Separable states and geometric phases of an interacting two-spin system

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It is known that an interacting bipartite system evolves as an entangled state in general, even if it is initially in a separable state. Due to the entanglement of the state, the geometric phase of the system is not equal to the sum of the geometric phases of its two subsystems. However, there may exist a set of states in which the nonlocal interaction does not affect the separability of the states, and the geometric phase of the bipartite system is then always equal to the sum of the geometric phases of its subsystems. In this article, we illustrate this point by investigating a well-known physical model. We give a necessary and sufficient condition in which a separable state remains separable so that the geometric phase of the system is always equal to the sum of the geometric phases of its subsystems.

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I. INTRODUCTION

The notion of geometric phase was first addressed by Pancharatnam for the comparison of the phases of two beams of polarized light in 1956 [1]. It was later shown to have important consequences for quantum systems. In 1984, Berry demonstrated that quantum system undergoing a cyclic adiabatic evolution acquires a phase with geometric nature [2]. Since then, geometric phase has attracted great interest. The original notion of Berry phase has been extended to nonadiabatic cyclic evolution by Aharonov and Anandan in 1987 [3], and to nonadiabatic and noncyclic evolution by Samuel and Bhandari in 1988 [4].

Although all these extensions of quantum systems are in pure states, another line of development has been toward extending the geometric phase to mixed states. The early extension to mixed states was given by Uhlmann within the mathematical context of purification [5]. In 2000, Sjöqvist *et al.* introduced an alternative definition of geometric phases for mixed states under unitary evolution based on quantum interferometry [6], and subsequently Singh *et al.* gave a kinematic description of the mixed state geometric phase and extended it to degenerate density operator [7]. The generalization of mixed geometric phases to quantum systems in nonunitary evolution was given by Tong *et al.* in 2004 [8]. Other discussions or experimental demonstrations of geometric phases for mixed states may be found in Refs. [9–24].

Another interesting issue of geometric phase is the relation of the bipartite or multipartite system with its subsystems. Sjöqvist calculated the geometric phase of a pair of entangled spin-half particles precessing in a time-independent uniform magnetic field [25], and the relative phase for polarization-entangled two-photon systems was considered by Hessmo *et al.* [26]. Tong *et al.* calculated the geometric phase of a bipartite entangled spin-half system in a rotating magnetic field [27] and investigated entangled bipartite systems with local unitary evolutions [28]. The effect of entanglement on the mutual geometric phase was recently studied by Williamson *et al.* [29]. Other discussions on geometric phases of composite systems and its applications may be found in Refs. [30–35].

All the previous discussions concerning the relation of the geometric phase of the composite system with its subsystems were of the systems under local unitary evolutions, $U(t) = U_a(t) \otimes U_b(t)$. It was shown that the geometric phase of the composite system, γ_{ab}, does not equal the sum of the geometric phases of its subsystems, γ_a and γ_b, in general [28,29]. The expression $\gamma_{ab} = \gamma_a + \gamma_b$ is valid only if the initial state is a separable one. This is because the entanglement of the state leads to an indecomposable geometric phase of the composite system. Since the interaction between two subsystems can lead to an entanglement of the subsystems, it is usually deemed that the geometric phase of the composite system in nonlocal unitary evolution does not equal the sum of the geometric phases of its subsystems in general, even if the initial state of the system is separable. In the present article, we investigate a well-known physical model, two interacting spin-half particles in a rotating magnetic field. We aim to show that there may exist a set of states in which the nonlocal interaction does not affect the separability of the states, and therefore the geometric phase of the bipartite system is always equal to the sum of the geometric phases of its subsystems. A necessary and sufficient condition for the set of separable states is given.

II. THE INTERACTING TWO–SPIN-HALF MODEL

Consider the system of two interacting spin-half particles in a rotating magnetic field, the Hamiltonian of which is described as

$$\hat{H}(t) = \hat{H}_a(t) \otimes I + I \otimes \hat{H}_b(t) + \hat{H}_{ab}(t), \tag{1}$$

where $\hat{H}_\mu(t) = \vec{B}(t) \cdot \vec{\sigma}_\mu \quad (\mu = a, b), \quad \hat{H}_{ab}(t) = J\vec{\sigma}_a \cdot \vec{\sigma}_b$. Here, $\vec{B}(t) = B(\sin\theta\cos\omega t, \sin\theta\sin\omega t, \cos\theta)$ is the rotating

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magnetic field. $\vec{\sigma}_a$ and $\vec{\sigma}_b$ are the Pauli operators of spins a and b, respectively. J denotes the interaction strength between a and b, and $J > 0$ describes antiferromagnetic coupling and $J < 0$ describes ferromagnetic coupling.

