

## Hermitian conjugate measurement

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We propose a class of probabilistic reversing operations on the state of a system that was disturbed by a weak measurement. It can approximately recover the original state from the disturbed state especially with an additional information gain using the Hermitian conjugate of the measurement operator. We illustrate the general scheme by considering a quantum measurement consisting of spin systems with an experimentally feasible interaction and show that the reversing operation simultaneously increases both the fidelity to the original state and the information gain with such a high probability of success that their average values increase simultaneously.

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### I. INTRODUCTION

Quantum measurement not only provides information about a physical system but also changes the state of the system because of its back-action. Although such a change in state was widely believed to be intrinsically irreversible [1], it has been shown that quantum measurement is not necessarily irreversible [2], because a certain class of measurements preserves all the information about the system during the measurement process. In recent work [2–14] on reversibility in quantum measurements, probabilistic reversing operations based on the inverse operator of  $\hat{M}$  [4,5,8–13] have been discussed, where  $\hat{M}$  is an operator describing the state change due to the measurement. That is, a second measurement is performed on the system so that it applies  $\hat{M}^{-1}$  to the system state to cancel the effect of  $\hat{M}$ , when a preferred outcome is obtained. However, if the premeasurement state is completely recovered using  $\hat{M}^{-1}$ , the information obtained by the first measurement is completely erased or neutralized by the information gain from the reversing operation (see Erratum of Ref. [4]). Recently, this type of reversing operation has been experimentally demonstrated using a superconducting phase qubit [15].

In this article, we consider a probabilistic reversing operation that can accomplish both approximate recovery of the premeasurement state and additional information gain. The operation is carried out with the Hermitian conjugate operator of  $\hat{M}$  rather than  $\hat{M}^{-1}$ . Note that  $\hat{M}^\dagger$  and  $\hat{M}^{-1}$  are different because  $\hat{M}$  is not unitary. However, the difference can be small if the interaction between the system and the measuring apparatus is sufficiently weak. In this case,  $\hat{M}^\dagger$  could approximately cancel the state change caused by the measurement. Moreover, a reversing operation using  $\hat{M}^\dagger$  has an advantage over that using  $\hat{M}^{-1}$  with respect to information gain. On observing the recovery by  $\hat{M}^{-1}$ , one might think that if the premeasurement state is approximately recovered, most of the information obtained is lost during the reversing operation. However, we show that if it is approximately recovered using  $\hat{M}^\dagger$ , the reversing operation *increases* rather than decreases information gain.

The additional information gain can be understood by polar decomposition of  $\hat{M}$ , i.e.,  $\hat{M} = \hat{U}\hat{N}$ , where  $\hat{U}$  is a unitary operator and  $\hat{N}$  is a nonunitary positive operator. As shown

below,  $\hat{N}$  carries information about the system, while  $\hat{U}$  does not. The reversing operation by  $\hat{M}^\dagger$  can thus increase information gain, because  $\hat{M}^\dagger$  cancels the unitary part  $\hat{U}$  but enhances the information-carrying nonunitary part  $\hat{N}$  as  $\hat{M}^\dagger\hat{M} = \hat{N}^2$ . This is in contrast with the reversing operation by  $\hat{M}^{-1}$ , where  $\hat{M}^{-1}$  cancels not only  $\hat{U}$  but also  $\hat{N}$  as  $\hat{M}^{-1}\hat{M} = \hat{I}$ . Of course, the premeasurement state cannot perfectly be recovered by  $\hat{M}^\dagger$ , because  $\hat{N}$  disturbs the state of the system. Nevertheless, the premeasurement state can approximately be recovered by  $\hat{M}^\dagger$  as long as the state disturbance by  $\hat{N}$  is much smaller than that by  $\hat{U}$ . We shall show such a physical example using spin systems with Ising-type interaction.

An approximate recovery with additional information gain was first discussed in Ref. [12]. However, the article did not identify the reason for the information gain because it focused on a reversing operation by  $\hat{M}^{-1}$ . Similarly, an approximate recovery with purity gain (instead of information gain) was discussed in Ref. [14] for a system weakly interacting with the environment by regarding the interaction with the environment as a measurement. However, the reversing operation in that case requires the average over the outcome of the “measurement,” because the environment does not refer to the outcome. This obscures the nature of the operator that contributes to the purity gain. Therefore, here we clarify the reason for the information gain, together with the property of the operator that is required to achieve the information gain.

This article is organized as follows: Section II describes the general formulation of quantum measurement and introduces fidelity loss and information gain due to measurement. Section III defines a Hermitian conjugate measurement together with the reversing measurement scheme. Section IV shows that in the case of weak measurement, the Hermitian conjugate measurement achieves both approximate recovery of the premeasurement state and additional information gain. Section V considers a quantum measurement of a spin- $s$  system using a spin- $j$  probe as an example. Section VI summarizes our results.

### II. QUANTUM MEASUREMENT, FIDELITY, AND INFORMATION GAIN

A quantum measurement is generally described [16,17] by a set of linear operators  $\{\hat{M}_m\}$ , called measurement operators,

that satisfy the completeness condition

$$\sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{I}, \quad (1)$$

where  $\hat{I}$  is the identity operator. If the system to be measured is in a state  $\hat{\rho}$ , the measurement yields outcome  $m$  with probability

$$p_m = \text{Tr}(\hat{\rho} \hat{M}_m^\dagger \hat{M}_m), \quad (2)$$

and for each outcome  $m$  the state of the system is changed into

$$\hat{\rho}_m = \frac{1}{p_m} \hat{M}_m \hat{\rho} \hat{M}_m^\dagger. \quad (3)$$

We can always construct a quantum measurement described by a given set of operators  $\{\hat{M}_m\}$ , using a measuring apparatus whose initial state, interaction, and observable are appropriately chosen [17].

Provided that the dimension of the support is finite, any linear operator  $\hat{M}_m$  can uniquely be decomposed by *left* polar decomposition into

$$\hat{M}_m = \hat{U}_m \hat{N}_m, \quad (4)$$

where  $\hat{U}_m$  is a unitary operator and  $\hat{N}_m \equiv \sqrt{\hat{M}_m^\dagger \hat{M}_m}$  is a positive operator. The operators  $\{\hat{N}_m\}$  also describe a quantum measurement because they are linear and satisfy  $\sum_m \hat{N}_m^\dagger \hat{N}_m = \hat{I}$ . The measurement described by  $\{\hat{N}_m\}$  gives the same amount of information gain as the measurement  $\{\hat{M}_m\}$  but changes the state as little as possible. This is because the probability  $p_m = \text{Tr}(\hat{\rho} \hat{N}_m^2)$  does not depend on  $\hat{U}_m$ . The unitary part,  $\hat{U}_m$ , is thus irrelevant to the information gain and contributes only to the state change. Unfortunately, we cannot always perform this optimal measurement  $\{\hat{N}_m\}$  because available interactions between the system and the measuring apparatus are subject to experimental constraints.

