Efficient optimal minimum error discrimination of symmetric quantum states

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This article deals with the quantum optimal discrimination among mixed quantum states enjoying geometrical uniform symmetry with respect to a reference density operator ρ_0. It is well known that the minimal error probability is given by the positive operator-valued measure obtained as a solution of a convex optimization problem, namely a set of operators satisfying geometrical symmetry, with respect to a reference operator Π_0 and maximizing $\mathrm{Tr}(\rho_0 \Pi_0)$. In this article, by resolving the dual problem, we show that the same result is obtained by minimizing the trace of a semidefinite positive operator X commuting with the symmetry operator and such that $X \geqslant \rho_0$. The new formulation gives a deeper insight into the optimization problem and allows to obtain closed-form analytical solutions, as shown by a simple but not trivial explanatory example. In addition to the theoretical interest, the result leads to semidefinite programming solutions of reduced complexity, allowing to extend the numerical performance evaluation to quantum communication systems modeled in Hilbert spaces of large dimension.

I. INTRODUCTION

In a quantum system Alice prepares the quantum channel into one of several quantum states. Bob measures the quantum channel by a set of measurement operators and, on the basis of the result, it guesses the choice made by the transmitter. These actions lead to a classical channel, and the problem arises of finding the measurement operators that provide optimal performance according to a predefined criterion; in this article, the minimum error probability. However, to solve the problem, except for some particular cases, has appeared to be a very difficult task since the pioneering contributions in the 1970s [1–3].

In recent years, particular attention has been paid to quantum states satisfying geometrical symmetry [4–7], in view of applications to optical communication systems. In some specific cases, including symmetric pure quantum states [4,5] and symmetric mixed quantum states with a characteristic structure [6], the solution named square root measurement (SRM) proves optimal. Nevertheless, in general, SRM represents a suboptimal strategy, although it provides pretty good performance in many scenarios.

In this article, we are concerned with the construction of optimal positive operator-valued measure (POVM) for the discrimination of symmetric *mixed* quantum states. We present some results that provide intuition into the problem and offer perspectives on its solution.

Optimal quantum state discrimination represents a convex optimization problem, and, as such, it can be formulated in a *primal* and in a *dual* form, with the latter having a reduced number of variables and constraints [3,6]. In this article we investigate how primal and dual problems simplify with symmetric quantum states. For the primal problem this study was already considered in Ref. [6]. Herein, we extend the analysis to the dual problem, where the optimal solution can be searched in a set of smaller dimension. The simplified formulation of the dual problem is illustrated with an example

where a closed-form solution is easily found. For problems of large dimension, that cannot be solved analytically, the reduced number of variables in the simplified dual statement becomes useful to the numerical solution by means of semidefinite programming (SDP) tools.

II. GENERAL FORMULATION OF MINIMUM ERROR DISCRIMINATION

The quantum decision problem is formalized in a N-dimensional complex Hilbert space $\mathcal{H}$ [1], where an ensemble of quantum states ρ_i, $i = 0, 1, \ldots, M-1$, with prior probabilities $q_i \geqslant 0$, $\sum_{i=0}^{M-1} q_i = 1$, is given. The quantum states ρ_i are density operators on $\mathcal{H}$, i.e., (self-adjoint) positive semidefinite (PSD) operators ($\rho_i \geqslant 0$), with unit trace, $\mathrm{Tr}(\rho_i) = 1$. For notation convenience, we denote by $\mathcal{P}$ the class of the PSD operators on $\mathcal{H}$. The measurement operators Π_i, $i = 0, 1, \ldots, M-1$, constitute a POVM having the properties $\Pi_i \in \mathcal{P}$ and $\sum_{i=0}^{M-1} \Pi_i = I$, where I is the identity operator on $\mathcal{H}$. The transition probabilities of the resulting quantum channel become $p(j|i) = \mathrm{Tr}(\rho_i \Pi_j)$, so the probability of correct detection is given by $P_c = \sum_{i=0}^{M-1} q_i \mathrm{Tr}(\rho_i \Pi_i)$.

