PHYSICAL REVIEW A **81**, 012105 (2010)

Entangling two unequal atoms through a common bath

F. Benatti,[1,2] R. Floreanini,[2] and U. Marzolino[1,2]
[1]*Dipartimento di Fisica Teorica, Università di Trieste, I-34014 Trieste, Italy*
[2]*Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, I-34014 Trieste, Italy*

(Received 15 July 2009; published 7 January 2010)

The evolution of two, noninteracting, two-level atoms immersed in a weakly coupled bath can be described by a refined, time-coarse-grained Markovian evolution, still preserving complete positivity. We find that this improved, reduced dynamics is able to entangle the two atoms even when their internal frequencies are unequal, an effect that appears impossible in the standard weak-coupling-limit approach. We study in detail this phenomenon for an environment made of quantum fields.

DOI: 10.1103/PhysRevA.81.012105 PACS number(s): 03.65.Yz, 03.67.Bg

I. INTRODUCTION

Independent, non-interacting atoms immersed in a common bath represent an example of an open quantum system (i.e., a subsystem that interacts with a large environment) [1–6]. Their time evolution can be given as a reduced dynamics, obtained by eliminating the degrees of freedom of the environment and by subsequently performing a Markovian, memoryless approximation, justified by the very rapid decay of correlations in the bath. The resulting atom evolution is irreversible and incorporates dissipative and noisy effects induced by the environment.

In many physical instances, the atoms can be treated within a nonrelativistic approximation as independent finite-level systems with negligibly small size. On the other hand, the environment can be modeled by a set of weakly coupled quantum fields (typically the electromagnetic field) in a given temperature state, interacting with the atoms through a dipole-type coupling [7]. Although this simplified setting ignores the internal atom dynamics and the full vectorial structure of the electromagnetic field, it is nevertheless perfectly adequate for studying the behavior of physical systems like ions in traps, atoms in optical cavities and fibers, and impurities in phonon fields [7–9].

The derivation of an acceptable subdynamics for the atoms is notoriously tricky, [1–3,10] and time evolutions that are not even positive have been adopted in the literature in order to describe their physical properties [5,11,12]. Instead, a physically consistent time evolution for the atom subsystem can be obtained through a suitable coarse-grained procedure within the weak coupling approximation [1,3,13–15]. The resulting subdynamics are described by a one parameter ($\equiv$time) family of completely positive maps that form a quantum dynamical semigroup.

Below, we shall explicitly discuss such a derivation for a subsystem composed of two, unequal, mutually non-interacting atoms. For simplicity, we shall restrict our attention to two-level atoms weakly coupled to a collection of independent, free, massless scalar fields in $3+1$ space-time dimensions, assumed to be in a state at temperature $T \equiv 1/\beta$.

As is well known, interaction with an environment usually leads to decoherence and noise, typical mixing enhancing phenomena. Therefore, one generally expects that quantum correlations between the two atoms that might have been present at the beginning be destroyed when they are immersed in the bath.

However, an external environment can also mediate indirect interactions between otherwise totally decoupled subsystems and therefore constitute a means to correlate them. This phenomenon is generic on a short, microscopic time scale where the dynamics are unitary and reversible; however, its persistence is not expected in general on longer time scales where irreversible, dissipative, and decohering effects described by Markovian master equations appear.

Nevertheless, there are instances where purely dissipative, non-Hamiltonian contributions to the master equation can lead to entanglement generation [8,16–23]. This is due to the particular form of the (Kraus) operators appearing there, which couple indirectly (i.e., not dynamically) the two subsystems. This phenomenon has been established in the case of subsystems formed by two, identical two-level systems [24–27] or harmonic oscillators [28,29] evolving with reduced dynamics obtained via the *weak-coupling limit* [30]. This technique is applicable when the time scale over which the dissipative effects become visible is so large that the free dynamics of the subsystems can be effectively averaged out, thus eliminating very rapid oscillations. This phenomenon typically occurs for environments with very fast-decaying correlations [2,3,5]. Nevertheless, it turns out that, in the case of atoms with unequal frequencies, this procedure of averaging out fast oscillations prevents the generation of entanglement; i.e., the environments for which the weak-coupling-limit procedure is justified are unable to correlate two atoms with different internal frequencies, while they are able to correlate two atoms when the atoms are identical.

In the following, we shall study in detail the conditions that allow the two otherwise independent, unequal atoms to become initially entangled through the action of the environment, when the weak-coupling-limit procedure is not applicable.[1] Following [15], we shall instead derive a time evolution for the two atoms that allows a finite, coarse-grained time interval. Remarkably, the results are still expressible in terms of a completely positive quantum dynamical semigroup, which reduces to the standard one obtained through the weak-coupling

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[1] A preliminary investigation on these topics in the case of two qubits weakly coupled to an Ohmic bath has been reported in [31].

limit in the limit of an infinitely large coarse-grained time interval. We shall see that in this refined framework the sharp dependence on the atom frequencies of the entanglement capability of the environment mentioned above looks more like a mathematical artifact than a real physical effect.

II. TWO-ATOM REDUCED DYNAMICS

As explained above, we shall study the behavior of a system composed of two, unequal, two-level atoms that starts interacting at time $t = 0$ with an environment made of a collection of independent, massless, scalar quantum fields at temperature $1/\beta$.[2] We are interested in the evolution of the atoms as open quantum systems and not in the details of their internal dynamics, so we model them in a nonrelativistic way as simple qubits described in terms of a two-dimensional Hilbert space.