In making the polar decomposition (4) of the measurement operator, we have assumed that the system's Hilbert space is finite dimensional, because a linear operator on an infinite-dimensional Hilbert space cannot always be decomposed by polar decomposition [18]. This assumption is not particularly restrictive, owing to the existence of a physical cutoff. For example, in photon counting [19], the measurement process that detects one photon with a photodetector is described by the annihilation operator,  $\hat{a}$ , of the photon; however, it has been shown that such an annihilation operator does not have polar decomposition [20]. Note that the Hilbert space of the photon field is infinite-dimensional, because it is spanned by the eigenstates  $|n\rangle$  of the photon-number operator  $\hat{a}^\dagger \hat{a}$  with  $n = 0, 1, 2, \dots$ . Even in this case, an effective upper bound on the photon number  $n_{\max}$  can be introduced by considering an actual experimental setup. Truncating the Hilbert space  $\{|n\rangle\}$  to finite dimensions  $n = 0, 1, 2, \dots, n_{\max}$ , we can consider an approximate polar decomposition as in Eq. (4).

To evaluate the amount of information obtained by a *single* measurement outcome, suppose that the premeasurement state  $\hat{\rho}$  is known to be one of the predefined states  $\{\hat{\rho}(a)\}$  with equal probability,  $p(a) = 1/N$ , where  $a = 1, \dots, N$ . Because the premeasurement state is usually an arbitrary unknown state in quantum measurement,  $\{\hat{\rho}(a)\}$  is essentially an infinite set ( $N \rightarrow \infty$ ). This contrasts with the case of quantum state

discrimination [21,22], in which  $N$  cannot be greater than the dimension of the Hilbert space due to the linear independence of  $\{\hat{\rho}(a)\}$ . The Shannon entropy associated with the system is initially

$$H_0 = - \sum_a p(a) \log_2 p(a) = \log_2 N, \quad (5)$$

which is a measure of the lack of information about the system.

The measurement  $\{\hat{M}_m\}$  is then performed to obtain information about the system. If the premeasurement state is  $\hat{\rho}(a)$ , the measurement yields an outcome  $m$  with probability

$$p(m|a) = \langle \hat{M}_m^\dagger \hat{M}_m \rangle_a = \langle \hat{N}_m^2 \rangle_a, \quad (6)$$

where the bracket with subscript  $a$  denotes

$$\langle \hat{O} \rangle_a \equiv \text{Tr}[\hat{\rho}(a) \hat{O}]. \quad (7)$$

The total probability for outcome  $m$  is thus

$$p(m) = \sum_a p(m|a) p(a) = \frac{1}{N} \sum_a \langle \hat{N}_m^2 \rangle_a = \overline{\langle \hat{N}_m^2 \rangle}, \quad (8)$$

where the overline denotes the average over  $a$ ,

$$\overline{f} \equiv \frac{1}{N} \sum_a f(a). \quad (9)$$

Conversely, given outcome  $m$ , we can find the probability that the premeasurement state is  $\hat{\rho}(a)$  by

$$p(a|m) = \frac{p(m|a) p(a)}{p(m)} \quad (10)$$

from Bayes's rule. This indicates that the Shannon entropy after measurement with outcome  $m$  is

$$H(m) = - \sum_a p(a|m) \log_2 p(a|m). \quad (11)$$

Therefore, the amount of information obtained from outcome  $m$  is evaluated by

$$I(m) = H_0 - H(m) = \frac{\overline{\langle \hat{N}_m^2 \rangle \log_2 \langle \hat{N}_m^2 \rangle} - \overline{\langle \hat{N}_m^2 \rangle \log_2 \overline{\langle \hat{N}_m^2 \rangle}}}{\overline{\langle \hat{N}_m^2 \rangle}}, \quad (12)$$

due to the assumption that  $p(a) = 1/N$  does not depend on  $a$ . The mean information gain after the measurement is given by

$$I = \sum_m p(m) I(m). \quad (13)$$

On the other hand, the state change caused by the measurement can be evaluated in terms of the fidelity [17,23] between the premeasurement and postmeasurement states. If the premeasurement state is  $\hat{\rho}(a)$  and the measurement outcome is  $m$ , the postmeasurement state is given by

$$\hat{\rho}(m, a) = \frac{1}{p(m|a)} \hat{M}_m \hat{\rho}(a) \hat{M}_m^\dagger. \quad (14)$$

The fidelity between the premeasurement and postmeasurement states then becomes

$$F(m, a) = \text{Tr} \sqrt{\sqrt{\hat{\rho}(a)} \hat{\rho}(m, a) \sqrt{\hat{\rho}(a)}}, \quad (15)$$

with  $0 \leq F(m, a) \leq 1$ . The more drastically the measurement changes the state of the system, the smaller the fidelity

becomes. Because  $a$  is unknown to us, the fidelity after the measurement with outcome  $m$  is evaluated using the probability in Eq. (10) by

$$F(m) = \sum_a p(a|m)F(m, a). \quad (16)$$

The mean fidelity after measurement is given by

$$F = \sum_m p(m)F(m). \quad (17)$$

### III. HERMITIAN CONJUGATE MEASUREMENT

To undo the state change caused by measurement, a reversing measurement scheme was proposed in Ref. [5] based on the inverse of the measurement operator. In this scheme, depending on the outcome  $m$  of the measurement, another measurement, called a reversing measurement, is performed on the postmeasurement state (3) of the system. The reversing measurement is described by a set of measurement operators  $\{\hat{R}_v^{(m)}\}$  that satisfy [5]

$$\sum_v \hat{R}_v^{(m)\dagger} \hat{R}_v^{(m)} = \hat{I} \quad (18)$$

and

$$\hat{R}_{v_0}^{(m)} = \lambda_m \hat{M}_m^{-1}, \quad 0 < |\lambda_m|^2 \leq \inf_{\hat{\rho}} p_m \quad (19)$$

for a particular  $v_0$ , where  $v$  denotes the outcome of the reversing measurement and  $\lambda_m$  is a complex number. The upper bound for  $\lambda_m$  is determined by the condition (18), namely  $\langle \hat{R}_{v_0}^{(m)\dagger} \hat{R}_{v_0}^{(m)} \rangle \leq 1$  for any  $\hat{\rho}$  [9]. Thus, the reversing measurement restores the premeasurement state if the measurement outcome is  $v_0$ .

In our situation with the predefined states  $\{\hat{\rho}(a)\}$ , when an outcome  $v$  is obtained from the reversing measurement on the state (14), the state of the system becomes

$$\hat{\rho}(m, v, a) = \frac{1}{p(m, v|a)} \hat{R}_v^{(m)} \hat{M}_m \hat{\rho}(a) \hat{M}_m^\dagger \hat{R}_v^{(m)\dagger}, \quad (20)$$

where

$$p(m, v|a) \equiv \langle \hat{M}_m^\dagger \hat{R}_v^{(m)\dagger} \hat{R}_v^{(m)} \hat{M}_m \rangle_a \quad (21)$$

is the joint probability for obtaining the set of outcomes  $(m, v)$  for the two successive measurements  $\{\hat{M}_m\}$  and  $\{\hat{R}_v^{(m)}\}$ . Conversely, given outcomes  $(m, v)$ , we can find the probability that the premeasurement state is  $\hat{\rho}(a)$ , with

$$p(a|m, v) = \frac{p(m, v|a)p(a)}{p(m, v)}, \quad (22)$$

where  $p(m, v)$  is the total probability for the set of outcomes  $(m, v)$ :

$$p(m, v) = \sum_a p(m, v|a)p(a). \quad (23)$$

The information gain then becomes

$$I(m, v) = H_0 - H(m, v), \quad (24)$$

with  $H(m, v)$  being the Shannon entropy after the reversing measurement:

$$H(m, v) = - \sum_a p(a|m, v) \log_2 p(a|m, v). \quad (25)$$

On the other hand, the fidelity after the reversing measurement is expressed as

$$F(m, v) = \sum_a p(a|m, v)F(m, v, a), \quad (26)$$

where  $p(a|m, v)$  is given in Eq. (22) and  $F(m, v, a)$  is the fidelity defined by

$$F(m, v, a) \equiv \text{Tr} \sqrt{\sqrt{\hat{\rho}(a)} \hat{\rho}(m, v, a) \sqrt{\hat{\rho}(a)}}. \quad (27)$$