The state of the system, $|\psi(t)\rangle$, satisfies the Schrödinger equation,

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle, \tag{2}$$

with initial state being $|\psi(0)\rangle$. $|\psi(t)\rangle$ may be expressed as

$$|\psi(t)\rangle = f_1(t)|00\rangle + f_2(t)|01\rangle + f_3(t)|10\rangle + f_4(t)|11\rangle, \tag{3}$$

where $|ij\rangle$ $(i, j = 0, 1)$ are the abbreviations of $|i\rangle \otimes |j\rangle$ with $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$, and $f_k(t)(k = 1, 2, 3, 4)$ are functions of t to be determined, satisfying $\sum_{k=1}^{4} |f_k(t)|^2 = 1$. Substituting Eq. (3) into Eq. (2), we have

$$i\frac{d}{dt}\begin{pmatrix} f_1(t)\\ f_2(t)\\ f_3(t)\\ f_4(t)\end{pmatrix} = \begin{pmatrix} J+2B\cos\theta & B\sin\theta e^{-i\omega t} & B\sin\theta e^{-i\omega t} & 0\\ B\sin\theta e^{i\omega t} & -J & 2J & B\sin\theta e^{-i\omega t}\\ B\sin\theta e^{i\omega t} & 2J & -J & B\sin\theta e^{-i\omega t}\\ 0 & B\sin\theta e^{i\omega t} & B\sin\theta e^{i\omega t} & J-2B\cos\theta \end{pmatrix}\begin{pmatrix} f_1(t)\\ f_2(t)\\ f_3(t)\\ f_4(t)\end{pmatrix}, \tag{4}$$

that is,

$$\begin{cases} i\dot{f}_1 = (J+2B\cos\theta)f_1 + (B\sin\theta)e^{-i\omega t}f_2 + (B\sin\theta)e^{-i\omega t}f_3,\\ i\dot{f}_2 = (B\sin\theta)e^{i\omega t}f_1 - Jf_2 + 2Jf_3 + (B\sin\theta)e^{-i\omega t}f_4,\\ i\dot{f}_3 = (B\sin\theta)e^{i\omega t}f_1 + 2Jf_2 - Jf_3 + (B\sin\theta)e^{-i\omega t}f_4,\\ i\dot{f}_4 = (B\sin\theta)e^{i\omega t}f_2 + (B\sin\theta)e^{i\omega t}f_3 + (J-2B\cos\theta)f_4. \end{cases} \tag{5}$$

To resolve the above differential equations, we further let $f_1(t) = \bar{f}_1(t)e^{-i\omega t}$, $f_2(t) = \bar{f}_2(t)$, $f_3(t) = \bar{f}_3(t)$, $f_4(t) = \bar{f}_4(t)e^{i\omega t}$. Then, Eq. (5) becomes

$$\begin{cases} i\dot{\bar{f}}_1 = (J+2B\cos\theta-\omega)\bar{f}_1 + B\sin\theta\bar{f}_2 + B\sin\theta\bar{f}_3,\\ i\dot{\bar{f}}_2 = B\sin\theta\bar{f}_1 - J\bar{f}_2 + 2J\bar{f}_3 + B\sin\theta\bar{f}_4,\\ i\dot{\bar{f}}_3 = B\sin\theta\bar{f}_1 + 2J\bar{f}_2 - J\bar{f}_3 + B\sin\theta\bar{f}_4,\\ i\dot{\bar{f}}_4 = B\sin\theta\bar{f}_2 + B\sin\theta\bar{f}_3 + (J-2B\cos\theta+\omega)\bar{f}_4. \end{cases} \tag{6}$$