In the absence of any interaction with the external fields, the single atom internal dynamics can be taken to be driven by a generic 2×2 Hamiltonian matrix. As a result, the total atom Hamiltonian H_S can be expressed as

$$H_S = H_S^{(1)} + H_S^{(2)}$$

$$H_S^{(\alpha)} = \frac{\omega_\alpha}{2} \vec{n} \cdot \vec{\sigma}^{(\alpha)} \equiv \frac{\omega_\alpha}{2} \sum_{i=1}^{3} n_i \sigma_i^{(\alpha)}, \quad \alpha = 1, 2, \tag{1}$$

where $\sigma_i^{(1)} = \sigma_i \otimes \mathbf{1}$ and $\sigma_i^{(2)} = \mathbf{1} \otimes \sigma_i$ are the basis operators pertaining to the two different atoms with σ_i, $i = 1, 2, 3$ denoting the Pauli matrices, n_i are the components of a unit vector, and ω_α represent the gaps between the two energy eigenvalues of the two atoms.

As mentioned in the introductory remarks, the coupling of the atoms with the external fields is assumed to be weak, so that the dipole approximation results appropriate [7]. In our simplified settings that ignore spinorial indices, the interaction term can then be described by a Hamiltonian H' that is linear in both atom and field variables:

$$H' = \sum_{i=1}^{3} \left[\sigma_i^{(1)} \otimes \Phi_i(f^{(1)}) + \sigma_i^{(2)} \otimes \Phi_i(f^{(2)})\right]. \tag{2}$$

The operators $\Phi_i(t, \vec{x})$ represent the set of external quantum fields, taken to be spinless and massless for simplicity. They evolve in time as free relativistic fields with a standard Hamiltonian H_Φ [32]. The atoms are assumed to have a spatial extension described by the two functions $f^{(\alpha)}(\vec{x})$, $\alpha = 1, 2$. To be more specific, we shall choose for the atoms a common profile $f(\vec{x})$ of spherically symmetric shape, with size ε:

$$f(\vec{x}) = \frac{1}{\pi^2} \frac{(\varepsilon/2)}{[|\vec{x}|^2 + (\varepsilon/2)^2]^2}, \tag{3}$$

and position the first atom at the origin of the reference frame so that $f^{(1)}(\vec{x}) \equiv f(\vec{x})$, while the second is displaced by an amount $\vec{\ell}$ with respect to the first atom, $f^{(2)}(\vec{x}) = f(\vec{x} + \vec{\ell})$. Since the atom-field interaction takes place over the whole region occupied by the atoms, the field operators entering the interaction Hamiltonian above are smeared over spatial extension of the atoms:

$$\Phi_i\left(f^{(\alpha)}\right) = \int d^3x f^{(\alpha)}(\vec{x}) \Phi_i(0, \vec{x}), \quad \alpha = 1, 2. \tag{4}$$

The total Hamiltonian H describing the complete system (i.e., the two atoms together with the external fields Φ_i) can thus be written as

$$H = H_S + H_\Phi + \lambda H' \equiv H_0 + \lambda H', \tag{5}$$

where λ is a small coupling constant.[3] Through the standard Liouville-von Neumann equation, $\partial_t \rho_{\mathrm{tot}}(t) = -i[H, \rho_{\mathrm{tot}}(t)]$, it generates the evolution in time of the state of the total system, described in general by a density matrix ρ_{tot}, starting at $t = 0$ from the initial configuration of $\rho_{\mathrm{tot}}(0)$.

We shall assume the atom and the fields to be initially prepared in an uncorrelated state, with the fields in the temperature state ρ_β and the atoms in a generic initial state $\rho(0)$, so that $\rho_{\mathrm{tot}}(0) = \rho(0) \otimes \rho_\beta$. The reduced time evolution of the two atoms is then obtained by integrating over the unobserved field degrees of freedom and is formally given by the transformation map $\rho(0) \mapsto \rho(t) \equiv \mathrm{Tr}_\Phi[\rho_{\mathrm{tot}}(t)]$. This map is in general very complicated because of nonlinearities and memory effects; nevertheless, it can be approximated by a linear, memoryless map when the coupling with the environment is small and its own internal dynamics are sufficiently fast [1–5]. Indeed, in such cases the details of the result of the internal environmental dynamics are irrelevant, because the time scale of the subsystem evolution is typically very long compared with the decay time of the correlations in the bath.[4]

In order to derive the equation obeyed by the reduced density matrix $\rho(t)$ in the case at hand, it is convenient to work in the interaction representation:

$$\tilde{\rho}_{\mathrm{tot}}(t) = e^{itH_0}\, \rho_{\mathrm{tot}}(t)\, e^{-itH_0}, \tag{6}$$

so that

$$\frac{\partial \tilde{\rho}_{\mathrm{tot}}(t)}{\partial t} = -i\lambda[H'(t), \tilde{\rho}_{\mathrm{tot}}(t)], \quad H'(t) = e^{itH_0}\, H'\, e^{-itH_0}. \tag{7}$$

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[2] As we shall see in the following, the choice of modeling the environment in terms of relativistic quantum fields (at finite temperature) allows an analytic treatment of the reduced two-atom dynamics, without additional approximations besides the Born and Markov approximations (see below).

[3] For simplicity, we have assumed that the environment couples with the same strength to the two atoms, which makes the following analytic derivation of the master equation more transparent, without compromising its generality.

[4] A discussion on the validity of this so-called Markovian approximation is reported in [13]. There, a non-Markovian, weak coupling approximation of the reduced dynamics is also introduced; it leads to a two-parameter family of dynamical maps, with a time-dependent generator [3]. We stress that this approach is completely different from the one discussed below, which instead describes the reduced two-atom dynamics in terms of a Markovian, one-parameter semigroup. In particular, whereas in [13] the standard weak-coupling limit can be reached only in the asymptotic, long-time regime, in the treatment presented below it can always be obtained for any time by letting the coarse-graining parameter become large.