If outcome  $v$  is that  $v_0$  for which the premeasurement state is recovered, fidelity (26) and information gain (24) reduce to

$$F(m, v_0) = 1, \quad (28)$$

$$I(m, v_0) = 0, \quad (29)$$

because  $\hat{R}_{v_0}^{(m)}$  is proportional to the inverse operator of  $\hat{M}_m$ ,

$$\hat{R}_{v_0}^{(m)} \hat{M}_m \propto \hat{I}. \quad (30)$$

That is, if the particular outcome  $v_0$  is obtained by the reversing measurement, the unknown original state  $\hat{\rho}(a)$  is perfectly recovered because the inverse operator of  $\hat{M}_m$  is applied to the system's state. However, when perfect recovery is achieved, the information obtained by the first measurement is completely lost by the reversing measurement,  $p(a|m, v_0) = p(a)$ , because the information concerning the premeasurement state is not reflected in the joint probability distribution for the perfect recovery [5]; i.e.,  $p(m, v_0|a) = |\lambda_m|^2$  does not depend on  $\hat{\rho}(a)$ .

Now, we consider a reversing operation that is based on the Hermitian conjugate of the measurement operator. That is, instead of the reversing measurement  $\{\hat{R}_v^{(m)}\}$ , we perform a measurement  $\{\hat{C}_\mu^{(m)}\}$  satisfying

$$\sum_\mu \hat{C}_\mu^{(m)\dagger} \hat{C}_\mu^{(m)} = \hat{I} \quad (31)$$

and

$$\hat{C}_{\mu_0}^{(m)} = \kappa_m \hat{M}_m^\dagger, \quad 0 < |\kappa_m|^2 \leq \left( \sup_{\hat{\rho}} p_m \right)^{-1} \quad (32)$$

with a complex number  $\kappa_m$  for a particular outcome  $\mu_0$ . The upper bound for  $\kappa_m$  is determined by the condition  $\langle \hat{C}_{\mu_0}^{(m)\dagger} \hat{C}_{\mu_0}^{(m)} \rangle \leq 1$  for any  $\hat{\rho}$ , which is equivalent to the condition  $\langle \hat{C}_{\mu_0}^{(m)} \hat{C}_{\mu_0}^{(m)\dagger} \rangle \leq 1$  for any  $\hat{\rho}$  because of polar decomposition (4). We shall refer to  $\{\hat{C}_\mu^{(m)}\}$  as a Hermitian conjugate measurement.

In our situation with  $\{\hat{\rho}(a)\}$ , when outcome  $\mu$  is obtained by the Hermitian conjugate measurement on state (14), the state of the system becomes

$$\hat{\rho}(m, \mu, a) = \frac{1}{p(m, \mu|a)} \hat{C}_\mu^{(m)} \hat{M}_m \hat{\rho}(a) \hat{M}_m^\dagger \hat{C}_\mu^{(m)\dagger}, \quad (33)$$

where

$$p(m, \mu|a) \equiv \langle \hat{M}_m^\dagger \hat{C}_\mu^{(m)\dagger} \hat{C}_\mu^{(m)} \hat{M}_m \rangle_a \quad (34)$$

is the joint probability for the set of outcomes  $(m, \mu)$ . We define fidelity  $F(m, \mu)$  and information gain  $I(m, \mu)$  as in the

case of reversing measurement, replacing  $\hat{R}_v^{(m)}$  with  $\hat{C}_\mu^{(m)}$ . If the outcome  $\mu$  is the preferred one  $\mu_0$ , the fidelity and information gain reduce to

$$F(m, \mu_0) = \frac{1}{\langle \hat{N}_m^4 \rangle} \sqrt{\langle \hat{N}_m^4 \rangle \langle \hat{N}_m^2 \rangle}, \quad (35)$$

$$I(m, \mu_0) = \frac{\langle \hat{N}_m^4 \rangle \log_2 \langle \hat{N}_m^4 \rangle - \langle \hat{N}_m^4 \rangle \log_2 \langle \hat{N}_m^4 \rangle}{\langle \hat{N}_m^4 \rangle}, \quad (36)$$

because from Eqs. (32) and (4) we have

$$\hat{C}_{\mu_0}^{(m)} \hat{M}_m \propto \hat{N}_m^2. \quad (37)$$

In the next section, we show that if the preferred outcome  $\mu_0$  is obtained by the Hermitian conjugate measurement, the unknown original state  $\hat{\rho}(a)$  is approximately recovered with additional information gain for a weak measurement.

#### IV. SIMULTANEOUS STATE RECOVERY AND INFORMATION GAIN FOR A WEAK MEASUREMENT

We consider the case of a measurement  $\{\hat{M}_m\}$  that provides only a small amount of information, e.g., measurement by an apparatus having a weak interaction with the system. In this case,  $\hat{N}_m$  in Eq. (4) can be expressed as

$$\hat{N}_m \equiv q_m (\hat{I} + \hat{\epsilon}_m), \quad (38)$$

where  $q_m$  is a positive number and  $\hat{\epsilon}_m$  is a small Hermitian operator. It follows from Eq. (1) that  $\{q_m\}$  and  $\{\hat{\epsilon}_m\}$  satisfy

$$\sum_m q_m^2 = 1, \quad (39)$$

$$\sum_m q_m^2 (2\hat{\epsilon}_m + \hat{\epsilon}_m^2) = 0. \quad (40)$$

Then, up to the order of  $\hat{\epsilon}_m^2$ , the information gain in Eq. (12) and its mean in Eq. (13) are calculated to be

$$I(m) \simeq 2V_I(\hat{\epsilon}_m), \quad (41)$$

$$I \simeq 2 \sum_m q_m^2 V_I(\hat{\epsilon}_m), \quad (42)$$

where  $V_I(\hat{\epsilon}_m)$  is a variance defined by

$$V_I(\hat{\epsilon}_m) \equiv \overline{\langle \hat{\epsilon}_m^2 \rangle} - \overline{\langle \hat{\epsilon}_m \rangle}^2 = \overline{\langle (\hat{\epsilon}_m - \overline{\langle \hat{\epsilon}_m \rangle})^2 \rangle} \geq 0. \quad (43)$$

This is a classical variance with respect to  $a$  of the quantum average  $\langle \hat{\epsilon}_m \rangle_a$ .