Equation (6) is a set of first-order linear ordinary differential equations. Its solution can be obtained by solving the characteristic equation. The four characteristic roots are

$$\begin{aligned} \lambda_1 &= 3J,\\ \lambda_2 &= -J,\\ \lambda_3 &= -J + \sqrt{4B^2\sin^2\theta + (2B\cos\theta-\omega)^2},\\ \lambda_4 &= -J - \sqrt{4B^2\sin^2\theta + (2B\cos\theta-\omega)^2}, \end{aligned} \tag{7}$$

each of which corresponding to a characteristic solution with respect to $\bar{f}_k(t)$. With the help of the solutions of $\bar{f}_k(t)$, which directly give the solutions of $f_k(t)$, the general solution of Eq. (2) can be expressed as

$$|\psi(t)\rangle = c_1|\psi_1(t)\rangle + c_2|\psi_2(t)\rangle + c_3|\psi_3(t)\rangle + c_4|\psi_4(t)\rangle, \tag{8}$$

where the time-independent coefficients c_k $(k = 1, 2, 3, 4)$, $\sum_{k=1}^{4} |c_k|^2 = 1$, are to be determined by the initial condition, and the four particular solutions read

$$|\psi_1(t)\rangle = e^{i\lambda_1 t}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix},$$

$$|\psi_2(t)\rangle = e^{i\lambda_2 t}\frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{2B\sin\theta}{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}e^{-i\omega t}\\ \frac{2B\cos\theta-\omega}{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}\\ \frac{2B\cos\theta-\omega}{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}\\ \frac{2B\sin\theta}{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}e^{i\omega t} \end{pmatrix},$$

$$|\psi_3(t)\rangle = e^{i\lambda_3 t}\begin{pmatrix} -\frac{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}-(2B\cos\theta-\omega)}{2\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}e^{-i\omega t}\\ \frac{B\sin\theta}{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}\\ \frac{B\sin\theta}{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}\\ -\frac{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}+(2B\cos\theta-\omega)}{2\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}e^{i\omega t} \end{pmatrix},$$

$$|\psi_4(t)\rangle = e^{i\lambda_4 t}\begin{pmatrix} \frac{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}+(2B\cos\theta-\omega)}{2\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}e^{-i\omega t}\\ \frac{B\sin\theta}{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}\\ \frac{B\sin\theta}{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}\\ \frac{\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}-(2B\cos\theta-\omega)}{2\sqrt{4B^2\sin^2\theta+(2B\cos\theta-\omega)^2}}e^{i\omega t} \end{pmatrix}. \tag{9}$$

III. THE GEOMETRIC PHASES OF THE TWO–SPIN-HALF SYSTEM

If the two–spin-half system is initially in state $|\psi(0)\rangle$, the geometric phase obtained by the quantum system during the time $t \in [0, \tau]$ can be calculated by using the formula [36,37],

$$\gamma_{ab}(\tau) = \arg\langle\psi(0)|\psi(\tau)\rangle + i\int_0^\tau \langle\psi(t)|\dot{\psi}(t)\rangle dt. \quad (10)$$

However, both the subsystems a and b are generally in mixed states due to the nonlocal interaction, even if the initial state $|\psi(0)\rangle$ is separable. The mixed states of the subsystems can be expressed as density operators,

$$\rho_a(t) = \mathrm{tr}_b|\psi(t)\rangle\langle\psi(t)|,\ \rho_b(t) = \mathrm{tr}_a|\psi(t)\rangle\langle\psi(t)|. \quad (11)$$

The geometric phases of the mixed states in nonunitary evolutions are calculated by using the formula [8]

$$\gamma_\mu(\tau) = \arg\left(\sum_{m=1}^{2}\sqrt{\omega_m^\mu(0)\omega_m^\mu(\tau)}\langle\phi_m^\mu(0)|\phi_m^\mu(\tau)\rangle \times e^{-\int_0^\tau\langle\phi_m^\mu(t)|\dot{\phi^\mu}_m(t)\rangle dt}\right), \quad (12)$$

where $\omega_m^\mu(t)$ and $|\phi_m^\mu(t)\rangle$ are the eigenvalues and eigenstates of the density operators $\rho_\mu(t)$ $(\mu = a, b)$, respectively.