One then focuses on the changes of the reduced state $\tilde{\rho}(t) \equiv \mathrm{Tr}_\Phi[\tilde{\rho}_{\mathrm{tot}}(t)]$ over a time interval Δt. By taking the trace over the field variables of the integrated version of Eq. (7) one gets (to lowest order in λ)

$$\begin{aligned}\frac{\tilde{\rho}(t+\Delta t)-\tilde{\rho}(t)}{\Delta t} &= \frac{1}{\Delta t}\int_t^{t+\Delta t} ds\,\frac{\partial\tilde{\rho}(s)}{\partial s}\\ &= -\frac{\lambda^2}{\Delta t}\int_t^{t+\Delta t} dt_1\int_t^{t_1} dt_2 \mathrm{Tr}_\Phi\{[H'(t_1),[H'(t_2),\tilde{\rho}_{\mathrm{tot}}(t)]]\}\\ &\quad + O(\lambda^4). \end{aligned} \tag{8}$$

One notices that the variation of $\tilde{\rho}(t)$ starts to become relevant at order λ^2; i.e., on time scales of order $\tau = \lambda^2 t$. Then, one can equivalently write:

$$\frac{\tilde{\rho}(t+\Delta t)-\tilde{\rho}(t)}{\Delta t} = \frac{1}{\Delta t}\int_\tau^{\tau+\lambda^2\Delta t} ds\,\frac{\partial\tilde{\rho}(s/\lambda^2)}{\partial s}, \tag{9}$$

so that in the limit of small λ (and finite Δt) one can readily approximate the right-hand side of Eq. (9) with $\partial_t\tilde{\rho}(t)$. At this point, one further observes that the environment, containing an infinite number of degrees of freedom, is much larger than the subsystem immersed in it, so that its dynamics are hardly affected by its presence. We are therefore justified in replacing in the double integral of Eq. (8) the evolved total state $\tilde{\rho}_{\mathrm{tot}}(t)$ with the product state $\tilde{\rho}(t)\otimes\rho_\beta$, taking the initial state ρ_β as a reference state for the bath [1–3].

Returning to the Schrödinger representation, one finally gets the following linear, Markovian master equation for the two-atom state $\rho(t)$:

$$\frac{\partial\rho(t)}{\partial t} = -i[H_S,\rho] + \mathcal{D}[\rho(t)], \tag{10}$$

where the bath-dependent contribution $\mathcal{D}[\rho(t)]$ contains both a Hamiltonian and a dissipative term

$$\mathcal{D}[\rho(t)] = -i[H_{12},\rho(t)] + \mathcal{L}[\rho(t)], \tag{11}$$

with

$$\begin{aligned} H_{12} &= \frac{i\lambda^2}{2\Delta t}\int_0^{\Delta t} ds_1\int_0^{\Delta t} ds_2\theta(s_1-s_2)\\ &\quad\times \mathrm{Tr}_\Phi(\rho_\beta[H'(s_1),H'(s_2)]), \end{aligned} \tag{12}$$

$$\begin{aligned} \mathcal{L}[\rho(t)] &= \frac{\lambda^2}{\Delta t}\mathrm{Tr}_\Phi\big[L(\rho(t)\otimes\rho_\beta)L - \tfrac{1}{2}\{L^2,\rho(t)\otimes\rho_\beta\}\big],\\ L &= \int_0^{\Delta t} ds H'(s), \end{aligned} \tag{13}$$

where the curly brackets represent the anticommutator, and $\theta(s)$ is the step function.

It is important to observe that, for any interval Δt, the master equation (10)–(13) generates a quantum dynamical semigroup of completely positive maps. Indeed, the generator on the right-hand side of Eq. (11) (other than the Hamiltonian piece) contains a dissipative term that turns out to be completely positive because it is the composition of two completely positive maps, the trace over the environment's degrees of freedom, and a linear operator on the total system, written in canonical Stinespring form [6,33,34]. Notice that, on the contrary, in the usual weak-coupling-limit approach to the derivation of a Markovian master equation, complete positivity is ensured by an ergodic average prescription, that, as mentioned in Sec. I, eliminates fast oscillating terms [10,30]. In the present formalism, this corresponds to letting the time coarse-graining parameter Δt go to infinity, and therefore it is applicable only to environments with sharply decaying correlations. In the following, we shall instead keep Δt finite and consider therefore more general situations.

III. MASTER EQUATION

For the case at hand, a more explicit expression for the generator in (11) can be obtained by recalling (2) and (4). Indeed, after straightforward manipulations, the master equation driving the dissipative dynamics of the two-atom state takes the following Kossakowski-Lindblad form [35,36]:

$$\frac{\partial\rho(t)}{\partial t} = -i[H_{\mathrm{eff}},\rho(t)] + \mathcal{L}[\rho(t)], \tag{14}$$

with

$$H_{\mathrm{eff}} = H_S - \frac{i}{2}\sum_{\alpha,\beta=1}^{2}\sum_{i,j=1}^{3} H_{ij}^{(\alpha\beta)}\sigma_i^{(\alpha)}\sigma_j^{(\beta)}, \tag{15}$$

and

$$\mathcal{L}[\rho] = \sum_{\alpha,\beta=1}^{2}\sum_{i,j=1}^{3} C_{ij}^{(\alpha\beta)}\big(\sigma_j^{(\beta)}\rho\sigma_i^{(\alpha)} - \tfrac{1}{2}\{\sigma_i^{(\alpha)}\sigma_j^{(\beta)},\rho\}\big). \tag{16}$$