On the other hand, a weak measurement does not necessarily imply a small change in the system state, because the state change depends not only on  $\hat{N}_m$  but also on  $\hat{U}_m$  in Eq. (4). In general,  $\hat{U}_m$  can be written as

$$\hat{U}_m \equiv e^{i\gamma_m e^{i\hat{\Gamma}_m}}, \quad (44)$$

where  $\gamma_m$  is a real number and  $\hat{\Gamma}_m$  is a Hermitian operator. Note that, even if the interaction between the system and the measuring apparatus is weak,  $\hat{\Gamma}_m$  can be large if the degrees of freedom of the system or those of the measuring apparatus are large [14], as shown below. When all  $\hat{\rho}(a)$ 's are pure,  $\hat{\rho}(a) = |\psi(a)\rangle\langle\psi(a)|$ , we obtain the fidelity from Eq. (16) and

its mean from Eq. (17) as

$$F(m) \simeq \overline{|\langle \psi | e^{i\hat{\Gamma}_m} | \psi \rangle|} [1 + O(\hat{\epsilon}_m)], \quad (45)$$

$$F \simeq \sum_m q_m^2 \overline{|\langle \psi | e^{i\hat{\Gamma}_m} | \psi \rangle|} [1 + O(\hat{\epsilon}_m)]. \quad (46)$$

Equations (45) and (46) show that the fidelity can almost vanish if  $\hat{\Gamma}_m$  is large enough, even though large  $\hat{\Gamma}_m$  does not always imply small  $F(m)$ . Below, we consider a measurement that provides a small amount of information through Eq. (38), despite the fact that it drastically changes the state of the system, such that

$$\frac{1 - F(m)}{1 - F_{\text{opt}}(m)} > 4, \quad (47)$$

where  $F_{\text{opt}}(m)$  would be the fidelity if the measurement were optimal, i.e.,  $\hat{\Gamma}_m = 0$ . The explicit form of  $F_{\text{opt}}(m)$  is

$$F_{\text{opt}}(m) = \frac{1}{\langle \hat{N}_m^2 \rangle} \sqrt{\langle \hat{N}_m^2 \rangle \langle \hat{N}_m \rangle} \simeq 1 - \frac{1}{2} V_F(\hat{\epsilon}_m), \quad (48)$$

with  $V_F(\hat{\epsilon}_m)$  being a variance defined by

$$V_F(\hat{\epsilon}_m) \equiv \overline{\langle \hat{\epsilon}_m^2 \rangle} - \overline{\langle \hat{\epsilon}_m \rangle}^2 = \overline{\langle (\hat{\epsilon}_m - \overline{\langle \hat{\epsilon}_m \rangle})^2 \rangle} \geq 0. \quad (49)$$

This is a classical average over  $a$  of the quantum variance  $\langle (\hat{\epsilon}_m - \overline{\langle \hat{\epsilon}_m \rangle}_a)^2 \rangle_a$ .

From Eqs. (35) and (36), the fidelity and information gain after the Hermitian conjugate measurement with the preferred outcome  $\mu_0$  can be calculated up to the order of  $\hat{\epsilon}_m^2$  to be

$$F(m, \mu_0) \simeq 1 - 2V_F(\hat{\epsilon}_m), \quad (50)$$

$$I(m, \mu_0) \simeq 8V_I(\hat{\epsilon}_m). \quad (51)$$

Note that as long as higher-order terms can be ignored,

$$F(m, \mu_0) > F(m) \quad (52)$$

by the assumption made in Eq. (47). This means that the Hermitian conjugate measurement approximately recovers the original state  $\hat{\rho}(a)$ . Moreover, it follows from Eqs. (41) and (51) that the Hermitian conjugate measurement *simultaneously* enhances the information gain by a factor of 4, because

$$I(m, \mu_0) \simeq 4I(m). \quad (53)$$

Such an approximate recovery occurs because  $\hat{U}_m^\dagger$  in  $\hat{C}_{\mu_0}^{(m)}$  cancels the large disturbance caused by the unitary part  $\hat{U}_m$  in  $\hat{M}_m$ , while the additional information gain is obtained because the composition of  $\hat{M}_m$  and  $\hat{C}_{\mu_0}^{(m)}$  results in the optimal measurement  $\hat{N}_m$  being applied twice, as shown in Eq. (37). The state recovery of Hermitian conjugate measurement presents a sharp contrast to that of the reversing measurement shown in Eqs. (28) and (29), in which the reversing measurement perfectly recovers the original state  $\hat{\rho}(a)$  but completely obliterates the information  $I(m)$ . The recovery with information loss occurs because  $\hat{R}_{\mu_0}^{(m)}$  contains not only  $\hat{U}_m^\dagger$ , which cancels  $\hat{U}_m$ , but also  $\hat{N}_m^{-1}$ , which cancels the nonunitary part  $\hat{N}_m$  in  $\hat{M}_m$ , as in Eq. (30).

One might think that the probability for an approximate recovery is very low, and if an average over the outcome  $\mu$  is taken, the fidelity increases with a decrease in information gain. However, the preferred outcome  $\mu_0$  is more probable

when the outcome  $m$  of the measurement  $\{\hat{M}_m\}$  occurs with high probability. In fact, given outcome  $m$ , the conditional probability for outcome  $\mu$  of the Hermitian conjugate measurement  $\{\hat{C}_\mu^{(m)}\}$  is given by  $p(\mu|m) = p(m, \mu)/p(m)$ , which, for the preferred outcome  $\mu_0$ , reduces to

$$p(\mu_0|m) \simeq |\kappa_m|^2 \{ p(m) + 4q_m^2 [V_F(\hat{\epsilon}_m) + V_I(\hat{\epsilon}_m)] \}. \quad (54)$$

This indicates that, when  $p(m)$  is large,  $p(\mu_0|m)$  is also large. Discussing the mean fidelity and information gain conditioned by outcome  $m$ ,

$$F'(m) \equiv \sum_{\mu} p(\mu|m) F(m, \mu), \quad (55)$$

$$I'(m) \equiv \sum_{\mu} p(\mu|m) I(m, \mu), \quad (56)$$

we must specify  $\hat{C}_\mu^{(m)}$ 's other than  $\mu = \mu_0$ . Here, we consider a minimal model, where the only two possible outcomes of the Hermitian conjugate measurement are  $\mu = \mu_0$  and  $\mu = \mu_1$ . Then, the measurement operator for  $\mu = \mu_1$  is chosen as

$$\hat{C}_{\mu_1}^{(m)} = \sqrt{1 - a_m^2} \left[ \hat{I} - \frac{a_m^2}{1 - a_m^2} \hat{\epsilon}_m - \frac{a_m^2}{2(1 - a_m^2)^2} \hat{\epsilon}_m^2 \right] \hat{U}_m^\dagger, \quad (57)$$

where  $a_m^2 \equiv |\kappa_m|^2 q_m^2$ , and we assume that  $a_m^2 \hat{\epsilon}_m / (1 - a_m^2)$  is small, so that condition (31) is satisfied up to the order of  $\hat{\epsilon}_m^2$ . When the outcome of the Hermitian conjugate measurement  $\{\hat{C}_\mu^{(m)}\}$  is  $\mu_1$ , the fidelity and the information gain become

$$F(m, \mu_1) \simeq 1 - \frac{1}{2} \left( 1 - \frac{a_m^2}{1 - a_m^2} \right)^2 V_F(\hat{\epsilon}_m), \quad (58)$$

$$I(m, \mu_1) \simeq 2 \left( 1 - \frac{a_m^2}{1 - a_m^2} \right)^2 V_I(\hat{\epsilon}_m). \quad (59)$$