Hence, the problem of finding the maximum probability of correct state discrimination can be concisely stated as follows.

Primal problem (PP1). Find the maximum of the probability of correct detection $P_c = \sum_{i=0}^{M-1} q_i \mathrm{Tr}(\rho_i \Pi_i)$ over the class of the POVM on $\mathcal{H}$.

The analytical solution of PP1 is in general difficult since P_c has to be maximized over the whole M-tuple of measure operators Π_i. As a matter of fact, closed-form results to the primal problem are available only for some particular quantum mechanical systems, e.g., the binary case [1]. Nevertheless, since the objective is to search a global maximum of a linear function into a convex set, the problem can be faced by means of numerical tools such as SDP. Besides, according to classical results in convex optimization theory [8], in place of the primal problem, it is in general more convenient to consider its corresponding dual problem, since it presents a smaller number of variables and constraints [3,6].

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Dual problem (DP1). Minimize the trace of the optimization operator X over the class $\mathcal{P}$, subject to the constraints $X \geqslant q_i \rho_i$, $i = 0, 1, \ldots, M - 1$. Once found a minimum trace operator X_{opt}, its trace gives the maximum probability of correct detection, $P_c = \text{Tr}(X_{\text{opt}})$.

The equalities $(X_{\text{opt}} - q_i \rho_i)\Pi_i = \Pi_i(X_{\text{opt}} - q_i \rho_i) = 0$, $i = 0, 1, \ldots, M - 1$, are necessary conditions on the optimal POVM. These conditions become sufficient, once the searched measure operators are constrained to belong to $\mathcal{P}$ and to solve the identity on $\mathcal{H}$.

III. DISCRIMINATION OF SYMMETRIC QUANTUM STATES

In quantum detection an important role is played by *geometrically uniform symmetry* [4–6]. Among the several generalizations, we consider the basic case of symmetric mixed quantum states generated from a reference density operator ρ_0 as

$$\rho_i = S^i \rho_0 S^{-i}, \quad i = 0, 1, \ldots, M - 1, \tag{1}$$

where the *symmetry operator* S is unitary ($SS^\dagger = S^\dagger S = I$) and such that $S^M = I$. The geometry implicitly requires that the mixed states are equiprobable, i.e., $q_i = 1/M$, $i = 0, 1, \ldots, M - 1$. In Ref. [6] it was shown that optimal POVM having the same symmetry can always be found. Hence, without loss of generality, we can assume that

$$\Pi_i = S^i \Pi_0 S^{-i}, \quad i = 0, 1, \ldots, M - 1, \tag{2}$$

where Π_0 is the reference measure operator. Consequently, for a fixed M, the knowledge of ρ_0, Π_0 and S is sufficient to fully describe the state ensemble and the POVM.

The density operators (1) have all the same rank, and the same holds for the measure operators (2). As proved in Ref. [9], the optimal measure operators can be assumed to have rank no higher than that of the corresponding density operators, namely $\text{rank}(\Pi_0) \leqslant \text{rank}(\rho_0)$.

A. Primal problem for symmetric quantum states

The specific geometry of the state ensemble can be exploited to get insight into how solving the state discrimination problem. In particular, the primal problem PP1 can be rewritten in a simpler form as follows.

Primal problem for symmetric quantum states (PP2). Find the maximum of $P_c = \text{Tr}(\rho_0 \Pi_0)$, with $\Pi_0 \in \mathcal{P}$ and such that $\sum_{i=0}^{M-1} S^i \Pi_0 S^{-i} = I$.