The coefficients of the Kossakowski matrix $C_{ij}^{(\alpha\beta)}$ and of the effective Hamiltonian H_{eff} are determined by the field correlation functions in the thermal state ρ_β:

$$G_{ij}^{(\alpha\beta)}(t-t') = \int d^3x d^3y f^{(\alpha)}(\vec{x}) f^{(\beta)}(\vec{y})\langle\Phi_i(t,\vec{x})\Phi_j(t',\vec{y})\rangle, \tag{17}$$

through their Fourier transform

$$\mathcal{G}_{ij}^{(\alpha\beta)}(z) = \int_{-\infty}^{\infty} dt e^{izt}\, G_{ij}^{(\alpha\beta)}(t), \tag{18}$$

and Hilbert transform

$$\mathcal{K}_{ij}^{(\alpha\beta)}(z) = \int_{-\infty}^{\infty} dt\,\mathrm{sgn}(t)e^{izt}G_{ij}^{(\alpha\beta)}(t) = \frac{\mathrm{P}}{\pi i}\int_{-\infty}^{\infty} dw\frac{\mathcal{G}_{ij}^{(\alpha\beta)}(w)}{w-z}, \tag{19}$$

respectively (P indicates the principle value).

More specifically, one finds that the Kossakowski matrix reads

$$\begin{aligned} C_{ij}^{(\alpha\beta)} &= \lambda^2\sum_{\xi,\xi'=\{+,-,0\}}\sum_{k,l=1}^{3} e^{i(\xi\omega_\alpha+\xi'\omega_\beta)\Delta t/2}\psi_{ki}^{(\xi)}\psi_{lj}^{(\xi')}\\ &\quad\times\frac{\Delta t}{2\pi}\int_{-\infty}^{\infty} d\omega\,\mathcal{G}_{kl}^{(\alpha\beta)}(\omega)\frac{\sin[(\xi\omega-\omega_\alpha)\Delta t/2]}{(\xi\omega-\omega_\alpha)\Delta t/2}\\ &\quad\times\frac{\sin[(\xi'\omega+\omega_\beta)\Delta t/2]}{(\xi'\omega+\omega_\beta)\Delta t/2}, \end{aligned} \tag{20}$$

where

$$\psi_{ij}^{(0)} = n_i n_j, \qquad \psi_{ij}^{(\pm)} = \tfrac{1}{2}(\delta_{ij} - n_i n_j \pm i\epsilon_{ijk}n_k) \tag{21}$$

are the components of auxiliary three-dimensional tensors, giving the free evolution of the atom operators:

$$\sigma_i^{(\alpha)}(t) = e^{itH_S}\sigma_i^{(\alpha)}e^{-itH_S} = \sum_{\xi=\{+,-,0\}}\sum_{j=1}^{3} e^{i\xi\omega_\alpha t}\psi_{ij}^{(\xi)}\sigma_j^{(\alpha)}.$$

The 6×6 matrix $C_{ij}^{(\alpha\beta)}$ turns out to be non-negative, since, as already mentioned, the evolution generated by (9) is completely positive.[5] An expression similar to the one in Eq. (20) also holds for $H_{ij}^{(\alpha\beta)}$ in Eq. (15), with $\mathcal{G}_{kl}^{(\alpha\beta)}(\omega)$ replaced by $\mathcal{K}_{kl}^{(\alpha\beta)}(\omega)$.

For simplicity, the fields giving rise to the environment are taken to be independent and are further assumed to obey a free evolution. In this case, one finds

$$\langle\Phi_i(x)\Phi_j(y)\rangle \equiv \mathrm{Tr}[\Phi_i(x)\Phi_j(y)\rho_\beta] = \delta_{ij}G(x-y), \tag{22}$$

where $G(x-y)$ is the standard four-dimensional Wightmann function for a single relativistic scalar field in a state at the inverse temperature β [32] that, with the usual $i\varepsilon$ prescription, can be written as

$$G(x) = \int \frac{d^4k}{(2\pi)^3}\theta(k^0)\delta(k^2)\{[1+\mathcal{N}(k^0)]e^{-ikx} + \mathcal{N}(k^0)e^{ikx}\}e^{-\varepsilon k^0}, \tag{23}$$

where

$$\mathcal{N}(k^0) = \frac{1}{e^{\beta k^0}-1}. \tag{24}$$

Although the $i\varepsilon$ prescription, assuring the convergence of the integral in (23), originates from causality requirements, in the present setting it can be related to the finite size of the two atoms. Indeed, the correlations in Eq. (17) actually involve the Fourier transform $\hat{f}(\vec{k}) = \int d^3x e^{i\vec{k}\cdot\vec{x}} f(\vec{x})$ of the shape function $f(\vec{x})$ in Eq. (3); it can be easily computed to be $\hat{f}(\vec{k}) = e^{-|\vec{k}|\varepsilon/2}$. Inserting it back in Eq. (17), this contribution can be conveniently attached to the definition of the Wightmann function $G(x)$, so that the integrand in Eq. (23) gets an extra $e^{-\varepsilon k^0}$ overall factor.