In this case, the Hermitian conjugate measurement decreases the information gain  $I(m, \mu_1) < I(m)$  from Eq. (41). The mean fidelity (55) and information gain (56) after the Hermitian conjugate measurement are then given by

$$F'(m) \simeq 1 - \frac{1}{2(1 - a_m^2)} V_F(\hat{\epsilon}_m), \quad (60)$$

$$I'(m) \simeq \frac{2}{1 - a_m^2} V_I(\hat{\epsilon}_m), \quad (61)$$

which imply  $I'(m) > I(m)$  and  $F'(m) > F(m)$  if  $a_m^2 < 3/4$  from Eq. (47). Therefore, the Hermitian conjugate measurement, on average, increases both the fidelity and information gain. We can obtain the same conclusion even after the averages over  $m$  are taken:

$$F' \equiv \sum_m p(m) F'(m) > F, \quad (62)$$

$$I' \equiv \sum_m p(m) I'(m) > I. \quad (63)$$

### V. EXAMPLE: ISING-TYPE INTERACTION

As an example, we consider a quantum measurement on a spin- $s$  system described by spin operators  $\{\hat{S}_x, \hat{S}_y, \hat{S}_z\}$ . We

assume that we have no *a priori* information about the state of the system except that it is a pure state. This means that the set of predefined states,  $\{\hat{\rho}(a)\}$ , consists of all possible pure states. That is,  $\hat{\rho}(a)$  can be written as  $\hat{\rho}(a) = |\psi(a)\rangle\langle\psi(a)|$  by a state vector

$$|\psi(a)\rangle = \sum_{\sigma} c_{\sigma}(a) |\sigma\rangle, \quad (64)$$

where  $|\sigma\rangle$  is the eigenstate of  $\hat{S}_z$  with eigenvalue  $\sigma$  ( $= -s, -s+1, \dots, s-1, s$ ) and  $c_{\sigma}(a)$ 's obey the normalization condition  $\sum_{\sigma} |c_{\sigma}(a)|^2 = 1$ .

To obtain information about the system's state, we perform a measurement using a spin- $j$  probe (measuring apparatus) described by spin operators  $\{\hat{J}_x, \hat{J}_y, \hat{J}_z\}$ . The measurement proceeds as follows. The probe is first prepared in a coherent spin state  $|\theta, \pi/2\rangle$  [24], which is the eigenstate of the spin component  $\hat{J}_y \sin \theta + \hat{J}_z \cos \theta$  with eigenvalue  $j$ . The probe then interacts with the system via an interaction Hamiltonian

$$H_{\text{int}} = \alpha \hat{J}_z \hat{S}_z, \quad (65)$$

where  $\alpha$  is a real constant. This  $\hat{J}_z \hat{S}_z$ -type interaction has direct relevance to the experimental situations in Refs. [25–29]. After interaction during time  $t$ , a unitary operator

$$\hat{U}_p = e^{-i\pi \hat{J}_y / 2} \quad (66)$$

is applied to the probe. Finally, we obtain outcome  $m$  ( $= -j, -j+1, \dots, j-1, j$ ) by performing a projective measurement on the probe observable  $\hat{J}_z$ . The outcome  $m$  then provides some information about the state  $\hat{\rho}(a)$ . The measurement process is described by the set of measurement operators [12]

$$\hat{M}_m = \hat{T}_m(\theta) \equiv \sum_{\sigma} a_{m\sigma}^{(j)}(\theta) |\sigma\rangle\langle\sigma|, \quad (67)$$

where

$$\begin{aligned} a_{m\sigma}^{(j)}(\theta) &= \frac{e^{-ij\pi/2}}{2^j} \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} \\ &\times \left( e^{-ig\sigma} \cos \frac{\theta}{2} + i e^{ig\sigma} \sin \frac{\theta}{2} \right)^{j-m} \\ &\times \left( e^{-ig\sigma} \cos \frac{\theta}{2} - i e^{ig\sigma} \sin \frac{\theta}{2} \right)^{j+m} \end{aligned} \quad (68)$$

with  $g \equiv \alpha t/2$  being the effective strength of the interaction. When the interaction is weak,  $\hat{N}_m$  in the decomposition of  $\hat{M}_m$  in Eq. (4) can be written as in Eq. (38), with

$$q_m = \frac{1}{2^j} \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}, \quad (69)$$

$$\hat{\epsilon}_m \simeq 2gm \sin \theta \hat{S}_z + O(g^2), \quad (70)$$

and  $\hat{U}_m$  can be written as in Eq. (44), with

$$\gamma_m = -\frac{j\pi}{2} - m\theta, \quad (71)$$

$$\hat{\Gamma}_m \simeq -2gj \cos \theta \hat{S}_z + O(g^2). \quad (72)$$

Because the probability for outcome  $m$  is  $p(m) \simeq q_m^2 + O(g)$  from Eq. (8), the expectation value and variance of the outcome

are given by

$$\bar{m} \equiv \sum_m p(m)m \simeq 0 + O(g), \quad (73)$$

$$(\delta m)^2 \equiv \sum_m p(m)(m - \bar{m})^2 \simeq \frac{j}{2} + O(g), \quad (74)$$

respectively. Comparing Eqs. (73) and (74) with Eq. (70), we find that  $\hat{\epsilon}_m \sim O(g\sqrt{j})$ . In contrast, Eq. (72) shows that  $\hat{\Gamma}_m \sim O(gj)$ . Therefore, even if  $\hat{\epsilon}_m$  is small,  $\hat{\Gamma}_m$  can be large for large values of  $j$ . In the following discussion, we shall consider such a situation by assuming that  $g$  is so small that

$$\frac{2}{3}g^2s(s+1)j\sin^2\theta \ll 1, \quad (75)$$

but  $j$  is so large that  $\hat{U}_m$  differs greatly from the identity operator,

$$|2gj\cos\theta| \sim \pi. \quad (76)$$

Substituting Eq. (70) into Eqs. (41) and (42), we obtain the information gain and its mean to the order of  $g^2$  as

$$I(m) \simeq \frac{4}{3}g^2s m^2 \sin^2\theta, \quad (77)$$

$$I \simeq \frac{2}{3}g^2sj\sin^2\theta, \quad (78)$$

where we have used

$$V_I(\hat{S}_z) = \frac{1}{6}s \quad (79)$$

(see Appendix). On the other hand, we cannot expand the fidelity in Eq. (16) and its mean in Eq. (17) in terms of  $g$  when  $\hat{\Gamma}_m$  is large. If we *formally* expand them, they are given by

$$F(m) \simeq 1 - \frac{1}{3}g^2s(2s+1)(j^2\cos^2\theta + m^2\sin^2\theta), \quad (80)$$

$$F \simeq 1 - \frac{1}{3}g^2s(2s+1)\left(j^2\cos^2\theta + \frac{j}{2}\sin^2\theta\right), \quad (81)$$

respectively, because the variance  $V_F(\hat{S}_z)$  is calculated to be

$$V_F(\hat{S}_z) = \frac{1}{6}s(2s+1) \quad (82)$$

(see Appendix). Compared to Eq. (80), the optimal fidelity (48) can be expanded in terms of  $g$  as

$$F_{\text{opt}}(m) \simeq 1 - \frac{1}{3}g^2s(2s+1)m^2\sin^2\theta, \quad (83)$$

without the term of order  $g^2j^2$  originating from  $\hat{\Gamma}_m$ .