Proof: This formulation was first given in Ref. [6]. It can straightforwardly be proved by using (1) and (2) in PP1, so that $P_c = \sum_{i=0}^{M-1} q_i \text{Tr}(\rho_i \Pi_i) = \sum_{i=0}^{M-1} \frac{1}{M} \text{Tr}(S^i \rho_0 S^{-i} S^i \Pi_0 S^{-i}) = \text{Tr}(\rho_0 \Pi_0)$. Moreover, if $\Pi_0 \geqslant 0$ then $\Pi_i = S^i \Pi_0 S^{-i} \geqslant 0$. ■

B. Dual problem for symmetric quantum states

The optimization problem PP2 is comparatively simple, and, perhaps, this is the reason why no particular attention has been paid in the literature to the study of the dual theorem to obtain an alternative formulation. In the following we investigate this point.

Dual problem for symmetric quantum states (DP2). Minimize the trace of the optimization operator X over the class $\mathcal{P}$, subject to the constraints $X \geqslant \frac{1}{M} \rho_0$ and $XS = SX$. Once found a minimum trace operator X_{opt}, its trace gives the maximum probability of correct detection, $P_c = \text{Tr}(X_{\text{opt}})$.

Proof: Define $\rho_i' = S^i(\frac{1}{M} \rho_0)S^{-i}$. Let $\mathcal{V}$ be the feasible set according to the general dual problem in Sec. II, i.e., the set of PSD operators X such that $X \geqslant \rho_i'$, $i = 0, 1, \ldots, M - 1$, and let $\mathcal{V}'$ be the set of PSD operators X' such that $X' \geqslant \rho_0'$ and $X'S = SX'$. The proof is organized in two steps. In the first step it is shown that $\mathcal{V}' \subset \mathcal{V}$, while in the second step it is proved that for any $X \in \mathcal{V}$ there exist $X' \in \mathcal{V}'$ such that $\text{Tr}(X') = \text{Tr}(X)$. Then, the search of X can be confined into $\mathcal{V}'$.

Step 1: If $X' \in \mathcal{V}'$, for the commutativity between X' and S we get $X' = SX'S^{-1}$ and recursively $X' = S^i X'S^{-i}$. Hence, for any i

$$X' - \rho_i' = X' - S^i \rho_0' S^{-i} = S^i X'S^{-i} - S^i \rho_0' S^{-i}$$
$$= S^i(X' - \rho_0')S^{-i} \geqslant 0,$$

since $X' \geqslant \rho_0'$ for assumption.

Step 2: For each $X \in \mathcal{V}$ we consider

$$X' = \frac{1}{M} \sum_{i=0}^{M-1} S^{-i} X S^i.$$

Being $X \geqslant \rho_i'$ for each i, it follows that

$$X' \geqslant \frac{1}{M} \sum_{i=0}^{M-1} S^{-i} \rho_i' S^i = \rho_0'.$$

Moreover, recalling that $S^M = I$

$$SX'S^{-1} = \frac{1}{M} \sum_{i=0}^{M-1} S^{-(i-1)} X S^{i-1} = X'$$

and then X' commutes with S. Finally,

$$\text{Tr}(X') = \frac{1}{M} \sum_{i=0}^{M-1} \text{Tr}(S^{-i} X S^i) = \sum_{i=0}^{M-1} \text{Tr}(X S^i S^{-i})$$
$$= \frac{1}{M} \sum_{i=0}^{M-1} \text{Tr}(X) = \text{Tr}(X),$$

and the proof is complete. ■

Therefore, the search of the unknown optimization operator X can be restricted to the subclass of $\mathcal{P}$ composed by the PSD operators that commute with the symmetry operator S.

The optimal Π_0 can be found by the relations $(X_{\text{opt}} - \frac{1}{M} \rho_0)\Pi_0 = \Pi_0(X_{\text{opt}} - \frac{1}{M} \rho_0) = 0$ subject to $\Pi_0 \in \mathcal{P}$ and $\sum_{i=0}^{M-1} S^i \Pi_0 S^{-i} = I$.

In the next section, we develop a formulation, where the commutation condition $XS = SX$ is replaced by an alternative constraint.

C. Alternative formulation of the dual problem

Since the symmetry operator S is known a priori, it is possible to exploit its spectral characterization to rewrite the dual problem DP2 as follows.