Using Eqs. (23) and (24), the Fourier transform in Eq. (18) can now be explicitly evaluated. By taking for simplicity the limit of point-like atoms (the size ε can be taken to vanish since it does not play any more the role of a regularization parameter), one gets

$$\mathcal{G}_{ij}^{(\alpha\beta)}(\omega) = \delta_{ij}\,\mathcal{G}^{(\alpha\beta)}(\omega), \tag{25}$$

with

$$\begin{aligned} \mathcal{G}^{(11)}(\omega) = \mathcal{G}^{(22)}(\omega) &= \frac{1}{2\pi}\frac{\omega}{1-e^{-\beta\omega}}, \\ \mathcal{G}^{(12)}(\omega) = \mathcal{G}^{(21)}(\omega) &= \frac{1}{2\pi}\frac{\omega}{1-e^{-\beta\omega}}\frac{\sin(\ell\omega)}{\ell\omega}, \end{aligned} \tag{26}$$

where ℓ denotes the modulus of the displacement vector $\vec{\ell}$. Then, recalling Eq. (19), for the Hilbert transform one similarly finds

$$\mathcal{K}_{ij}^{(\alpha\beta)}(z) = \delta_{ij}\mathcal{K}^{(\alpha\beta)}(z), \quad \mathcal{K}^{(\alpha\beta)}(z) = \frac{\mathrm{P}}{\pi i}\int_{-\infty}^{\infty} dw\,\frac{\mathcal{G}^{(\alpha\beta)}(w)}{w-z}. \tag{27}$$

With these results, and by taking into account that $\sum_k \psi_{ki}^{(\xi)}\psi_{kj}^{(\xi')} = \psi_{ij}^{(-\xi)}\delta(\xi+\xi')$, the Kossakowski matrix takes the more explicit form

$$C_{ij}^{(\alpha\beta)} = C_{+}^{(\alpha\beta)}\delta_{ij} - iC_{-}^{(\alpha\beta)}\sum_{k=0}^{3}\epsilon_{ijk}n_k + \big[C_0^{(\alpha\beta)} - C_{+}^{(\alpha\beta)}\big]n_i n_j, \tag{28}$$

where

$$\begin{aligned} C_{\pm}^{(\alpha\beta)} &= I_{\pm}^{(\alpha\beta)}\cos(\omega_{\alpha\beta}\Delta t/2) + i\,I_{\mp}^{(\alpha\beta)}\sin(\omega_{\alpha\beta}\Delta t/2), \\ \omega_{\alpha\beta} &\equiv \omega_\alpha - \omega_\beta, \end{aligned} \tag{29}$$

with

$$I_{\pm}^{(\alpha\beta)} = \frac{\Delta t}{4\pi}\int_{-\infty}^{\infty} d\omega\big[\mathcal{G}^{(\alpha\beta)}(\omega) \pm \mathcal{G}^{(\alpha\beta)}(-\omega)\big] \times \frac{\sin[(\omega-\omega_\alpha)\Delta t/2]}{(\omega-\omega_\alpha)\Delta t/2}\,\frac{\sin[(\omega-\omega_\beta)\Delta t/2]}{(\omega-\omega_\beta)\Delta t/2}, \tag{30}$$

while

$$C_0^{(\alpha\beta)} \equiv I_0^{(\alpha\beta)} = \frac{\Delta t}{4\pi}\int_{-\infty}^{\infty} d\omega\big[\mathcal{G}^{(\alpha\beta)}(\omega) + \mathcal{G}^{(\alpha\beta)}(-\omega)\big] \times \left[\frac{\sin(\omega\Delta t/2)}{\omega\Delta t/2}\right]^2. \tag{31}$$

Only the combinations $\mathcal{G}_{\pm}^{(\alpha\beta)}(\omega) \equiv \mathcal{G}^{(\alpha\beta)}(\omega) \pm \mathcal{G}^{(\alpha\beta)}(-\omega)$ actually occur in the previous integrals, and from the explicit expressions in Eq. (26) one obtains

$$\begin{aligned} \mathcal{G}_{+}^{(11)} = \mathcal{G}_{+}^{(22)} &= \frac{\omega}{2\pi}\left[\frac{1+e^{-\beta\omega}}{1-e^{-\beta\omega}}\right], \\ \mathcal{G}_{+}^{(12)} = \mathcal{G}_{+}^{(21)} &= \frac{\omega}{2\pi}\left[\frac{1+e^{-\beta\omega}}{1-e^{-\beta\omega}}\right]\frac{\sin(\omega\ell)}{\omega\ell}, \end{aligned} \tag{32}$$

$$\mathcal{G}_{-}^{(11)} = \mathcal{G}_{-}^{(22)} = \frac{\omega}{2\pi}, \quad \mathcal{G}_{-}^{(12)} = \mathcal{G}_{-}^{(21)} = \frac{\omega}{2\pi}\,\frac{\sin(\omega\ell)}{\omega\ell}; \tag{33}$$

they contain the dependence on the bath temperature $1/\beta$ and on the separation ℓ between the two atoms. Because of the presence of the Boltzmann factors, the integrals $I_{\pm,0}^{(\alpha\beta)}$ in Eqs. (30) and (31) cannot in general be expressed in terms of elementary functions. However, in the case of a bath at high temperature (i.e., for small β), the factor in square brackets in (32) behaves as $2/\beta\omega$ and the above integrals can be explicitly evaluated (see the Appendix). In the physical

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[5] On the other hand, let us remark that direct use of the standard second order perturbative approximation (e.g., see [8,9]) often leads to physically inconsistent results [5,11,12], giving a finite time evolution for $\rho(t)$ that in general does not preserve the positivity of probabilities.