By substituting Eqs. (8) and (9) into Eqs. (10) and (11), and further using Eq. (12), one can calculate the geometric phase of the two–spin-half system and the geometric phases of its two subsystems. It is easy to show that γ_{ab} is not equal to the sum of γ_a and γ_b in general, even if the initial state $|\psi(0)\rangle$ is a separable one.

To illustrate this point, we take $|\psi(0)\rangle = |01\rangle$ as an example. In this case, the state of the system at time t reads

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|\psi_1(t)\rangle + \frac{1}{\sqrt{2}}\cos\eta|\psi_2(t)\rangle + \frac{1}{2}\sin\eta|\psi_3(t)\rangle + \frac{1}{2}\sin\eta|\psi_4(t)\rangle, \quad (13)$$

and the geometric phase obtained by the system during the time $t \in [0, \tau]$ is

$$\gamma_{ab} = \arctan\frac{\sin 4J\tau}{\cos^2\eta + \sin^2\eta\cos\alpha\tau + \cos 4J\tau} - 2J\tau, \quad (14)$$

where

$$\alpha = \sqrt{4B^2\sin^2\theta + (2B\cos\theta - \omega)^2}, \quad (15)$$

and

$$\tan\eta = \frac{2B\sin\theta}{2B\cos\theta - \omega}. \quad (16)$$

The reduced density operators of the subsystems a and b are

$$\rho_\mu = \begin{pmatrix}\rho_{11}^\mu & \rho_{12}^\mu\\ \rho_{21}^\mu & \rho_{22}^\mu\end{pmatrix}, \quad \mu = a, b, \quad (17)$$

where

$$\rho_{11}^a = 1 - \rho_{22}^a = \tfrac{1}{2}[1 + (\cos^2\eta + \sin^2\eta\cos\alpha t)\cos 4Jt],$$
$$\rho_{12}^a = \rho_{21}^{a\,*} = \tfrac{1}{2}[\sin\eta\cos\eta(1-\cos\alpha t) + i\sin\eta\sin\alpha t]e^{-i\omega t}\cos 4Jt; \quad (18)$$

$$\rho_{11}^b = 1 - \rho_{22}^b = \tfrac{1}{2}[1 - (\cos^2\eta + \sin^2\eta\cos\alpha t)\cos 4Jt],$$
$$\rho_{12}^b = \rho_{21}^{b\,*} = -\tfrac{1}{2}[\sin\eta\cos\eta(1-\cos\alpha t) + i\sin\eta\sin\alpha t]e^{-i\omega t}\cos 4Jt. \quad (19)$$

The geometric phases obtained by the subsystems during the time $t \in [0, \tau]$ are, respectively,

$$\gamma_a(\tau) = \arctan\frac{-\cos\eta\sqrt{1-\cos\alpha\tau}}{\sqrt{1+\cos\alpha\tau}} + \frac{\omega\sin^2\eta}{2\alpha}\sin\alpha\tau + \frac{1}{2}\alpha\tau\cos\eta - \frac{1}{2}\omega\tau\sin^2\eta, \quad (20)$$

and

$$\gamma_b(\tau) = \arctan\left(\cos\eta\tan\frac{\alpha\tau}{2}\right) - \frac{\omega\sin^2\eta}{2\alpha}\sin\alpha\tau - \frac{1}{2}\alpha\tau\cos\eta + \frac{1}{2}\omega\tau\sin^2\eta. \quad (21)$$

Clearly, the geometric phase of the large system is not equal to the sum of the geometric phases of the two subsystems, $\gamma_{ab} \neq \gamma_a + \gamma_b$, even if the initial state $|\psi(0)\rangle$ is a separable one.