Other form for the dual problem for symmetric quantum states (DP3). Let $\lambda_0, \lambda_1, \ldots, \lambda_{\bar{N}-1}$ be the $\bar{N} \leqslant N$ *distinct* eigenvalues of S, and N_i be the multiplicity of λ_i. Let U be a basis of eigenvectors of S, with the first N_0 eigenvectors corresponding to λ_0, the next N_1 eigenvectors corresponding to λ_1, and so on. Then, minimize the trace of the optimization operator $\tilde{X}$ over the subclass of $\mathcal{P}$ consisting of *block-diagonal* operators with blocks of dimension N_i, under the constraint $\tilde{X} \geqslant \frac{1}{M} U^\dagger \rho_0 U$. Once found a minimum trace operator $\tilde{X}_{\text{opt}}$, its trace gives the maximum probability of correct detection, $P_c = \text{Tr}(\tilde{X}_{\text{opt}})$.

Proof: By DP2 the optimization operator X can be chosen to commute with S. Therefore, given an eigenbasis U for S, we can write $X = U\tilde{X}U^\dagger$ where, by well-known results on simultaneous diagonalization of commuting self-adjoint operators [10], $\tilde{X}$ turns out to be block diagonal with the size of the blocks given by the multiplicity of the eigenvalues of S. Moreover, we find that $\text{Tr}(X) = \text{Tr}(U\tilde{X}U^\dagger) = \text{Tr}(\tilde{X})$ and the constraint $X \geqslant \frac{1}{M}\rho_0$ becomes $\tilde{X} \geqslant \frac{1}{M} U^\dagger \rho_0 U$. ∎

The commutative requirement $XS = SX$ is then replaced by fixing the block-diagonal structure on $\tilde{X}$. Given an optimal $\tilde{X}_{\text{opt}}$, the reference optimal operator Π_0 is solution of $(U\tilde{X}_{\text{opt}}U^\dagger - \frac{1}{M}\rho_0)\Pi_0 = \Pi_0(U\tilde{X}_{\text{opt}}U^\dagger - \frac{1}{M}\rho_0) = 0$, subject to $\Pi_0 \in \mathcal{P}$ and $\sum_{i=0}^{M-1} S^i \Pi_0 S^{-i} = I$.

The new formulation of the dual problem is given in a form that is particularly suitable for SDP computational tools [11] and, moreover, it is analytically more tractable that the previous version. We now quantify the complexity of the different approaches.

D. Problem dimension and number of constraints

The complexity of a linear program, to get the solution of a convex optimization problem, is hard to evaluate in terms of arithmetic operations (see Ref. [8] for details). Nevertheless, since we are seeking a global optimum, the dimension of the feasible region for the objective function gives an order of the complexity of the problem [8].

The set of self-adjoint operators on $\mathcal{H}$ forms an N^2 dimensional real vector space. Therefore, for a given optimization problem we can find the dimension d of the correspondent real space on which the considered objective function is defined. In other terms, we find the number of real variables d in a given objective function. The results are summarized in Table I where C_e and C_i represent, respectively, the number of equality and inequality constraints for a given problem. Note that the PSD condition on self-adjoint operators is counted as an inequality, e.g., the relation $\Pi_0 \in \mathcal{P}$ is counted as the inequality $\Pi_0 \geqslant 0$.

TABLE I. Number of real decision variables d, equality constraints C_e, and inequality constraints C_i for the optimization problems.

	d	C_e	C_i
PP1	MN^2	1	M
PP2	N^2	1	1
DP1	N^2	0	M
DP2	N^2	0	2
DP3	$\sum_{i=0}^{\bar{N}} N_i^2$	0	1

The dual problem DP3 presents the smaller number of variables and constraints among the considered optimization problems and, in particular, $d = \sum_{i=0}^{\bar{N}-1} N_i^2$ is smaller than N^2, depending on the spectrum of the symmetry operator S. If S has all distinct eigenvalues, then $\bar{N} = N$, $N_i = 1$ for each i, and the optimization operator $\tilde{X}$ in DP3 becomes diagonal, giving $d = N < N^2$.