situation for which $\ell \leqslant \Delta t$,[6] one finds

$$\begin{aligned}
I_+^{(\alpha\beta)} = {}& \frac{1}{\pi\beta\omega_{\alpha\beta}\Delta t}\left(\frac{\sin(\ell\omega_\alpha/2)}{\ell\omega_\alpha/2}\right. \\
& \times \sin\left\{[\omega_\alpha(1-\ell/\Delta t)-\omega_\beta]\frac{\Delta t}{2}\right\} \\
& \left. + \frac{\sin(\ell\omega_\beta/2)}{\ell\omega_\beta/2}\sin\left\{[\omega_\alpha-\omega_\beta(1-\ell/\Delta t)]\frac{\Delta t}{2}\right\}\right),
\end{aligned} \tag{34}$$

$$I_-^{(\alpha\beta)} = \frac{1}{\pi\ell\omega_{\alpha\beta}\Delta t}\sin\left[\frac{\omega_{\alpha\beta}(\Delta t-\ell)}{2}\right]\sin\left[\frac{\ell(\omega_\alpha+\omega_\beta)}{2}\right], \tag{35}$$

$$I_0^{(\alpha\beta)} = \frac{1}{4\pi\beta}\left(2-\frac{\ell}{\Delta t}\right). \tag{36}$$

Inserting these results back in Eqs. (29) and (31), one finally obtains the explicit expression for the Kossakowski matrix $C_{ij}^{(\alpha\beta)}$ in the large-temperature limit.[7]

Coming now to the Hamiltonian contribution to the master equation, one sees that the effective Hamiltonian H_{eff} in Eq. (15) can be split into two parts, $H_{\text{eff}} = \tilde{H}_S + H_{\text{eff}}^{(12)}$. The first term is just a renormalization of the starting system Hamiltonian, while the second term represents an environment-induced direct coupling term for the two atoms. The term $\tilde{H}_S$ has the same form as the Hamiltonian in Eq. (1) but with the frequencies redefined as

$$\begin{aligned}
\tilde{\omega}_\alpha = {}& \omega_\alpha - i\frac{\Delta t}{2\pi}\int_{-\infty}^{\infty} d\omega\big[\mathcal{K}^{(\alpha\alpha)}(\omega) - \mathcal{K}^{(\alpha\alpha)}(-\omega)\big] \\
& \times \left\{\frac{\sin[(\omega-\omega_\alpha)\Delta t/2]}{(\omega-\omega_\alpha)\Delta t/2}\right\}^2.
\end{aligned} \tag{37}$$

Recalling the definition of $\mathcal{K}^{(\alpha\alpha)}(\omega)$ in Eq. (27), one sees that it can be split into a vacuum- and a temperature-dependent piece:

$$\begin{aligned}
\mathcal{K}^{(\alpha\alpha)}(\omega) = {}& \frac{1}{2\pi^2 i}\left[\mathrm{P}\int_0^\infty dz\frac{z}{z-\omega}\right. \\
& \left. + \mathrm{P}\int_0^\infty dz\frac{z}{1-e^{\beta z}}\left(\frac{1}{z+\omega}-\frac{1}{z-\omega}\right)\right].
\end{aligned} \tag{38}$$

Although not expressible in terms of simple functions, the temperature-dependent second term is a finite, odd function of ω; on the contrary, the remaining vacuum contribution in Eq. (38) is divergent, and therefore so are the shifted frequencies $\tilde{\omega}_\alpha$. As a consequence, the definition of the effective Hamiltonian H_{eff} requires the introduction of a suitable cutoff and a renormalization procedure. This is not a surprise because the appearance of the divergences is due to the nonrelativistic treatment of the two-level atoms, while any sensible calculation of energy shifts would have required the use of quantum field theory techniques [7]. In order to make H_{eff} well defined we follow the simple prescription of performing a suitable temperature independent subtraction, so that the expressions in Eq. (37) reproduce the correct quantum field theory result obtained by considering the external fields in the vacuum state.

The induced two-atom interaction term $H_{\text{eff}}^{(12)}$ can instead be expressed as

$$H_{\text{eff}}^{(12)} = \sum_{i,j=1}^{3}\mathcal{H}_{ij}^{(12)}\sigma_i^{(1)}\otimes\sigma_j^{(2)}, \tag{39}$$

where

$$\begin{aligned}
\mathcal{H}_{ij}^{(12)} = {}& \left[\cos\left(\frac{\omega_{12}\Delta t}{2}\right)\delta_{ij} + \sin\left(\frac{\omega_{12}\Delta t}{2}\right)\sum_{k=0}^{3}\varepsilon_{ijk}n_k\right]J_+ \\
& + \left[J_0 - \cos\left(\frac{\omega_{12}\Delta t}{2}\right)J_+\right]n_i n_j,
\end{aligned} \tag{40}$$

with

$$\begin{aligned}
J_+ = {}& -i\frac{\Delta t}{4\pi}\int_{-\infty}^{\infty} d\omega\big[\mathcal{K}^{(12)}(\omega) + \mathcal{K}^{(12)}(-\omega)\big] \\
& \times \frac{\sin[(\omega-\omega_1)\Delta t/2]}{(\omega-\omega_1)\Delta t/2}\,\frac{\sin[(\omega-\omega_2)\Delta t/2]}{(\omega-\omega_2)\Delta t/2},
\end{aligned} \tag{41}$$

$$J_0 = -i\frac{\Delta t}{8\pi}\int_{-\infty}^{\infty} d\omega\big[\mathcal{K}^{(12)}(\omega) + \mathcal{K}^{(12)}(-\omega)\big]\left[\frac{\sin(\omega\Delta t/2)}{\omega\Delta t/2}\right]^2. \tag{42}$$

In addition, $\mathcal{K}^{(12)}(\omega)$ can be split as in Eq. (38) into a temperature-dependent term, odd in ω, and a vacuum term. Clearly, only this second contribution enters the above integrals $J_{+,0}$. It is finite (for nonvanishing atom separation) and, with the help of Eq. (32), can be explicitly computed to be

$$\begin{aligned}
\mathcal{K}^{(12)}(\omega) + \mathcal{K}^{(12)}(-\omega) &= \frac{\mathrm{P}}{2\pi^2 i}\int_{-\infty}^{\infty} dz\frac{z}{z+\omega}\frac{\sin\ell z}{\ell z} \\
&= \frac{1}{2\pi i}\frac{\cos\ell\omega}{\ell}.
\end{aligned} \tag{43}$$