Figures 1, 2, and 3 show  $p(m)$ ,  $F(m)$ , and  $I(m)$ , respectively, as functions of  $m$  for  $s = 1/2$ ,  $j = 7$ ,  $g = 0.25$ , and  $\theta = \pi/6$ , where the assumptions in Eqs. (75) and (76) are satisfied. In Fig. 3,  $I(m)$  deviates from Eq. (77) for large  $|m|$ , because higher-order terms in  $g$  are not negligible there. Note that  $\hat{\epsilon}_m$  for  $|m| \simeq j$  is not necessarily small even if Eq. (75) is assumed, though the probability for such  $m$  is very small, as shown in Fig. 1. The mean fidelity and information gain are  $F = 0.535$  and  $I = 0.045$ , respectively. In this example, Eq. (47) is satisfied when  $-5 \leq m \leq 5$ .

To recover the original state  $\hat{\rho}(a)$ , we next perform a Hermitian conjugate measurement on the state  $\hat{\rho}(m, a)$  after measurement  $\{\hat{M}_m\}$ . It is chosen independently of  $m$  as

$$\hat{C}_\mu^{(m)} = \hat{T}_m(\pi - \theta) = \sum_\sigma a_{\mu\sigma}^{(j)}(\pi - \theta) |\sigma\rangle\langle\sigma|, \quad (84)$$

which can be achieved in the same way as the measurement  $\{\hat{M}_m\}$ , by replacing the initial probe state  $|\theta, \pi/2\rangle$  with

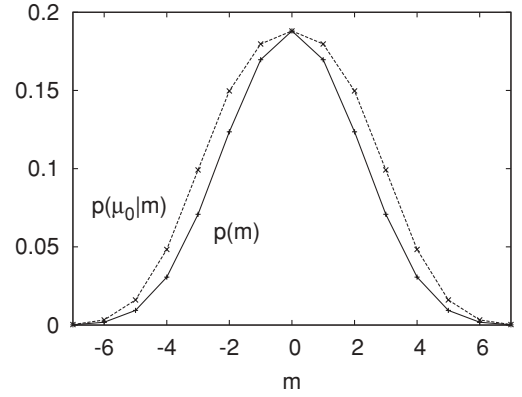


FIG. 1. Probability  $p(m)$  of obtaining outcome  $m$  for measurement  $\{\hat{M}_m\}$  and probability  $p(\mu_0|m)$  of obtaining the preferred outcome  $\mu_0 = m$  for the Hermitian conjugate measurement  $\{\hat{C}_\mu^{(m)}\}$  conditioned by the first outcome  $m$ , with  $s = 1/2$ ,  $j = 7$ ,  $g = 0.25$ , and  $\theta = \pi/6$ .

$|\pi - \theta, \pi/2\rangle$ . The preferred outcome  $\mu_0$  is equal to  $m$ , because

$$\hat{T}_m(\pi - \theta) = (-1)^{j+m} \hat{T}_m^\dagger(\theta). \quad (85)$$

Note that this measurement  $\{\hat{C}_\mu^{(m)}\}$  can also be regarded as a reversing measurement with the preferred outcome  $\nu_0 = -m$  if  $s = 1/2$  [12], because

$$\hat{T}_{-m}(\pi - \theta) \propto \hat{T}_m^{-1}(\theta); \quad (86)$$

this relation holds only approximately if  $s > 1/2$ . In fact, an approximate recovery with additional information gain was first reported [12] regarding the reversing measurement without identifying the origin of the information gain. The origin is now clarified in terms of the Hermitian conjugate measurement. If the initial probe state for  $\hat{M}_m$  is the more general  $|\theta, \phi\rangle$  [12], that for the Hermitian conjugate measurement is  $|\pi - \theta, \phi\rangle$  with  $\mu_0 = m$  or  $|\pi - \theta, \phi + \pi\rangle$  with  $\mu_0 = -m$ , while that for the reversing measurement of  $s = 1/2$  is  $|\pi - \theta, \pi - \phi\rangle$  with  $\nu_0 = -m$  or  $|\pi - \theta, -\phi\rangle$  with  $\nu_0 = m$ .

If the Hermitian conjugate measurement  $\{\hat{C}_\mu^{(m)}\}$  yields an outcome  $\mu$  ( $= -j, -j+1, \dots, j-1, j$ ), the fidelity and

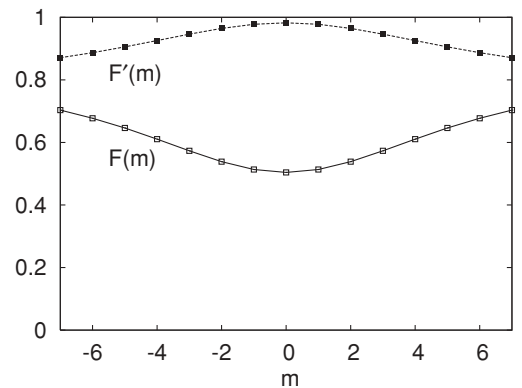


FIG. 2. Fidelity  $F(m)$  after measurement  $\{\hat{M}_m\}$  and mean fidelity  $F'(m)$  after the Hermitian conjugate measurement  $\{\hat{C}_\mu^{(m)}\}$  as functions of the first outcome  $m$ , with  $s = 1/2$ ,  $j = 7$ ,  $g = 0.25$ , and  $\theta = \pi/6$ .

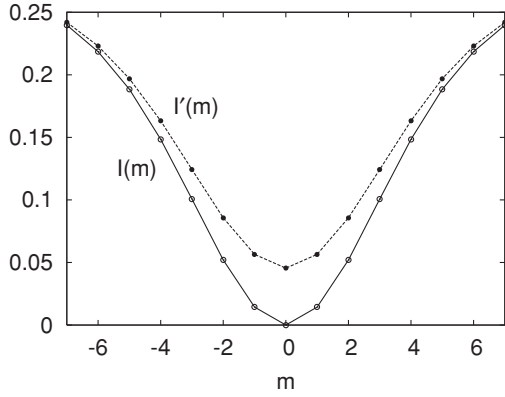


FIG. 3. Information  $I(m)$  after measurement  $\{\hat{M}_m\}$  and mean information  $I'(m)$  after the Hermitian conjugate measurement  $\{\hat{C}_\mu^{(m)}\}$  as functions of the first outcome  $m$ , with  $s = 1/2$ ,  $j = 7$ ,  $g = 0.25$ , and  $\theta = \pi/6$ .

