

## Relativistic massless harmonic oscillator

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A detailed study of the relativistic classical and quantum mechanics of the massless harmonic oscillator is presented.

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### I. INTRODUCTION

One of the most important physical systems for both classical and quantum mechanics is the harmonic oscillator. In contrast to the nonrelativistic harmonic oscillator that is discussed in most textbooks, the theory of the relativistic harmonic oscillator is far from complete. The reason for this is the complexity of the problem related to the nonlinearity of differential equations of motion for the classical relativistic oscillator. No wonder that even in the simple case of the massive one-dimensional relativistic oscillator there are problems with the identification of periodic solutions to equations of motion [1]. The problem of a quantum relativistic harmonic oscillator is usually formulated in one of three different frameworks: the Klein-Gordon, Dirac, or Salpeter equations. The first one uses the spinless Klein-Gordon equation with a Lorentz invariant oscillatory potential [2]. However, the solutions of that equation are blamed by pathologies such as the appearance of ghost states. The second approach, referred to by Moshinsky [3] as the “Dirac oscillator” and describing spin one-half particles utilizes the Dirac equation with an appropriate combination of the scalar, vector, and tensor couplings with an external field [4]. It can be successively applied to the analysis of relativistic symmetries, which recently were recognized experimentally in both nuclear and hadron spectroscopy [5]. Unfortunately, this approach has no classical relativistic counterpart. Finally, the third approach follows from the relativistic Hamiltonian dynamics for a scalar particle and on the quantum level it is based on the spinless Salpeter equation [6]. The Salpeter equation [6–16] is a “square root” of the Klein-Gordon equation [17] and can be regarded as its alternative [16]. The serious advantages of the Salpeter scheme are the lack of problems with probabilistic interpretation on the quantum level as well as the classically well-defined physical content of this theory. This last framework is frequently used as a phenomenological description of the quark-antiquark-gluon system as a hadron model.

Surprisingly, to our best knowledge, the simplest case of the massless relativistic harmonic oscillator was not discussed in the literature. In this work we perform a detailed analysis of the massless relativistic harmonic oscillator. In particular, we find the exact solutions to the classical Hamilton equations as well as to the corresponding quantum Salpeter equation and discuss their basic properties. The article is organized as follows. In Sec. II, by integrating the corresponding Hamilton system we identify all kinds of possible motion of the oscillator as well as find its quantitative characteristics. For an easy illustration of the dynamics of the relativistic massless harmonic oscillator we also provide a graphical presentation of the numerical inte-

gration of the equations of motion. Section III is devoted to the quantization of the massless relativistic harmonic oscillator.

### II. THE ANALYSIS OF THE CLASSICAL RELATIVISTIC MASSLESS HARMONIC OSCILLATOR

The Hamiltonian of the relativistic massless particle subject to the potential  $\frac{1}{2}\kappa^2\mathbf{x}^2$  is given by

$$H = c|\mathbf{p}| + \frac{1}{2}\kappa^2\mathbf{x}^2, \quad (2.1)$$

where  $\mathbf{x}$  and  $\mathbf{p}$  are the position and the momentum of a particle,  $|\mathbf{p}| = \sqrt{\mathbf{p}^2}$  is the norm of the vector  $\mathbf{p}$  (so  $c|\mathbf{p}|$  is the kinetic energy of the particle),  $\kappa$  is a constant, and  $c$  is the speed of light. Therefore, the Hamilton's equations are

$$\begin{aligned} \dot{\mathbf{x}} &= c \frac{\mathbf{p}}{|\mathbf{p}|}, \\ \dot{\mathbf{p}} &= -\kappa^2\mathbf{x}. \end{aligned} \quad (2.2)$$

We point out that an immediate consequence of Eq. (2.2) is  $\dot{\mathbf{x}}^2 = c^2$ , that is, the length of the velocity is  $c$ , as it should be for a massless particle. The familiar integrals of the motion in a central field [18] are the energy  $E$  and the angular momentum  $\mathbf{J}$

$$E = c|\mathbf{p}| + \frac{\kappa^2}{2}\mathbf{x}^2, \quad (2.3)$$

$$\mathbf{J} = \mathbf{x} \times \mathbf{p}. \quad (2.4)$$

As a result of the conservation of the angular momentum  $\mathbf{J}$  the motion is planar and we can restrict, without loss of generality, to the case of a particle moving in the  $(x^1, x^2)$  plane. On passing to the polar coordinates  $\mathbf{x} = (x^1, x^2) = (r \cos \varphi, r \sin \varphi)$  and  $\mathbf{p} = (p^1, p^2) = (p \cos \theta, p \sin \theta)$ , where  $r = |\mathbf{x}|$  and  $p = |\mathbf{p}|$ , we obtain from Eq. (2.2) the following system

$$\begin{aligned} \dot{r} &= c \cos(\theta - \varphi), \\ \dot{\varphi} &= \frac{c}{r} \sin(\theta - \varphi), \\ \dot{p} &= -\kappa^2 r \cos(\theta - \varphi), \\ \dot{\theta} &= \kappa^2 \frac{r}{p} \sin(\theta - \varphi). \end{aligned} \quad (2.5)$$