Inserting this result into Eqs. (41) and (42), one finally obtains, again for $\ell \leqslant \Delta t$ (see Appendix):

$$\begin{aligned}
J_+ &= -\frac{1}{2\pi\ell\omega_{12}\Delta t}\cos\left[\frac{(\omega_1+\omega_2)\ell}{2}\right]\sin\left[\frac{\omega_{12}(\Delta t-\ell)}{2}\right], \\
J_0 &= \frac{1}{8\pi}\left(\frac{1}{\Delta t}-\frac{1}{\ell}\right).
\end{aligned} \tag{44}$$

We are now ready to discuss the entanglement properties of the time evolution generated by the master equation (14)–(16).

IV. ENVIRONMENT ENTANGLEMENT GENERATION

In order to study the entanglement power of the thermal bath made of free quantum fields, we shall focus on the small-t behavior of the dynamics generated by Eq. (14). Our aim is to investigate whether the two independent atoms can get entangled by the action of the environment in which they are immersed at the beginning of their dissipative evolution. Then,

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[6] This condition assures that the two atoms actually feel the presence of the quantum fields. Because of relativistic causality [32], the fields would not be able to interact with the atoms in the time interval Δt if they were too far apart.

[7] Since this is an approximate result, positivity of the matrix is not *a priori* guaranteed and should be formally imposed in order to preserve the properties of the exact expression (20). In particular, positivity of the two diagonal submatrices $C_{ij}^{(\alpha\alpha)}$ requires $\beta\omega_\alpha/2 \leqslant 1$, which is satisfied by the requirement of small β.

without loss of generality, one can limit the considerations to pure, separable initial states and therefore take

$$\rho(0) = |\varphi\rangle\langle\varphi| \otimes |\psi\rangle\langle\psi| \tag{45}$$

with $|\varphi\rangle$ and $|\psi\rangle$ given single-atom states. Indeed, if the environment is unable to create entanglement out of pure states, it will certainly not correlate their mixtures.

Since we are dealing with a couple of two-level systems, one can use partial transposition as a criterion for entanglement creation [37,38]. More precisely, the environment is able to create quantum correlations between the two atoms if and only if the operation of partial transposition spoils the positivity of the state $\rho(t)$.

The presence of negative eigenvalues in the partially transposed reduced density matrix $\hat{\rho}(t)$ can be ascertained by looking at the sign of the average

$$\mathcal{A}(t) = \langle\chi|\hat{\rho}(t)|\chi\rangle \tag{46}$$

with $|\chi\rangle$ a four-dimensional vector. Indeed, choose $|\chi\rangle$ to be orthogonal to $|\varphi\rangle \otimes |\psi\rangle$, so that the above average initially vanishes [$\mathcal{A}(0) = 0$]. Then the two atoms, initially prepared in a state $\rho(0) \equiv \hat{\rho}(0)$ as in Eq. (45), will surely become entangled if $|\chi\rangle$ can be further chosen so that $\partial_t\mathcal{A}(0) < 0$. From this condition, a simple test for entanglement creation involving the elements of the Kossakowski matrix (28) and of the effective interaction Hamiltonian (39) can then be extracted [24,39]. Explicitly, it reads

$$\langle u|C^{(11)}|u\rangle\langle v|\big(C^{(22)}\big)^T|v\rangle < \big|\langle u|\mathrm{Re}\big(C^{(12)} + i\,H_{\rm eff}^{(12)}\big)|v\rangle\big|^2, \tag{47}$$

where T means matrix transposition. The three-dimensional vectors $|u\rangle$ and $|v\rangle$ contain information about the starting factorized state (45). In fact, their components can be expressed as

$$u_i = \langle\varphi|\sigma_i|\varphi_\perp\rangle, \qquad v_i = \langle\psi_\perp|\sigma_i|\psi\rangle, \tag{48}$$

where $|\varphi_\perp\rangle$ and $|\psi_\perp\rangle$ are the orthonormal complement to the initial atom states $|\varphi\rangle$ and $|\psi\rangle$, respectively. Therefore, the external quantum fields will be able to initially entangle the two atoms evolving with the Markovian dynamics generated by Eq. (14) if there exists an initial state of the form (45) for which the inequality (47) is satisfied.

In order to obtain a manageable expression for it, we first note that, without loss of generality, the unit vector $\vec{n}$ that defines the atom Hamiltonian in Eq. (1) can be oriented along the third axis. Furthermore, as the initial atom state we shall choose $\rho(0) = |-\rangle\langle-| \otimes |+\rangle\langle+|$, which is constructed from the eigenstates of the single atom Hamiltonian $\sigma_3\,|\pm\rangle = \pm|\pm\rangle$. As a consequence, recalling Eq. (48), one finds that the three-dimensional vector $|u\rangle$ has components $u_i = \{1, -i, 0\}$, and furthermore that $v_i = u_i$. Then, using the explicit expressions for the elements of the Kossakowski matrix $C_{ij}^{(\alpha\beta)}$ and of the induced interaction Hamiltonian $H_{\rm eff}^{(12)}$, the inequality (47) reduces to

$$\left(1 - \frac{\beta\omega_1}{2}\right)\left(1 + \frac{\beta\omega_2}{2}\right) < \pi^2\beta^2\big[\big(I_+^{(12)}\big)^2 + 4(J_+)^2\big]. \tag{49}$$

Notice that the left-hand side of this expression is positive since, as discussed in the previous section, complete positivity requires $\beta\omega_\alpha/2 \leqslant 1$.