IV. EXAMPLE OF APPLICATION

In this section, the previous results are applied to the discrimination of an ensemble of M symmetric mixed quantum states on a two-dimensional ($N = 2$) complex Hilbert space. The symmetry operator is

$$S = \begin{bmatrix} \cos\left(\frac{\pi}{M}\right) & -\sin\left(\frac{\pi}{M}\right) \\ \sin\left(\frac{\pi}{M}\right) & \cos\left(\frac{\pi}{M}\right) \end{bmatrix}, \quad (3)$$

and it represents a linear transformation given by a counter-clockwise rotation through angle π/M. We assume that the reference density operator has the general form

$$\rho_0 = \begin{bmatrix} \alpha & \beta \\ \beta & 1 - \alpha \end{bmatrix}, \quad (4)$$

where α and β are real numbers. Since ρ_0 is PSD, the feasible values of α and β are constrained as $0 \leqslant \alpha \leqslant 1$ and $|\beta| \leqslant \sqrt{\alpha(1-\alpha)}$, respectively. Without loss of generality we assume $\alpha \geqslant 1/2$.

The operator S has two nondegenerate eigenvalues equal to $\lambda_1(S) = e^{i\pi/M}$ and $\lambda_2(S) = e^{-i\pi/M}$. Therefore, the corresponding eigenvectors define the basis

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}. \quad (5)$$

Consequently, for the dual problem DP3 the optimization operator $\tilde{X}$ has to be diagonal

$$\tilde{X} = \begin{bmatrix} \tilde{x}_1 & 0 \\ 0 & \tilde{x}_2 \end{bmatrix}, \quad (6)$$

where both $\tilde{x}_1$ and $\tilde{x}_2$ are real and non-negative, being $\tilde{X}$ PSD. The constraint $\tilde{X} \geqslant \frac{1}{M} U^\dagger \rho_0 U$ reads

$$\tilde{X} \geqslant \frac{1}{2M} \begin{bmatrix} 1 & (2\alpha - 1) + i2\beta \\ (2\alpha - 1) - i2\beta & 1 \end{bmatrix}, \quad (7)$$

and after some simple algebra,[1] we find that it can be rewritten as

$$(2M\tilde{x}_1 - 1)(2M\tilde{x}_2 - 1) - [(2\alpha - 1)^2 + (2\beta)^2] \geqslant 0, \quad (8)$$

with $\tilde{x}_1 \geqslant (1/2M)$ and $\tilde{x}_2 \geqslant (1/2M)$. Hence, the minimum of $\text{Tr}(\tilde{X}) = \tilde{x}_1 + \tilde{x}_2$ is obtained for $\tilde{x}_1 = \tilde{x}_2 = (1/2M)(1 + \sqrt{(2\alpha - 1)^2 + (2\beta)^2}$. In conclusion, the minimum error probability $P_e = 1 - P_c = 1 - \text{Tr}(\tilde{X})$ is

$$P_e = \frac{M-1}{M} - \frac{1}{M}\sqrt{(2\alpha - 1)^2 + (2\beta)^2}. \quad (9)$$

We note that the first term on the right-hand side of (9) corresponds to a blind guessing on the equiprobable elements

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[1]It is recalled that a Hermitian matrix is PSD if and only if its principal minors are all non-negative.

