lambda_3 + \lambda_4\cos 4\theta \geqslant 0, \tag{51}$$

where

$$\lambda_3 = (\tau_1 + \tau_2 - 1)\left[u_1^2(1-2\tau_1) + u_2^2(1-2\tau_2)\right] + (1-2\tau_1)(1-2\tau_2)\left(u_1^2u_2^2+1\right), \quad (52)$$

$$\lambda_4 = (\tau_2 - \tau_1)\left[u_1^2(1-2\tau_1) - u_2^2(1-2\tau_2)\right]. \quad (53)$$

When $\tau_1 = \tau_2 = \tau$, $\lambda_4 = 0$ and the inequality (51) becomes

$$\left(1-u_1^2\right)\left(1-u_2^2\right)(1-2\tau)^2 \geqslant 0. \quad (54)$$

It cannot be violated. Therefore, the cavity field never be entangled if the initial single-mode Gaussian states have the same nonclassical depth. This conclusion is valid no matter whether the cavity field is initially mixed or pure.

When $\lambda_4 < 0$ and $\theta = n\pi/2$ with $n = 1, 2, \ldots,$ the left side of inequality (51) takes its minimum value and inequality (51) becomes

$$\left(1-u_1^2\right)\left(1-u_2^2\right)(1-2\tau_1)(1-2\tau_2) \geqslant 0. \quad (55)$$

In this case, the inequality is always valid and the cavity field is separable.

When $\lambda_4 > 0$ and $\theta = (2n+1)\pi/4$ with $n = 0, 1, 2, \ldots,$ the left side of inequality (51) takes its minimum value and inequality (51) becomes

$$\left[1-2\tau_1 - u_2^2(1-2\tau_2)\right]\left[1-2\tau_2 - u_1^2(1-2\tau_1)\right] \geqslant 0. \quad (56)$$

When either $\tau_1 < \tau_2$ with the conditions $u_1^2 > (1-2\tau_2)/(1-2\tau_1)$ and $u_2^2 < (1-2\tau_1)/(1-2\tau_2)$ or $\tau_1 > \tau_2$ with the conditions $u_1^2 < (1-2\tau_2)/(1-2\tau_1)$ and $u_2^2 > (1-2\tau_1)/(1-2\tau_2)$, inequality (56) can be violated and the cavity field is entangled.

When $\lambda_4 > 0$ and $\theta = n\pi/2$ with $n = 1, 2, \ldots,$ the left side of inequality (51) takes its maximum value and the inequality becomes

$$\left(1-u_1^2\right)\left(1-u_2^2\right) \geqslant 0. \quad (57)$$

Obviously, the inequality is always valid and the cavity field is never entangled.

To summarize, when $\lambda_4 > 0$, the cavity field can oscillate periodically in time between entangled and separable states since $\theta[= K\gamma\Omega t/(\Omega^2 - 4\Delta^2)]$ is proportional to time.

IV. NUMERICAL RESULTS

In this section, we investigate the influence of the Rabi frequency of the driving field, cavity loss, and the purity and nonclassicality of the initial state on the entanglement property of the cavity field for the case of $\phi_1 - \phi_2 + 2\Phi = 0$ by numerically finding the eigenvalues of the matrix Γ_c defined on the left side of the inequality (43). When one of the eigenvalues of the matrix Γ_c is negative, inequality (43) must be violated and the cavity field certainly is in an entangled