information gain become

$$F(m, \mu) \simeq 1 - \frac{1}{3}g^2s(2s+1)(\mu+m)^2 \sin^2 \theta, \quad (87)$$

$$I(m, \mu) \simeq \frac{4}{3}g^2s(\mu+m)^2 \sin^2 \theta. \quad (88)$$

Figure 4 plots the sets of outcomes  $(m, \mu)$  for which  $F(m, \mu) > F(m)$  and  $I(m, \mu) > I(m)$  with  $s = 1/2$ ,  $j = 7$ ,  $g = 0.25$ , and  $\theta = \pi/6$ . The conditional probability for the preferred outcome  $\mu_0 = m$  in Eq. (54) is shown in Fig. 1. Taking the average over outcome  $\mu$ , we obtain the mean fidelity and mean information defined in Eqs. (55) and (56), respectively, as

$$F'(m) \simeq 1 - \frac{1}{3}g^2s(2s+1)\left(m^2 + \frac{j}{2}\right) \sin^2 \theta, \quad (89)$$

$$I'(m) \simeq \frac{4}{3}g^2s\left(m^2 + \frac{j}{2}\right) \sin^2 \theta. \quad (90)$$

Figures 2 and 3 also show  $F'(m)$  and  $I'(m)$ , respectively, as functions of  $m$ . Note that, in this example,  $F'(m) > F(m)$  and  $I'(m) > I(m)$  for any value of  $m$ . If the average over outcome  $m$  is taken, the total mean fidelity in Eq. (62) and total mean

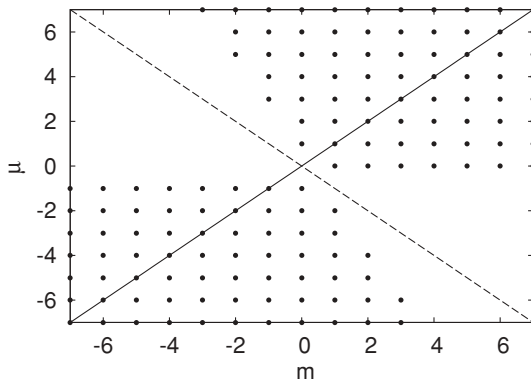


FIG. 4. Sets of outcomes  $(m, \mu)$  for which  $F(m, \mu) > F(m)$  and  $I(m, \mu) > I(m)$ , with  $s = 1/2$ ,  $j = 7$ ,  $g = 0.25$ , and  $\theta = \pi/6$ . The solid line ( $\mu = m$ ) denotes the Hermitian conjugate measurement with the preferred outcome, while the dashed line ( $\mu = -m$ ) corresponds to the reversing measurement with the preferred outcome,  $F(m, -m) = 1$  and  $I(m, -m) = 0$ .

information in Eq. (63) are given by

$$F' \simeq 1 - \frac{1}{3}g^2s(2s+1)j \sin^2 \theta, \quad (91)$$

$$I' \simeq \frac{4}{3}g^2sj \sin^2 \theta. \quad (92)$$

Assumption (75) ensures that  $F'$  is close to 1. Unlike Eq. (81), no term of order  $g^2j^2$  appears in the fidelity expression in Eq. (91), because the effect of large  $\hat{\Gamma}_m$  is canceled out by the Hermitian conjugate measurement. When  $s = 1/2$ ,  $j = 7$ ,  $g = 0.25$ , and  $\theta = \pi/6$ ,  $F' = 0.966 > F$  and  $I' = 0.081 > I$ . Thus, the Hermitian conjugate measurement increases both fidelity and information gain when the particular outcomes are obtained, as well as when averages over the outcomes are taken.

## VI. CONCLUSION AND DISCUSSION

We have discussed a probabilistic reversing operation on a system subjected to a state change caused by a weak measurement. The reversing operation can increase not only the fidelity to its original state but also the information gain. The essential feature of the operation is to utilize the Hermitian conjugate of the measurement operator rather than its inverse. The Hermitian conjugate operator cancels the unitary part of the measurement operator, which does not carry information, and enhances the information-carrying nonunitary part because the composition of  $\hat{M}_m$  and  $\hat{C}_{\mu_0}^{(m)}$  results in the optimal measurement  $\hat{N}_m$  being applied twice, as shown in Eq. (37). In contrast, the inverse operator cancels both unitary and nonunitary parts. As an explicit example, we considered a quantum measurement of a spin- $s$  system using a spin- $j$  probe and demonstrated that the reversing operation can increase not only the fidelity and information gain with a high probability but also their average values. The measurement and its reversing operation described in Sec. V can be implemented [12] using an ensemble of  $2s$  two-level atoms as a system and a collection of  $2j$  photons with two polarizations (horizontal or vertical) as a probe. The interaction in Eq. (65) is then realized via a Faraday rotation [25–28].

The Hermitian conjugate measurement  $\{\hat{C}_\mu^{(m)}\}$  is more feasible than the reversing measurement  $\{\hat{R}_\nu^{(m)}\}$ . Consider a quantum measurement in which a probe with initial state  $|i\rangle$  interacts with the system via an interaction  $\hat{U}_{\text{int}}$  and then it is measured with respect to a certain observable. The measurement operator for this measurement is written as  $\hat{M}_m = \langle m | \hat{U}_{\text{int}} | i \rangle$ , where  $|m\rangle$  is the final state of the probe corresponding to outcome  $m$ . Because its Hermitian conjugate operator is given by  $\hat{M}_m^\dagger = \langle i | \hat{U}_{\text{int}}^\dagger | m \rangle$ , the Hermitian conjugate measurement can be performed by a probe with initial state  $|m\rangle$  together with the time-reversed interaction  $\hat{U}_{\text{int}}^\dagger$ . The preferred outcome is the one that corresponds to the probe state  $|i\rangle$ . The implementation of the Hermitian conjugate measurement can be complicated in more general situations. Nevertheless, in photon counting [19], the standard photon counter implements the annihilation operator  $\hat{a}$  of the photon, while the quantum counter [2,5,30–32] implements its Hermitian conjugate operator, i.e., the creation operator  $\hat{a}^\dagger$ .

Note that, while the Hermitian conjugate of an operator always exists, unlike the inverse, it does not always increase

the fidelity and information gain. For example, a projection operator  $\hat{P}$  does not have an inverse  $\hat{P}^{-1}$ , but it does have the Hermitian conjugate  $\hat{P}^\dagger = \hat{P}$ . However, when the measurement operator  $\hat{M}_m$  is a projection operator, the Hermitian conjugate measurement leaves the fidelity and information gain unchanged. Moreover, in the case of an optimal measurement  $\{\hat{N}_m\}$ , its Hermitian conjugate measurement increases the information gain but decreases the fidelity. Thus, our approximate recovery with additional information gain relies on assumptions in Eqs. (38) and (47), which mean that the measurement provides little information but drastically changes the state of the system because  $\hat{\epsilon}_m$  is small and  $\hat{\Gamma}_m$  is large.

It might appear that our conclusion is due to the choice of information measure in Eq. (12). However, the same conclusion could be drawn from another appropriate measure of information, such as the measure proposed in Ref. [33]. This is because Eq. (37) states that the combined effect of operations of  $\hat{M}_m$  and  $\hat{C}_{\mu_0}^{(m)}$  amounts to applying the optimal measurement  $\hat{N}_m$  twice. If we perform a measurement twice and obtain the same outcome, our knowledge about the state of the system becomes more accurate than for a single measurement outcome.