The integrals of the motion take the form

$$E = cp + \frac{1}{2}\kappa^2 r^2, \quad (2.6)$$

$$J \equiv J_3 = rp \sin(\theta - \varphi). \quad (2.7)$$

From Eqs. (2.5), (2.6), and (2.7) we find

$$r \sqrt{r^2 \left( E - \frac{\kappa^2}{2} r^2 \right)^2 - (Jc)^2} d\varphi = \pm |J| c dr. \quad (2.8)$$

We point out that the “+” and “−” signs correspond to the two possible orientations of the angular momentum. We choose, without loss of generality, the sign “+” and  $J > 0$  throughout this work. Now, from Eq. (2.8) we find that the trajectories should satisfy

$$r \left( E - \frac{\kappa^2}{2} r^2 \right) \geq Jc, \quad (2.9)$$

and we can classify the types of motion as follows. The first two possibilities refer to  $J \neq 0$ . Namely,

1. For  $r(E - \frac{\kappa^2}{2} r^2) = Jc$ , we get

$$\begin{aligned} r &= R = \sqrt{\frac{2E}{3\kappa^2}}, & \varphi &= \omega t + \varphi_0, \\ p &= p_0 = \frac{2E}{3c}, & \theta &= \omega t + \varphi_0 + \frac{\pi}{2}, \end{aligned} \quad (2.10)$$

where  $\omega = \frac{c}{R} = \frac{\kappa^2 R}{p_0} = c\sqrt{\frac{3\kappa^2}{2E}}$ . So in this case we have a uniform motion in a circle with the linear speed  $|\mathbf{v}| = \omega R = c$ . This solution can also be obtained from Eq. (2.2) by demanding  $\mathbf{x} \cdot \mathbf{p} = 0$ . Indeed, it can be easily checked that Eqs. (2.2) and (2.3) imply the system

$$\begin{aligned} \frac{d}{dt} \mathbf{x} \cdot \mathbf{p} &= E - \frac{3}{2} \kappa^2 \mathbf{x}^2, \\ \frac{d}{dt} \mathbf{p}^2 &= -2\kappa^2 \mathbf{x} \cdot \mathbf{p}, \\ \frac{d}{dt} \mathbf{x}^2 &= \frac{2c}{|\mathbf{p}|} \mathbf{x} \cdot \mathbf{p}. \end{aligned} \quad (2.11)$$

From Eq. (2.11) it follows easily that the orthogonality of  $\mathbf{x}$  and  $\mathbf{p}$  refers to the motion of a particle in a circle with radius  $|\mathbf{x}| = R = \sqrt{\frac{2E}{3\kappa^2}}$ .

2. For  $r(E - \frac{\kappa^2}{2} r^2) > Jc$ , the path lies entirely within the annulus bounded by the circles  $r = r_{\min}$  and  $r = r_{\max}$ , that is, we have

$$r_{\min} \leq r \leq r_{\max}, \quad (2.12)$$

where  $r_{\min}$  and  $r_{\max}$  are the real positive solutions of the equation

$$\frac{\kappa^2}{2} r^3 - rE + Jc = 0. \quad (2.13)$$

We find after some calculation

$$r_{\min} = 2\sqrt{\frac{2E}{3\kappa^2}} \sin \frac{\alpha}{3}, \quad (2.14)$$

$$r_{\max} = \sqrt{\frac{2E}{3\kappa^2}} \left( \sqrt{3} \cos \frac{\alpha}{3} - \sin \frac{\alpha}{3} \right), \quad (2.15)$$

where  $\sin \alpha = \frac{Jc}{\kappa^2} \left( \frac{3\kappa^2}{2E} \right)^{\frac{3}{2}}$  and  $0 \leq \alpha \leq \frac{\pi}{2}$ .

We now return to Eq. (2.8). An immediate consequence of the integration of Eq. (2.8) is the relation

$$\varphi = \varphi_0 + \frac{Jc}{2} \int_{r_0^2}^{r^2} \frac{dx}{x \sqrt{x(E - \frac{\kappa^2}{2} x)^2 - (Jc)^2}}. \quad (2.16)$$

In the particular case of  $r_0 = r_{\min} \neq 0$  and  $r(t) > r_{\min}$ ,  $t > 0$ , the integral from the right-hand side of

Eq. (2.16), can be expressed by means of the elliptic integral of the third kind  $\Pi(\phi, n, k)$  (see Ref. [19], 3.137, Eq. (3)), namely we have

$$\begin{aligned} \varphi &= \varphi_0 + \frac{2Jc}{\kappa^2 r_{\min}^2 \sqrt{r_-^2 - r_{\min}^2}} \Pi \left( \arcsin \sqrt{\frac{r^2 - r_{\min}^2}{r_{\max}^2 - r_{\min}^2}}, \right. \\ &\quad \left. 1 - \frac{r_{\max}^2}{r_{\min}^2}, \sqrt{\frac{r_{\max}^2 - r_{\min}^2}{r_-^2 - r_{\min}^2}} \right), \quad r_0 = r_{\min}, \end{aligned} \quad (2.17)$$

where  $r_-$  is the negative root of the polynomial from the left-hand side of Eq. (2.13) satisfying

$$r_{\min} + r_{\max} + r_- = 0, \quad r_-^2 > r_{\max}^2 > r_{\min}^2. \quad (2.18)$$

Clearly Eq. (2.17) defines  $r$  as an implicit function of  $\varphi$ .