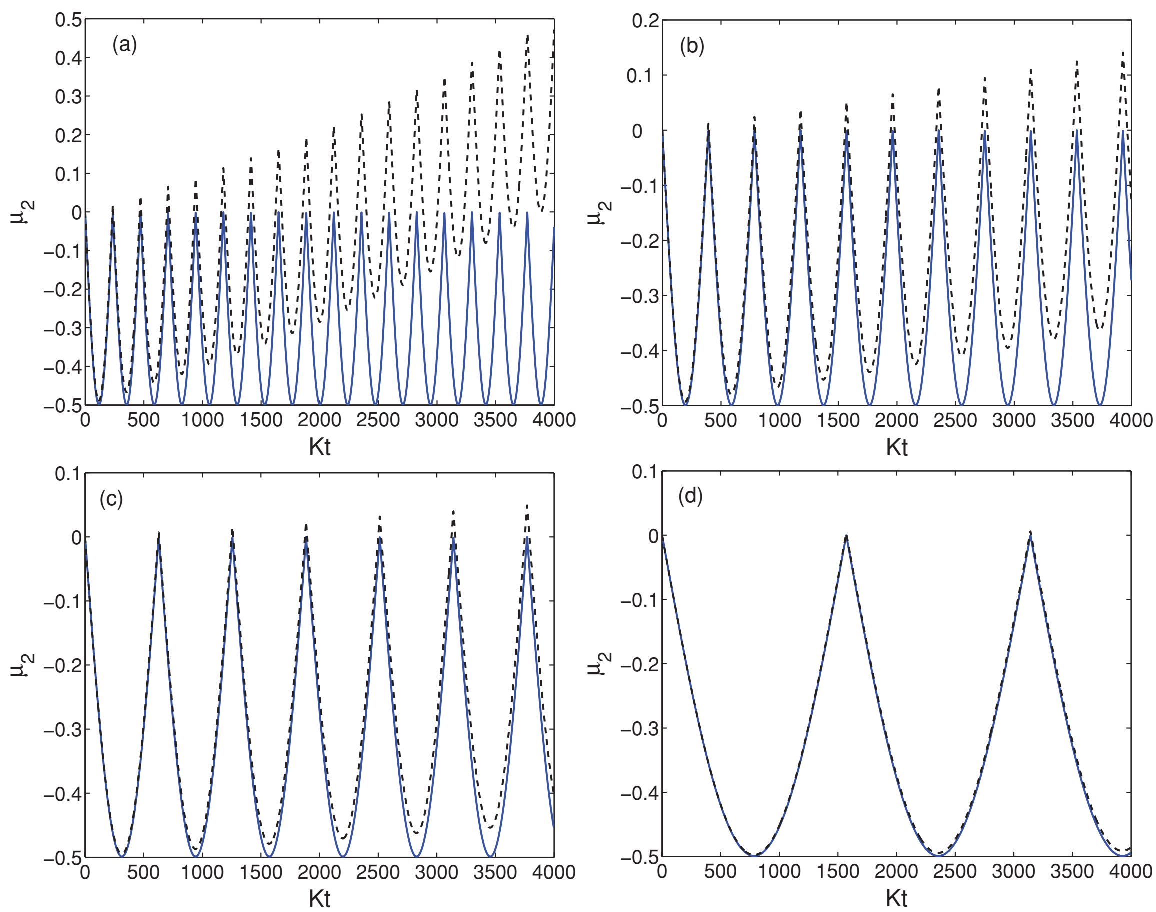

FIG. 2. (Color online) Time evolution of the minimum eigenvalue of the matrix Γ_c against Kt for $\Delta = \gamma$, $\kappa = 0$, $\tau_1 = \tau_2 = 0.499$, $u_1 = u_2 = 1$. The value of the Rabi frequency Ω is (a) 150γ; (b) 250γ; (c) 400γ; (d) 1000γ. The blue-solid and black-dashed lines represent the results obtained from the approximated solutions (35)–(41) and the numerical solutions of (24)–(30), respectively.

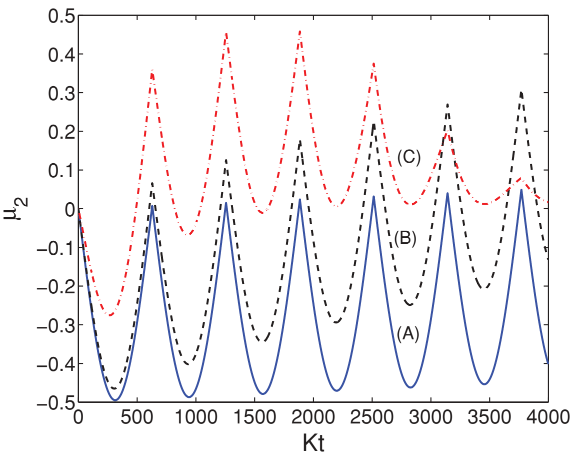

FIG. 3. (Color online) Time evolution of the minimum eigenvalue of the matrix Γ_c against Kt for $\Omega = 400\gamma, \Delta = \gamma, \tau_1 = \tau_2 = 0.499, u_1 = u_2 = 1$. The curves (A), (B), and (C) represent the results for $\kappa = 0, 0.0001K$, and $0.001K$, respectively.