IV. CONDITION FOR GEOMETRIC PHASE OF THE SYSTEM BEING EQUAL TO THE SUM OF THOSE OF ITS SUBSYSTEMS

Geometric phase is useful in quantum calculation, but a real quantum system may comprise two or more subsystems with interactions between them. In this case when interactions appear, the geometric phase of the composite system is not equal to the sum of the geometric phases of its subsystems. The relations among the geometric phases of the large system and the subsystems are complicated, and therefore they are not easy to be synchronously controlled. It is interesting to find a condition in which the geometric phase of the composite system equals the sum of the geometric phases of its subsystems.[1] The formulas (10) and (12) show that the value of the geometric phase of a quantum system not only depends on the initial state $|\psi(0)\rangle$ and the final state $|\psi(\tau)\rangle$ but also depends on all the instantaneous states $|\psi(t)\rangle (t \in [0, \tau])$. It is determined completely by the path traced by the states. If we require that the geometric phase of the composite system is equal to the sum of the geometric phases of its subsystems for all time, the sufficient condition is that $|\psi(t)\rangle$ remains separable at all time, that is

$$|\psi(t)\rangle = |\phi_a(t)\rangle \otimes |\phi_b(t)\rangle. \quad (22)$$

One may demonstrate this point by substituting expression (22) into geometric phase formulas. Indeed, if there is

[1]If it is only required that the geometric phase of the composite system equals the sum of those of its subsystems at some special time $t = \tau$, there should be no difficulties in choosing initial states. However, here we require that the geometric phase of the composite system is always equal to the sum of those of its subsystems at all the time $t \in \tau$.

$|\psi(t)\rangle = |\phi_a(t)\rangle \otimes |\phi_b(t)\rangle$ for $t \in [0, \tau]$, one then has

$$\begin{aligned}\arg\langle\psi(0)|\psi(\tau)\rangle &= \arg\langle\phi_a(0)|\phi_a(\tau)\rangle\langle\phi_b(0)|\phi_b(\tau)\rangle \\ &= \arg\langle\phi_a(0)|\phi_a(\tau)\rangle \\ &\quad + \arg\langle\phi_b(0)|\phi_b(\tau)\rangle(\mathrm{mod}2\pi),\end{aligned} \tag{23}$$

and

$$\begin{aligned} i\int_0^\tau \langle\psi(t)|\dot\psi(t)\rangle dt &= i\int_0^\tau \langle\phi_a(t)|\dot\phi_a(t)\rangle dt \\ &\quad + i\int_0^\tau \langle\phi_b(t)|\dot\phi_b(t)\rangle dt,\end{aligned} \tag{24}$$

where the normalized relations $\langle\phi_\mu(t)|\phi_\mu(t)\rangle = 1(\mu = a, b)$ are used. Substituting them into Eq. (10), one further has

$$\gamma_{ab}(\tau) = \gamma_a(\tau) + \gamma_b(\tau), \tag{25}$$

where

$$\gamma_\mu(\tau) = \arg\langle\phi_\mu(0)|\phi_\mu(\tau)\rangle + i\int_0^\tau \langle\phi_\mu(t)|\dot\phi_\mu(t)\rangle dt, \quad \mu = a, b, \tag{26}$$

are 2π-modular geometric phases of the subsystems.

With the above knowledge that the geometric phase of the system is equal to the sum of those of its subsystems if the time-dependent state is always separable, we may now calculate the condition for the two interacting spin-half particles. To this end, we rewrite the general solution expressed by Eq. (8), with the bases $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, as

$$\begin{aligned}|\psi(t)\rangle = &\left(-\frac{1}{\sqrt2}c_2\sin\eta e^{i\lambda_2 t}e^{-i\omega t} - c_3\sin^2\frac{\eta}{2}e^{i\lambda_3 t}e^{-i\omega t}\right. \\ &\left.+ c_4\cos^2\frac{\eta}{2}e^{i\lambda_4 t}e^{-i\omega t}\right)|00\rangle + \left(\frac{1}{\sqrt2}c_1e^{i\lambda_1 t}\right. \\ &+ \frac{1}{\sqrt2}c_2\cos\eta e^{i\lambda_2 t} + \frac12 c_3\sin\eta e^{i\lambda_3 t} \\ &\left.+ \frac12 c_4\sin\eta e^{i\lambda_4 t}\right)|01\rangle + \left(-\frac{1}{\sqrt2}c_1e^{i\lambda_1 t}\right. \\ &+ \frac{1}{\sqrt2}c_2\cos\eta e^{i\lambda_2 t} + \frac12 c_3\sin\eta e^{i\lambda_3 t} \\ &\left.+ \frac12 c_4\sin\eta e^{i\lambda_4 t}\right)|10\rangle + \left(\frac{1}{\sqrt2}c_2\sin\eta e^{i\lambda_2 t}e^{i\omega t}\right. \\ &\left.- c_3\cos^2\frac{\eta}{2}e^{i\lambda_3 t}e^{i\omega t} + c_4\sin^2\frac{\eta}{2}e^{i\lambda_4 t}e^{i\omega t}\right)|11\rangle.\end{aligned} \tag{27}$$