Furthermore, for  $r_0 = r_{\max} \neq 0$  and  $r(t) < r_{\max}$ ,  $t > 0$ , the implicit equation for the trajectory can be obtained from Eq. (2.16) with the help of the elliptic functions of the third kind  $\Pi(\phi, n, k)$  and first kind  $F(\phi, k)$  (see Ref. [19], 3.137, Eq. (4)). It follows that

$$\begin{aligned} \varphi &= \varphi_0 - \frac{2Jc}{\kappa^2 r_-^2 r_{\max}^2 \sqrt{r_-^2 - r_{\min}^2}} \left[ (r_-^2 - r_{\max}^2) \right. \\ &\quad \times \Pi \left( \arcsin \sqrt{\frac{(r_-^2 - r_{\min}^2)(r_{\max}^2 - r^2)}{(r_{\max}^2 - r_{\min}^2)(r_-^2 - r^2)}}, \right. \\ &\quad \left. \frac{r_{\max}^2 - r_{\min}^2}{r_-^2 - r_{\min}^2} \frac{r_-^2}{r_{\max}^2}, \sqrt{\frac{r_{\max}^2 - r_{\min}^2}{r_-^2 - r_{\min}^2}} \right) \\ &\quad \left. + r_{\max}^2 F \left( \arcsin \sqrt{\frac{(r_-^2 - r_{\min}^2)(r_{\max}^2 - r^2)}{(r_{\max}^2 - r_{\min}^2)(r_-^2 - r^2)}}, \right. \right. \\ &\quad \left. \left. \sqrt{\frac{r_{\max}^2 - r_{\min}^2}{r_-^2 - r_{\min}^2}} \right) \right], \quad r_0 = r_{\max}. \end{aligned} \quad (2.19)$$

Now, the four-dimensional system (2.2), where  $\mathbf{x} = (x^1, x^2)$  and  $\mathbf{p} = (p^1, p^2)$  are completely integrable. Indeed, it possesses two integrals in involution  $E$  and  $J$ . Therefore, the motion between two circles with the radius  $r_{\min}$  and  $r_{\max}$  can be only quasiperiodic and periodic. Of course, the case of the periodic motion refers to a closed path. This means that an angle  $\Delta\varphi$  given by [see formula (2.17)]

$$\begin{aligned} \Delta\varphi &= \frac{Jc}{2} \int_{r_{\min}^2}^{r_{\max}^2} \frac{dx}{x \sqrt{x(E - \frac{\kappa^2}{2} x)^2 - (Jc)^2}} \\ &= \frac{2Jc}{\kappa^2 r_{\min}^2 \sqrt{r_-^2 - r_{\min}^2}} \\ &\quad \times \Pi \left( \frac{\pi}{2}, 1 - \frac{r_{\max}^2}{r_{\min}^2}, \sqrt{\frac{r_{\max}^2 - r_{\min}^2}{r_-^2 - r_{\min}^2}} \right), \end{aligned} \quad (2.20)$$

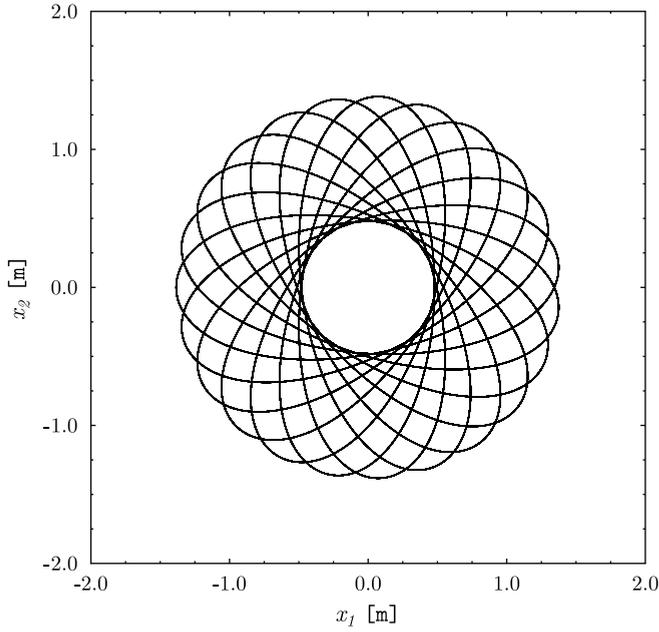


FIG. 1. The periodic solution of the system (2.2) obtained by numerical integration. The initial data are  $\mathbf{x}_0 = (0.479000, 0.000000)$  m,  $\mathbf{p}_0 = (0.000000, 1.290805)$  Jsm $^{-1}$ , the parameter  $\kappa^2 = 1$  Jm $^{-2}$ , and  $c = 1$  ms $^{-1}$ .