belonging to the quantum state ensemble, while the second term represents the gain due to the optimal quantum discrimination. It is interesting to observe that the minimum error probability is obtained for $\beta = \sqrt{\alpha(1-\alpha)}$ and it results

$$P_e = 1 - \frac{2}{M}. \tag{10}$$

For instance, (10) holds for $\alpha = 1$ and $\beta = 0$ which is the case of study considered by Helstrom [1] and Ban *et al.* [5], which models linearly dependent spin-1/2 quantum states, where the generating density operator has rank-one $\rho_0 = |\psi_0\rangle\langle\psi_0|$ with pure state $|\psi_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. When ρ_0 is diagonal, i.e., $\beta = 0$, (9) simplifies as

$$P_e = 1 - \alpha\frac{2}{M}. \tag{11}$$

The optimal reference measure operator Π_0 can be found from the conditions given in Sec. III C, $(U\tilde{X}_{\text{opt}}U^\dagger - \frac{1}{M}\rho_0)\Pi_0 = \Pi_0(U\tilde{X}_{\text{opt}}U^\dagger - \frac{1}{M}\rho_0) = 0$, that in this example simplify as $(\tilde{x}_1 I - \frac{1}{M}\rho_0)\Pi_0 = \Pi_0(\tilde{x}_1 I - \frac{1}{M}\rho_0) = 0$, being $\tilde{X}_{\text{opt}} = \tilde{x}_1 I$. Note that these conditions also imply that $\rho_0\Pi_0 = \Pi_0\rho_0$. The optimal Π_0 can then be numerically found by solving a linear system of equations including the additional requirements $\Pi_0 \in \mathcal{P}$ and $\sum_{i=0}^{M-1} S^i \Pi_0 S^{-i} = I$.

It is useful to observe that the condition $\sum_{i=0}^{M-1} S^i \Pi_0 S^{-i} = I$ implicitly fixes the value of the trace of Π_0. In fact, $\text{Tr}(\sum_{i=0}^{M-1} S^i \Pi_0 S^{-i}) = \sum_{i=0}^{M-1}\text{Tr}(S^i\Pi_0 S^{-i}) = \sum_{i=0}^{M-1}\text{Tr}(\Pi_0 S^{-i}S^i) = M\text{Tr}(\Pi_0)$ and being $\text{Tr}(I) = N$ it follows that $\text{Tr}(\Pi_0) = N/M$. By simple algebra it can be found a closed-form expression for Π_0 in the following two cases.

(a.) $\beta = 0$. The optimal Π_0 is given by

$$\Pi_0 = \frac{2}{M}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{12}$$

In particular, setting $\alpha = 1/3$ the same numerical results obtained in Ref. [12] are found.

(b.) $\beta = \sqrt{\alpha(1-\alpha)}$. The measure operator results

$$\Pi_0 = \frac{2}{M}\rho_0. \tag{13}$$

The constraint $\sum_{i=0}^{M-1} S^i \Pi_0 S^{-i} = I$ becomes $\frac{2}{M}\sum_{i=0}^{M-1}\rho_i = I$ showing that the quantum state ensemble has a particular structure. Indeed, such a specific geometry has been considered by Yuen *et al.* in Ref. [3], (IV.4)] and the results therein reported are in agreement with (10) and (13).

The proposed formulation of the dual problem has also proved useful to numerically solve systems of large dimensions, where the computational complexity sets a severe limit to the possibility of finding an optimal solution. This is the case of pulse position modulation (PPM), a modulation format candidate for deep space communications [13]. In quantum PPM the states are defined in a Hilbert space given by the tensorial product of M subspaces, each of dimension n, and, therefore, the overall space dimension N grows exponentially with the PPM order M, being $N = n^M$ [14]. In Ref. [15], DP3 is applied to quantum PPM using the software CVX for SDP [11]. The solution of DP3 results considerably faster than DP1.

V. CONCLUSIONS

We have studied the dual problem for minimum error probability discrimination of symmetric quantum states. It has been shown that the optimization operator, in the objective function, can be assumed to commute with the symmetry operator. This result leads to an alternative formulation of the dual problem that presents a reduced number of variables and constraints. The obtained dual statement is convenient to find analytical solutions to the discrimination problem, as we showed with an illustrative example. On the other hand, the new formulation also permits a computationally efficient numerical solution by means of SDP methods. This property is particularly useful to study the performance limits of quantum mechanical systems described by geometrically uniform quantum states on Hilbert spaces of large dimensions, such as modulated coherent states in optical communications.

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