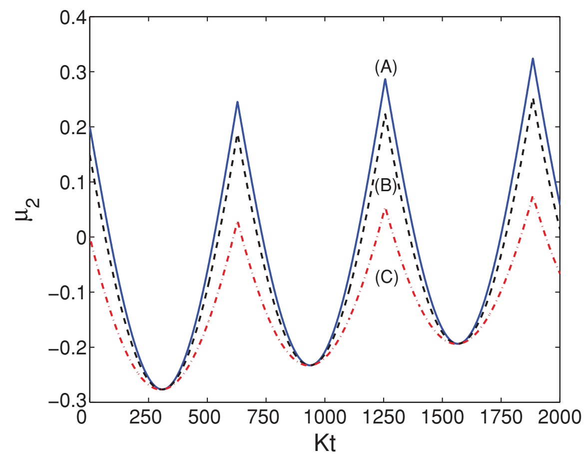

FIG. 5. (Color online) Time evolution of the minimum eigenvalue of the matrix Γ_c against Kt for $\Omega = 400\gamma, \Delta = \gamma, \kappa = 0.0001K, \tau_1 = \tau_2 = 0.499$. The curves (A), (B), and (C) represent the results for $u_1 = u_2 = 0.1, 0.5$, and 1, respectively.

state. In the our calculations, Ω and Δ are rescaled in the unit of γ.

In Fig. 2(a)–2(d), the minimum eigenvalue of Γ_c is plotted against Kt for various values of the Rabi frequency Ω. In these figures, the blue-solid and black-dashed lines represent the results obtained from the approximated solutions (35)–(41) in the limit of $\Omega \gg \gamma$ and the numerical solutions of Eqs. (24)–(30), respectively. We see that the entanglement is generated in the beginning period but gradually is dismissed if the Rabi frequency is not sufficiently large. The cavity field can periodically come into an entangled state in a long time interval as the Rabi frequency increases. In the limit of $\Omega \gg \gamma$, the eigenvalue oscillates between 0 and -0.5 and thus the cavity field periodically evolutes in an ideal entangled state as we have analytically shown in the preceding section.

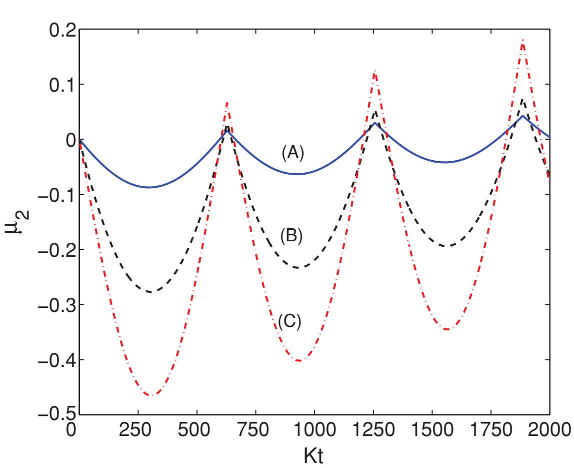

FIG. 4. (Color online) Time development of the minimum eigenvalue of the matrix Γ_c against Kt for $\Omega = 400\gamma, \Delta = \gamma, \kappa = 0.0001K, u_1 = u_2 = 1$. The curves (A), (B), and (C) represent the results for $\tau_1 = \tau_2 = 0.1, 0.3$, and 0.499, respectively.

In Fig. 3, the minimum eigenvalue of the matrix Γ_c is plotted against Kt for various values of the cavity loss rate κ. The cavity field can be entangled in the beginning period even if the cavity loss is large. However, the disentanglement process is greatly speeded up as the cavity loss increases.

In Fig. 4, the minimum eigenvalue of the matrix Γ_c is shown against Kt for various values of the nonclassical depth of the initial state. It is observed that the cavity field can be kept in an entangled state for a long time interval as the nonclassical depth increases.

In Fig. 5, the minimum eigenvalue of the matrix Γ_c vs Kt is shown for various values of the purity of the initial state. We see that the initial purity has the much weak effect on the time evolution of entanglement of the cavity field.

V. SUMMARY

The generation of two-mode Gaussian entangled states of the cavity field is investigated in a quantum beat laser. According to Simon's criterion that is a necessary and sufficient condition for the entanglement existence of two-mode Gaussian states with a general form of covariance matrices, we analytically study the influence of phase of the classical driving field, and the purity and nonclassicality of the initial Gaussian state of the cavity field on inseparability of the resulting two-mode Gaussian state in the limit of strong driving and weak cavity loss. We find that in the limit case the cavity field can evolute periodically in time into an ideal two-mode entangled Gaussian state. For the general case, we numerically investigate the effects of the cavity loss, Rabi frequency of the classical driving field, the purity and nonclassicality of the initial Gaussian state of the cavity field on the entanglement property of the resulting two-mode Gaussian state. We find that keeping the Rabi frequency of the classical field being much larger than the decay rate of the atomic levels to other states is necessary for maintaining the cavity field in an entangled state for a long time interval. It is noticed that the disentanglement process is greatly speeded up as the cavity loss increases. The results also show that the cavity field can be kept in an entangled state for a longer time period as the nonclassicality of the initial field becomes stronger. We also notice that the purity of the initial state has no obvious effects on the generation of the entangled states.

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