should be a rational function of  $\pi$  (i.e.,  $\Delta\varphi = \pi m/n$ , where  $m$  and  $n$  are integers). An example of a periodic motion between two circles is presented in Figs. 1 and 2. It should be noted that the length of momentum (kinetic energy of the particle  $c|\mathbf{p}|$ ) has maximum at  $r = r_{\min}$ , decreases (increases) as  $r$  approaches  $r_{\max}$  ( $r_{\min}$ ), and for  $r = r_{\max}$  has minimum. Clearly such behavior of the momentum of a massless particle is consistent with the

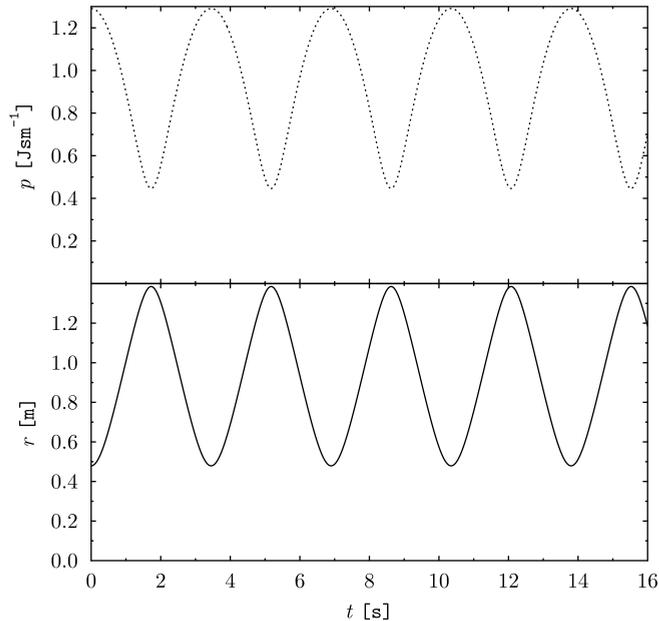


FIG. 2. The plot of the radius  $r = |\mathbf{x}|$  (solid line) and the length of momentum  $p = |\mathbf{p}|$  (dotted line) versus time obtained by the numerical integration of (2.11). The initial condition is the same as in Fig. 1.

form of Eq. (2.6). The values of  $r_{\min}$  and  $r_{\max}$  as well as the extrema of the length of the momentum can be expressed as a function of the energy by means of the implicit formulas (2.14) and (2.15). We finally remark that the case of the uniform motion in a circle discussed earlier [type 1) of the motion] refers to the condition

$$r_{\min} = r_{\max} = R = \sqrt{\frac{2E}{3\kappa^2}}.$$

The third type of the motion corresponds to  $J = 0$ , so we have

3.  $r(E - \frac{\kappa^2}{2}r^2) \geq 0$ . From this inequality we find  $0 \leq r \leq r_{\max} = \frac{\sqrt{2E}}{\kappa}$ . On the other hand, taking into account Eq. (2.7) we find that for  $J = 0$  the system (2.5) reduces to

$$\begin{aligned} \dot{r} &= \pm c, \\ \dot{\varphi} &= 0, \\ \dot{p} &= \mp \kappa^2 r, \\ \dot{\theta} &= 0, \end{aligned} \quad (2.21)$$

where  $\theta - \varphi = 0$  or  $|\theta - \varphi| = \pi$ . Therefore, a particle motion is uniform in a segment  $[0, \frac{\sqrt{2E}}{\kappa}]$ , more precisely, we have  $r = \pm ct + r_0$ , where  $0 \leq r \leq \frac{\sqrt{2E}}{\kappa}$  and the two signs correspond to two possible directions of motion. Assuming that a particle moves in the  $x$ -coordinate line, (i.e.,  $x = x^1$ ) we get

$$x = \pm ct + x_0, \quad -r_{\max} \leq x \leq r_{\max}, \quad (2.22)$$

where the turning points are  $x = r_{\max}$  and  $x = -r_{\max}$ . On setting  $x_0 = -r_{\max}$ , we can write the trajectory explicitly as

$$x(t) = (-1)^{\lfloor \frac{2t}{T} \rfloor} c \left\{ t - \left( 2 \left[ \frac{2t}{T} \right] + 1 \right) \frac{T}{4} \right\}, \quad (2.23)$$

where  $T = \frac{4r_{\max}}{c}$  is the period of oscillations of a massless particle between the turning points  $x = r_{\max}$  and  $x = -r_{\max}$  and  $[a]$  is the biggest integer in  $a$ . The trajectory (2.23) is illustrated in Fig. 3. Notice that, at the turning points, the momentum of a massless particle vanishes [see Eq. (2.6) for  $r = r_{\max}$ ] that is  $p_{\min} = 0$ . The maximum value of momentum  $p_{\max} = \frac{E}{c}$  is reached for  $x = 0$ . The time development of the momentum for  $p_0 = 0$  and  $x_0 = -r_{\max}$  can be written in the form