In quantum cryptography [34–37], our scheme could benefit eavesdroppers. If the available interactions are limited, the information obtained by eavesdropping would be lowered with respect to the disturbance of the state transferred between the sender and the receiver. However, the Hermitian conjugate measurement could make eavesdropping more efficient, because it approximately recovers the state with additional information gain. On the other hand, in quantum error correction [38–40], the Hermitian conjugate measurement scheme has less advantage than the reversing measurement scheme [9], because no information gain is required, and the emphasis is on perfect state recovery.

#### ACKNOWLEDGMENTS

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#### APPENDIX: CALCULATION OF VARIANCES

We here prove Eqs. (79) and (82). The variances are defined by

$$V_I(\hat{S}_z) = \overline{\langle \hat{S}_z \rangle^2} - (\overline{\langle \hat{S}_z \rangle})^2, \quad (\text{A1})$$

$$V_F(\hat{S}_z) = \overline{\langle \hat{S}_z^2 \rangle} - \overline{\langle \hat{S}_z \rangle}^2, \quad (\text{A2})$$

where the expectation values are given from Eqs. (9) and (64) by

$$\overline{\langle \hat{S}_z \rangle} = \frac{1}{N} \sum_a \sum_\sigma |c_\sigma(a)|^2 \sigma, \quad (\text{A3})$$

$$\overline{\langle \hat{S}_z^2 \rangle} = \frac{1}{N} \sum_a \sum_\sigma |c_\sigma(a)|^2 \sigma^2, \quad (\text{A4})$$

$$\overline{\langle \hat{S}_z \rangle^2} = \frac{1}{N} \sum_a \sum_{\sigma, \sigma'} |c_\sigma(a)|^2 |c_{\sigma'}(a)|^2 \sigma \sigma'. \quad (\text{A5})$$

Because index  $a$  runs over all pure states, there is no preferred  $\sigma$ . From this symmetry, we can set

$$\frac{1}{N} \sum_a |c_\sigma(a)|^2 \equiv C \quad (\text{A6})$$

and

$$\frac{1}{N} \sum_a |c_\sigma(a)|^2 |c_{\sigma'}(a)|^2 \equiv \begin{cases} D & (\text{if } \sigma = \sigma'); \\ E & (\text{if } \sigma \neq \sigma'), \end{cases} \quad (\text{A7})$$

where  $C$ ,  $D$ , and  $E$  are constants that do not depend on  $\sigma$  and  $\sigma'$ . Using these constants with the summations  $\sum_\sigma \sigma = 0$  and

$$\sum_\sigma \sigma^2 = - \sum_{\sigma \neq \sigma'} \sigma \sigma' = \frac{1}{3} s(s+1)(2s+1), \quad (\text{A8})$$

it can be shown that

$$\overline{\langle \hat{S}_z \rangle} = 0, \quad (\text{A9})$$

$$\overline{\langle \hat{S}_z^2 \rangle} = \frac{1}{3} s(s+1)(2s+1)C, \quad (\text{A10})$$

$$\overline{\langle \hat{S}_z \rangle^2} = \frac{1}{3} s(s+1)(2s+1)(D-E). \quad (\text{A11})$$

To calculate  $C$ ,  $D$ , and  $E$ , we introduce a parametrization of coefficients  $\{c_\sigma(a)\}$ . Let  $\alpha_\sigma(a)$  and  $\beta_\sigma(a)$  be the real and imaginary parts of  $c_\sigma(a)$ , respectively. The normalization condition then becomes

$$\sum_\sigma |c_\sigma(a)|^2 = \sum_\sigma [\alpha_\sigma(a)^2 + \beta_\sigma(a)^2] = 1, \quad (\text{A12})$$

which is the condition for a point to be on the unit sphere in  $2(2s+1)$  dimensions. Therefore, we parametrize  $\alpha_\sigma(a)$  and  $\beta_\sigma(a)$  using hyperspherical coordinates as

$$\begin{aligned} \alpha_s(a) &= \sin \theta_{4s} \sin \theta_{4s-1} \cdots \sin \theta_3 \sin \theta_2 \sin \theta_1 \cos \phi, \\ \beta_s(a) &= \sin \theta_{4s} \sin \theta_{4s-1} \cdots \sin \theta_3 \sin \theta_2 \sin \theta_1 \sin \phi, \\ \alpha_{s-1}(a) &= \sin \theta_{4s} \sin \theta_{4s-1} \cdots \sin \theta_3 \sin \theta_2 \cos \theta_1, \\ \beta_{s-1}(a) &= \sin \theta_{4s} \sin \theta_{4s-1} \cdots \sin \theta_3 \cos \theta_2, \\ &\vdots \\ \alpha_{-s}(a) &= \sin \theta_{4s} \cos \theta_{4s-1}, \\ \beta_{-s}(a) &= \cos \theta_{4s}, \end{aligned} \quad (\text{A13})$$

with  $0 \leq \phi < 2\pi$  and  $0 \leq \theta_p \leq \pi$  ( $p = 1, 2, \dots, 4s$ ). By replacing the summation over  $a$  with an integral,

$$\frac{1}{N} \sum_a \rightarrow \frac{(2s)!}{2\pi^{2s+1}} \int_0^{2\pi} d\phi \prod_{p=1}^{4s} \int_0^\pi d\theta_p \sin^p \theta_p, \quad (\text{A14})$$

and setting  $\sigma = s$  and  $\sigma' = -s$ , we find that

$$C = \frac{1}{N} \sum_a |c_s(a)|^2 = \frac{(2s)!}{\pi^{2s}} \prod_{p=1}^{4s} \int_0^\pi d\theta_p \sin^{p+2} \theta_p, \quad (\text{A15})$$

$$D = \frac{1}{N} \sum_a |c_s(a)|^4 = \frac{(2s)!}{\pi^{2s}} \prod_{p=1}^{4s} \int_0^\pi d\theta_p \sin^{p+4} \theta_p, \quad (\text{A16})$$



$$\begin{aligned}
E &= \frac{1}{N} \sum_a |c_s(a)|^2 |c_{-s}(a)|^2 \\
&= C - \frac{(2s)!}{\pi^{2s}} \prod_{p=4s-1}^{4s} \int_0^\pi d\theta_p \sin^{p+4} \theta_p \\
&\quad \times \prod_{p=1}^{4s-2} \int_0^\pi d\theta_p \sin^{p+2} \theta_p.
\end{aligned} \tag{A17}$$

Using the integral formula

$$\int_0^\pi d\theta \sin^n \theta = \sqrt{\pi} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \tag{A18}$$

for  $n > -1$  with the Gamma function  $\Gamma(n)$ , the constants are calculated to be

$$\begin{aligned}
C &= \frac{1}{2s+1}, \quad D = \frac{1}{(s+1)(2s+1)}, \\
E &= \frac{1}{2(s+1)(2s+1)}.
\end{aligned} \tag{A19}$$

Substituting these results into Eqs. (A9)–(A11), we finally obtain

$$\overline{\langle \hat{S}_z \rangle} = 0, \quad \overline{\langle \hat{S}_z^2 \rangle} = \frac{1}{3}s(s+1), \quad \overline{\langle \hat{S}_z^2 \rangle} = \frac{1}{6}s, \tag{A20}$$

which prove Eqs. (79) and (82) through definitions (A1) and (A2).

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