Noting that the concurrence of a quantum state provides a criterion for distinguishing between separable states and entangled states [38,39], we may obtain the necessary and sufficient condition for the separable states by calculating the concurrence of the above state. The concurrence of the state reads

$$\begin{aligned}C(t) &= \sqrt{2[1 - \mathrm{tr}[\mathrm{tr}_b|\psi(t)\rangle\langle\psi(t)|]^2]} \\ &= |c_2^2 + 2c_3c_4 - c_1^2 e^{i8Jt}|.\end{aligned} \tag{28}$$

The above equation shows that the concurrence is, if $J \neq 0$, dependent on the time t. The separability of an initial state does not guarantee separability of the state at time t. If the system is initially in a separable state, satisfying $c_2^2 + 2c_3c_4 - c_1^2 = 0$ with $c_1 \neq 0$, it will evolve to an entangled state at the late time and then go back to a separable state at each time $t = n\pi/4J$, $n = 1, 2, \ldots$. If we require that the geometric phase of the composite system is always equal to the sum of the geometric phases of its subsystems for all time, the sufficient condition is that $|\psi(t)\rangle$ is separable for all time. This requirement is fulfilled if and only if the concurrence $C(t)$ is zero for all time, that is

$$c_2^2 + 2c_3c_4 - c_1^2 e^{i8Jt} = 0, \tag{29}$$

which further leads to

$$\begin{aligned}c_1 &= 0, \\ c_2^2 + 2c_3c_4 &= 0.\end{aligned} \tag{30}$$

Equation (30) is the necessary and sufficient condition in which an initial separable state of the system keeps in a separable one. The nonlocal interaction between the two spins does not affect the separability of the states in the set defined by condition (30). In this case, the geometric phase of the composite system is always equal to the sum of the geometric phases of its subsystems.

To illustrate the above result, we consider an example. Let $c_1 = 0, c_2 = -\frac{1}{\sqrt2}\sin\eta, c_3 = -\sin^2\frac{\eta}{2}, c_4 = \cos^2\frac{\eta}{2}$, which means that the system is initially in the stats $|\psi(0)\rangle = |00\rangle$. At time t, the instantaneous state reads

$$\begin{aligned}|\psi(t)\rangle = &-\tfrac{1}{\sqrt2}\sin\eta|\psi_2(t)\rangle - \tfrac12(1-\cos\eta)|\psi_3(t)\rangle \\ &+ \frac12(1+\cos\eta)|\psi_4(t)\rangle,\end{aligned} \tag{31}$$

where α and η have been defined in Eqs. (15) and (16), respectively. The states of the subsystems a and b can be calculated by using Eq. (11), which gives $\rho_\mu = (\begin{smallmatrix}\rho_{11}^\mu & \rho_{12}^\mu \\ \rho_{21}^\mu & \rho_{22}^\mu\end{smallmatrix}), \mu = a, b$, with the elements

$$\begin{aligned}\rho_{11}^\mu &= 1 - \rho_{22}^\mu = \tfrac12(1 + \cos^2\eta + \sin^2\eta\cos\alpha t), \\ \rho_{12}^\mu &= \rho_{21}^{\mu\,*} = \tfrac12(\sin\eta\cos\eta(1-\cos\alpha t) + i\sin\eta\sin\alpha t)e^{-i\omega t}.\end{aligned} \tag{32}$$