$$p = -\frac{\kappa^2}{2} \left( x^2(t) - c \frac{T}{4} \right), \quad (2.24)$$

where  $x(t)$  is given by Eq. (2.23). The plot of  $p$  versus  $t$  is shown in Fig. 3. Because the momentum of a massless particle tends to zero as its position approaches the turning point we deal with a “red shift” similar to the gravitational one. It should also be noted that the motion in the segment can be easily obtained from Eq. (2.11) by setting  $\frac{x\mathbf{p}}{|\mathbf{x}||\mathbf{p}|} = \pm 1$ , that is,  $\mathbf{x}$  and  $\mathbf{p}$  are parallel or antiparallel and therefore satisfy  $\mathbf{x} \times \mathbf{p} = \mathbf{0}$ . Evidently, in the case of the system (2.2) this condition is equivalent to  $J = 0$ . We point out that the motion in a segment corresponds to the

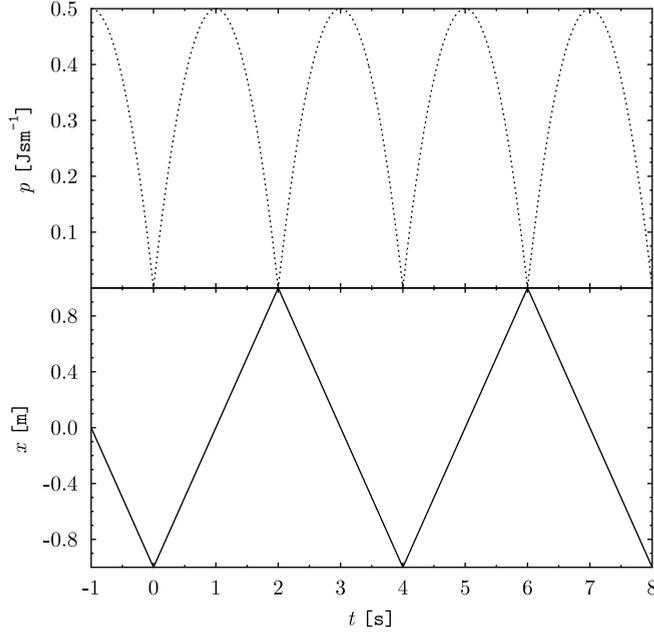


FIG. 3. The plot of the coordinate (solid line) and the momentum (dotted line) of a massless oscillating particle versus time given by Eqs. (2.23) and (2.24), respectively, where  $c = 1 \text{ ms}^{-1}$ ,  $\kappa^2 = 1 \text{ Jm}^{-2}$ ,  $r_{\text{max}} = 1 \text{ m}$ , and  $T = 4 \text{ s}$ .

condition  $r_{\text{min}} = 0$  and  $r_{\text{max}} = \frac{\sqrt{2E}}{\kappa}$  for the nonnegative solutions to Eq. (2.13). We finally remark that the type of motion is completely determined by the values of the energy  $E$  and the angular momentum  $J$ . Namely, using the parametrization of  $r_{\text{min}}$  and  $r_{\text{max}}$  defined by Eqs. (2.14) and (2.15) we find

$$0 \leq \frac{Jc}{\kappa^2} \left( \frac{3\kappa^2}{2E} \right)^{\frac{3}{2}} \leq 1, \quad (2.25)$$

where  $\frac{Jc}{\kappa^2} \left( \frac{3\kappa^2}{2E} \right)^{\frac{3}{2}} = 1$  refers to the motion in a circle,  $0 < \frac{Jc}{\kappa^2} \left( \frac{3\kappa^2}{2E} \right)^{\frac{3}{2}} < 1$  corresponds to the motion between two circles, and  $\frac{Jc}{\kappa^2} \left( \frac{3\kappa^2}{2E} \right)^{\frac{3}{2}} = 0$ , so  $J = 0$  is the condition for the motion in the segment.

### III. QUANTUM MECHANICS OF THE RELATIVISTIC MASSLESS HARMONIC OSCILLATOR

In relativistic quantum mechanics the massless harmonic oscillator defined by the Hamiltonian (2.1) is described by a massless version of the spinless Salpeter equation

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = \left( c\hbar \sqrt{-\Delta_{\mathbf{x}}} + \frac{\kappa^2}{2} \mathbf{x}^2 \right) \psi(\mathbf{x}, t), \quad (3.1)$$

where  $\Delta_{\mathbf{x}} = \left( \frac{\partial}{\partial \mathbf{x}} \right)^2$ . Therefore, the eigenvalue equation for the Hamiltonian  $\hat{H} \psi_E = E \psi_E$  takes the form of the pseudodifferential equation

$$\left( c\hbar \sqrt{-\Delta_{\mathbf{x}}} + \frac{\kappa^2}{2} \mathbf{x}^2 \right) \psi_E(\mathbf{x}) = E \psi_E(\mathbf{x}). \quad (3.2)$$

Performing the Fourier transformation

$$\psi(\mathbf{x}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int d^3 \mathbf{k} e^{i \frac{\mathbf{k} \cdot \mathbf{x}}{\hbar}} \tilde{\psi}(\mathbf{k}), \quad (3.3)$$

we get from Eq. (3.2) the following equation

$$\left( -\Delta_{\mathbf{k}} + \frac{2c}{(\kappa\hbar)^2} |\mathbf{k}| \right) \tilde{\psi}_E(\mathbf{k}) = \frac{2E}{(\kappa\hbar)^2} \tilde{\psi}_E(\mathbf{k}), \quad (3.4)$$

where  $\Delta_{\mathbf{k}} = \left( \frac{\partial}{\partial \mathbf{k}} \right)^2$ . Finally, switching over to the spherical coordinates  $\mathbf{k} = (k \sin \alpha \cos \beta, k \sin \alpha \sin \beta, k \cos \alpha)$ , where  $k = |\mathbf{k}|$  and making the ansatz

$$\tilde{\psi}_E(\mathbf{k}) = \frac{\chi(k)}{k} Y_l^m(\alpha, \beta), \quad (3.5)$$

where  $Y_l^m(\alpha, \beta)$  are the spherical functions, we obtain the ‘‘radial equation’’

$$\left( -\frac{d^2}{dk^2} + \frac{l(l+1)}{k^2} + \frac{2c}{(\kappa\hbar)^2} k \right) \chi(k) = \frac{2E}{(\kappa\hbar)^2} \chi(k). \quad (3.6)$$

To our best knowledge, in the case of  $l \neq 0$  the solution of Eq. (3.6) is not known. For  $l = 0$  the solution to Eq. (3.6) can be expressed by means of the Airy function  $\text{Ai}(x)$  [20], namely

$$\chi(k) = C \text{Ai} \left[ \frac{2c}{(2c\kappa\hbar)^{\frac{2}{3}}} \left( k - \frac{E}{c} \right) \right], \quad (3.7)$$

where  $C$  is constant. We point out that  $l = 0$  was also the case discussed in Ref. [6] where the recurrence was identified that is satisfied by coefficients of the formal power expansion for the solution to the spinless Salpeter equation corresponding to the massive relativistic harmonic oscillator. Clearly  $l = 0$  refers to the vanishing angular momentum, therefore we deal in this case with the quantization of the motion of a massless particle in the segment  $0 \leq r \leq r_{\text{max}} = \frac{\sqrt{2E}}{\kappa}$  discussed in the previous section and corresponding to the condition  $J = 0$  (third type of the motion). Furthermore, for  $l = 0$  the ansatz (3.5) takes the form

$$\tilde{\psi}_E(\mathbf{k}) = \frac{\chi(k)}{k} Y_0^0(\alpha, \beta) = \frac{1}{\sqrt{4\pi}} \frac{\chi(k)}{k}. \quad (3.8)$$

Demanding that  $\tilde{\psi}_E(\mathbf{k})$  is well defined for  $k = 0$  we find  $\chi(0) = 0$  (compare Ref. [21] Eq. (32.11)), which leads to  $\text{Ai}\left(-\frac{2E}{(2c\kappa\hbar)^{\frac{2}{3}}}\right) = 0$ . This quantization condition means that the values of the energy  $E_n$ ,  $n = 1, 2, \dots$ , are given by the zeros of the Airy function  $a_n$ . We have

$$E_n = -\frac{(2c\kappa\hbar)^{\frac{2}{3}}}{2} a_n, \quad n = 1, 2, \dots, \quad (3.9)$$

Using the fact that the functions  $\text{Ai}(x + a_n)/\text{Ai}'(a_n)$ ,  $n = 1, 2, \dots$ , where  $\text{Ai}'(x)$  designates the derivative of the Airy function  $\text{Ai}(x)$ , form an orthonormal basis on the interval  $[0, \infty)$  [16], we find that the normalized solutions (3.5) to (3.4) in the Hilbert space  $L^2(\mathbb{R}^3, d^3 \mathbf{k})$ , with  $l = 0$  can be written as

$$\begin{aligned} \tilde{\psi}_n(\mathbf{k}) \equiv \tilde{\psi}_{E_n}(\mathbf{k}) &= \sqrt{\frac{c}{2\pi}} \frac{1}{(2c\kappa\hbar)^{\frac{1}{3}}} \frac{1}{\text{Ai}'(a_n)} \frac{1}{k} \\ &\times \text{Ai} \left( \frac{2c}{(2c\kappa\hbar)^{\frac{2}{3}}} k + a_n \right). \end{aligned} \quad (3.10)$$