By using the formulas (10) and (12), we can calculate the geometric phases of the system and the subsystems, and we have

$$\begin{aligned}\gamma_{ab}(\tau) = &\arctan\frac{-2\cos\eta\sin\alpha\tau}{\sin^2\eta + (1+\cos^2\eta)\cos\alpha\tau} \\ &+ \frac{\omega\sin^2\eta}{\alpha}\sin\alpha\tau + \alpha\tau\cos\eta - \omega\tau\sin^2\eta,\end{aligned} \tag{33}$$

and

$$\begin{aligned}\gamma_a(\tau) = \gamma_b(\tau) = &\arctan\frac{-\cos\eta\sqrt{1-\cos\alpha\tau}}{\sqrt{1+\cos\alpha\tau}} \\ &+ \frac{\omega\sin^2\eta}{2\alpha}\sin\alpha\tau + \frac12\alpha\tau\cos\eta - \frac12\omega\tau\sin^2\eta.\end{aligned} \tag{34}$$

By comparing Eq. (33) with Eq. (34) we see that the geometric phase of the subsystem is half of the large system.

In passing, we would like to point out that all the states in the set defined by condition (30) are the eigenstates of the interaction Hamiltonian. There is no time-dependent state that is always separable but not an eigenstate of the interaction Hamiltonian. This is easy to be understood, since the interaction does not change the entanglement of an eigenstate but changes that of a noneigenstate. Noting that the time-dependent state of the system, initially in a separable state with $c_1 \neq 0$, will be cyclically separable with the period $t = \pi/4J$, one may argue whether the geometric phase holds the additivity cyclically, too, in the case where $c_2^2 + 2c_3c_4 - c_1^2 = 0$ but $c_1 \neq 0$. A further discussion can show that the additivity is not valid for the geometric phase in the case. This is because that geometric phase is equal to total phase minus dynamic phase, and dynamic phase is not only dependent on the initial and final states but also dependent on the states at all the evolutional time $t \in [0, \tau]$. The additivity is invalid for dynamic phase, although it is valid for total phase, which is only dependent on the initial and final states. Besides, it is worth noting that the form of condition (30) is based on the expression of the basis states $|\psi_k(t)\rangle (k = 1, 2, 3, 4)$ in Eq. (9). It is not gauge invariant. For example, if a π phase difference is introduced between $|\psi_3(t)\rangle$ and $|\psi_4(t)\rangle$, the coefficients c_3 and c_4 would acquire a relative sign and the condition would then read $c_2^2 - 2c_3c_4 = 0$. In general, there could be an arbitrary phase factor between c_2^2 and $2c_3c_4$ in Eq. (30) if an alternative expression of basis states are taken. The form of the condition depends on the choice of phase convention between the basis states.

V. SUMMARY AND REMARKS

An interacting bipartite system evolves into an entangled state in general, even if it is initially in a separable state. Due to the entanglement, the geometric phase of the system is not equal to the sum of the geometric phases of its two subsystems. However, there may exist a set of states in which the nonlocal interaction does not affect the separability of the states, and the geometric phase of the bipartite system is equal to the sum of the geometric phases of its subsystems. By considering a well-known physical model, two interacting spin-half particles in a rotating magnetic field, we illustrate this point. Indeed, our calculation shows that the geometric phase of the system is not equal to the sum of the geometric phases of the subsystems in general. They are not equal even if the system is initially in separable states, due to the nonlocal interaction between the subsystems. Yet, there is such a set of states for which the nonlocal interaction does not affect the separability of the states, and the geometric phase of the bipartite system is always equal to the sum of the geometric phases of its subsystems. We give a necessary and sufficient condition for an initial separable state to remain separable.

The geometric property of the geometric phase has stimulated many applications. It has been found that the geometric phase plays important roles in quantum phase transition, quantum information processing, etc. [40–42]. The geometric phase shift can be fault tolerant with respect to certain types of errors, thus several proposals using nuclear magnetic resonance (NMR), laser-trapped ions, etc. have been given to use geometric phase to construct fault-tolerant quantum information processor [43–45].

The geometric phase is useful in quantum computation, but real physical systems are usually composite and therefore the relations among the geometric phase of the large system and those of the subsystems are complicated. It is very difficult to control each of the values of them. Our result shows that it is possible to make the phases' relations simple if the initial states are properly chosen. In this sense, our finding may be useful both in the theory itself and in the applications of the geometric phase. The investigation on the current bipartite system involving two–spin-half particles implies that such kinds of states may exist in other interacting physical systems.

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