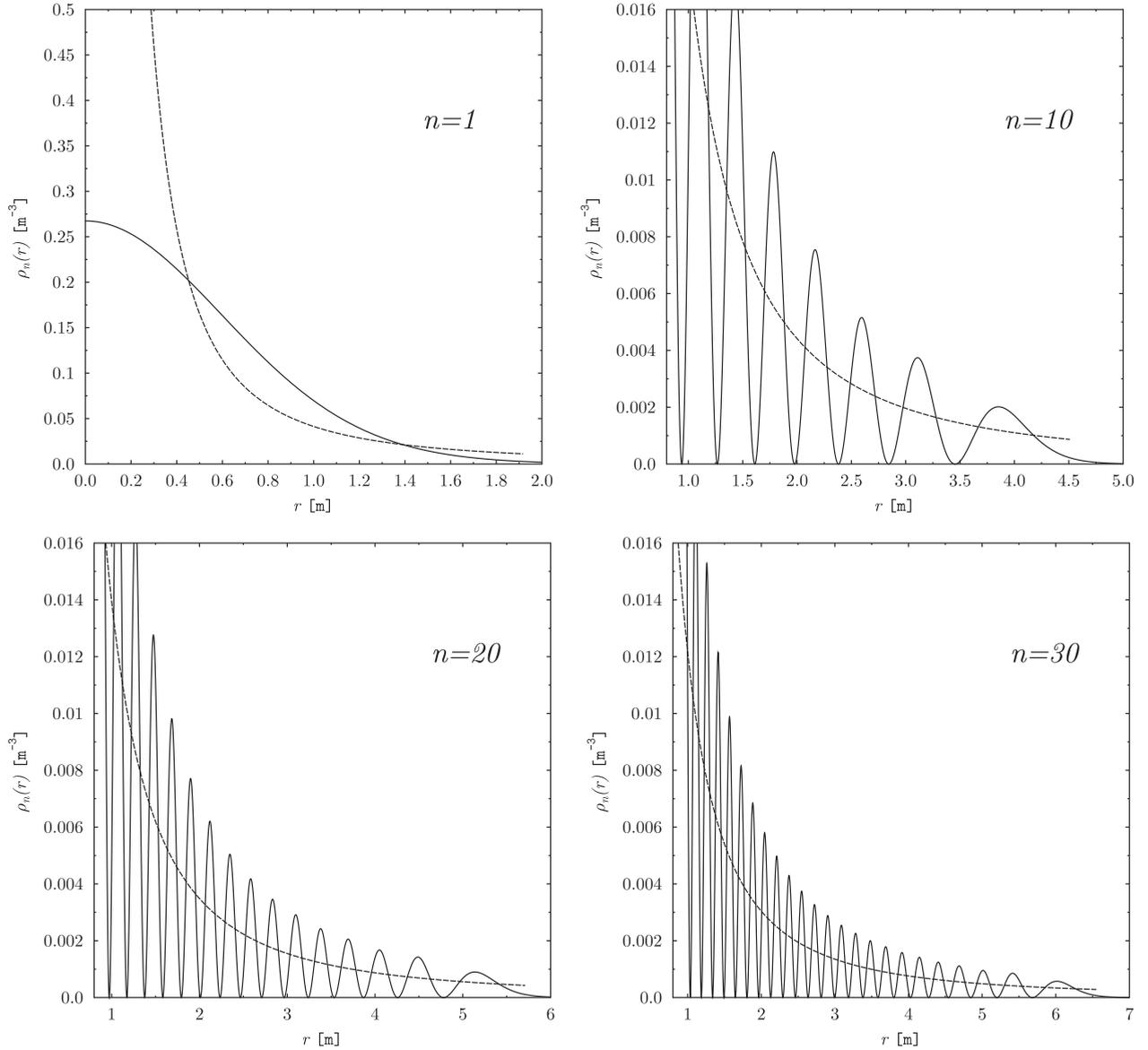


FIG. 4. The plot of the quantum probability density  $\rho_n(r) = |\psi_n(\mathbf{x})|^2$  (solid line), where  $\psi_n(\mathbf{x})$  is the wave function (3.12) and the classical probability density  $\rho_{cl}(r)$  (dashed line) given by (3.13), where  $r_{\max} = r_{\max}(E_n) = \frac{\sqrt{2E_n}}{\kappa}$ . We set  $c = 1 \text{ ms}^{-1}$ ,  $\kappa^2 = 1 \text{ Jm}^{-2}$ , and  $\hbar = 1 \text{ Js}$ .

From Eqs. (3.10) and (3.3) we finally obtain the normalized wave functions such that

$$\psi_n(\mathbf{x}) = \sqrt{\frac{c}{\hbar}} \frac{1}{\pi} \frac{1}{(2c\kappa\hbar)^{\frac{1}{3}}} \frac{1}{\text{Ai}'(a_n)} \frac{1}{r} \int_0^\infty dk \sin \frac{kr}{\hbar} \times \text{Ai} \left( \frac{2c}{(2c\kappa\hbar)^{\frac{2}{3}}} k + a_n \right), \quad (3.11)$$

where  $r = |\mathbf{x}|$ .

As in the case of the nonrelativistic harmonic oscillator with the probability density different from zero outside the turning points, the probability density  $\rho_n(r) = |\psi_n(\mathbf{x})|^2$  does not vanish for  $r > r_{\max}(E_n)$ , where  $r_{\max}(E_n) = \frac{\sqrt{2E_n}}{\kappa}$ ,  $n = 1, 2, \dots$ . However, it follows from the numerical calculation that  $\rho_n(r)$  has no maxima for  $r > r_{\max}(E_n)$  (see Fig. 4). Furthermore, taking into account all directions of the motion in the segment  $[0, r_{\max}]$  (the classical limit does not deal with

a single classical orbit but an ansamble of classical orbits [22]) and taking into account that the probability of finding a particle in the spherical layer  $r, r + dr$  is inverse proportional to the surface of the sphere with radius  $r$ , we find that the classical probability density is given by the formula

$$\rho_{cl}(\mathbf{x}) \equiv \rho_{cl}(r) = \frac{\theta(r_{\max} - r)}{4\pi r_{\max} r^2}, \quad (3.12)$$

where  $r_{\max} = \frac{\sqrt{2E}}{\kappa}$  and  $\theta(x)$  is the Heaviside step function. Clearly, the normalization condition is of the form

$$\int d^3\mathbf{x} \rho_{cl}(\mathbf{x}) = \int_0^\infty \rho_{cl}(r) d\mu(r) = 1, \quad (3.13)$$

where  $d\mu(r) = 4\pi r^2 dr$ . The comparison of the quantum probability density  $\rho_n(r)$  and the classical one  $\rho_{cl}(r)$  for  $r_{\max}(E_n)$  is shown in Fig. 4. As expected the differences

between the quantum and the classical descriptions decrease as the quantum number  $n$  increases.

We now discuss the expectation values of both the kinetic and potential energies. Using the identity [23]

$$\frac{1}{[\text{Ai}'(a_n)]^2} \int_0^\infty x \text{Ai}^2(x + a_n) dx = -\frac{2}{3} a_n, \quad (3.14)$$

Eqs. (3.10), and (3.9) we get

$$\langle \psi_n | c \hat{p} \psi_n \rangle = c \int d^3 \mathbf{k} |\mathbf{k}| |\psi_n(\mathbf{k})|^2 = \frac{2}{3} E_n, \quad (3.15)$$

where  $\hat{p} = \sqrt{\hat{\mathbf{p}}^2}$ . Hence, taking into account the form of the Hamiltonian in Eq. (3.1), we find

$$\left\langle \psi_n \left| \frac{\kappa^2}{2} \hat{p}^2 \psi_n \right. \right\rangle = \frac{1}{3} E_n, \quad (3.16)$$

where  $\hat{r} = \sqrt{\hat{\mathbf{x}}^2}$ . We conclude that the virial theorem takes the nonstandard form in the case of the massless relativistic harmonic oscillator. More precisely, the roles of the kinetic and potential energies are exchanged. Interestingly, we have the same formulas on average kinetic and potential energies in the classical case. Indeed, from Eq. (3.12) it follows easily that

$$\left\langle \frac{\kappa^2}{2} r^2 \right\rangle_{\text{cl}} = \frac{\kappa^2}{2} \int_0^\infty r^2 \rho_{\text{cl}}(r) d\mu(r) = \frac{1}{3} E. \quad (3.17)$$

Therefore, by virtue of the first equation of Eq. (2.3) we have

$$\langle cp \rangle_{\text{cl}} = \frac{2}{3} E. \quad (3.18)$$

We finally write down the following approximate relation obtained numerically

$$\langle \psi_n | \hat{r} \psi_n \rangle \approx \frac{r_{\max}(E_n)}{2} = \frac{\sqrt{2E_n}}{2\kappa}, \quad (3.19)$$

where the formula is exact in the limit  $n \rightarrow \infty$ . The approximation in Eq. (3.19) is very good. The maximal

relative error  $|\langle \psi_n | \hat{r} \psi_n \rangle - r_{\max}(E_n)/2| / \langle \psi_n | \hat{r} \psi_n \rangle$  arising in the case with  $n = 1$  is about 5% and is lesser than 1% for  $n = 2$ . The fact that the limit  $n \rightarrow \infty$  [when we have the exact equality in Eq. (3.19)] is the classical limit is confirmed by the classical formula

$$\langle r \rangle_{\text{cl}} = \int_0^\infty r \rho_{\text{cl}}(r) d\mu(r) = \frac{r_{\max}}{2} = \frac{\sqrt{2E}}{2\kappa}, \quad (3.20)$$

following directly from Eq. (3.12).

#### IV. CONCLUSION

In this work we study the relativistic massless harmonic oscillator in both classical and quantum cases. It seems that the obtained results concerning the classical oscillator are of importance not only from the physical point of view. Indeed, Eq. (2.2) is one of the simplest examples of a nonlinear Hamiltonian system with constant length of velocity. As far as we are aware such an interesting class of nonlinear dynamical systems was not discussed in the literature. Referring to the observations of this work related to the quantum mechanics of the relativistic massless harmonic oscillator we wish to point out that Eq. (3.11) is, to our best knowledge, the first example of the nontrivial exact solution to the Salpeter equation. We also stress the good behavior of the corresponding probability density and expectation values of observables, which confirms the correctness of the quantization based on the massless spinless Salpeter equation. Furthermore, we obtain the exact formula (3.9) on the spectrum of the Hamiltonian. It should be noted that for the Salpeter equation only energy bounds were analyzed in the literature so far (for the massive relativistic harmonic oscillator see Ref. [10]). Finally, we have obtained the interesting form of the virial theorem for the massless relativistic harmonic oscillator with the exchanged roles of the kinetic and potential energies.

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