As remarked at the end of Sec. II, the parameter Δt identifies the time scale over which the presence of the environment is felt by the two-atom system. Clearly, the weaker the coupling is with the environment, the longer one needs to wait for the bath-induced effects to become apparent.

Let us first discuss the standard weak-coupling-limit approximation. In this case, one actually lets the coupling constant λ approach zero, so that changes in the two-atom density matrix become visible only for infinitely large Δt. In this limit, however, the two integrals on the right-hand side of Eq. (49) become vanishingly small.[8] Thus, for atoms with unequal frequencies, the inequality can never be satisfied, and thus no entanglement is generated. Conversely, when the two frequencies coincide, $\omega_1 = \omega_2 = \omega$, the condition (49) becomes

$$1 - \left(\frac{\beta\omega}{2}\right)^2 < \left[\frac{\sin(\omega\ell)}{\omega\ell}\right]^2 + \frac{\beta^2}{4}\left[\frac{\cos(\omega\ell)}{\ell}\right]^2. \tag{50}$$

This result generalizes the one discussed in [26], where the contribution of the environment-induced interaction Hamiltonian [the second term in the right-hand side of Eq. (49)] was neglected. In particular, one sees that, in this case, for any given (small) inverse temperature β, there is always an atom separation ℓ below which the inequality (50) is satisfied, and therefore entanglement is created between the two atoms. This phenomenon is forbidden only for infinitely large separation or infinitely large temperature, in which case the environment-induced decoherence and noisy effects dominate.

However, the sharp dependence of the entanglement capability of the environment on the atom frequencies in the weak-coupling-limit approach is striking because it originates in the elimination of fast oscillating terms in the reduced two-atom dynamics through an ergodic average—a procedure that is justified only in the limit of a vanishing λ and very fast decay correlations in the environment.

Instead, if the coupling of the atoms to the bath is weak but not infinitesimally small, environment-induced changes in the atom density matrix $\rho(t)$ can be seen for finite time intervals Δt. In this case, it is the full condition (49) that regulates the entanglement capability of the thermal bath. One can check that indeed this inequality can be satisfied even for $\omega_1 \neq \omega_2$, and therefore that a bath made of thermal quantum fields can correlate two unequal atoms.

In order to show this, we first note that the Hamiltonian contribution in Eq. (49), being positive, can only enhance entanglement generation, which is the result of the hermiticity of the induced coupling term $H_{\rm eff}^{(12)}$ in Eq. (39).[9] One can

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[8] Indeed, in the limit $\Delta t \to \infty$ both integrals vanish since, for $\omega_1 \neq \omega_2$, the two functions $\sin[(\omega - \omega_1)\Delta t/2]/[(\omega - \omega_1)\Delta t/2]$ and $\sin[(\omega - \omega_2)\Delta t/2]/[(\omega - \omega_2)\Delta t/2]$ have disjoint supports.

[9] In general, the dissipative and Hamiltonian contributions in the right-hand side of Eq. (47) can destructively interfere, making the inequality more difficult to satisfy and thus reducing the entanglement power of the environment.

therefore limit the considerations to a simpler inequality, in which the term $(J_+)^2$ is neglected. When this reduced condition is satisfied the full condition in Eq. (49) will also clearly be satisfied. Recalling Eq. (34), and keeping only first order terms in ℓ for simplicity, the condition for environment-assisted entanglement generation reduces to

$$\left(1-\frac{\beta\omega_1}{2}\right)\left(1+\frac{\beta\omega_2}{2}\right) < \left[\frac{\sin(\omega_{12}\Delta t/2)}{\omega_{12}\Delta t/2}\right]^2 - \left(\frac{\ell}{\Delta t}\right)\frac{\sin(\omega_{12}\Delta t)}{\omega_{12}\Delta t}. \quad (51)$$

For a given bath temperature and atom frequencies, this condition is satisfied provided a sufficiently small time interval Δt results from the coupling with the environment. Furthermore, the smaller the atom separation is, the easier it will be to satisfy the condition of Eq. (51).

Finally, note that, in contrast to situation encountered in the weak-coupling limit-approximation, here there is no sharp change between the regime of entanglement generation and the region of solely decoherence. The transition is smoothly regulated by the coarse-graining parameter Δt, i.e., ultimately by the strength of the coupling of the atoms to the environment.

V. DISCUSSION

We have seen that two atoms, prepared initially in a separable state, can get entangled as a result of their independent interaction with a common bath made of thermal quantum fields, even when their internal frequencies are not equal. This result is based on a refined Markovian approximation of the reduced atom dynamics that allows an explicit dependence on the time scale Δt, measuring the interval over which the atoms feel the presence of the environment.

This conclusion contrasts with the conclusion obtained through the usual weak-coupling-limit approach to the atom reduced dynamics. In that case, the entanglement power of the external environment is reduced to zero for atoms with unequal frequencies as a consequence of the procedure of taking the ergodic average. In light of the results presented in the previous section, this conclusion appears to be a mathematical artifact that originates from letting λ go to zero and Δt go to infinity[10]—conditions that are hardly met in actual physical situations. Instead, for weakly coupled baths with finite Δt, environment-assisted entanglement generation is always allowed, and can be controlled through the external parameters, the bath inverse temperature β, and the atom spatial separation ℓ.

In the high temperature case (i.e., small β) and for arbitrary Δt, we have shown explicitly that this conclusion holds because of the condition (51). Similarly, in situations allowing a large but finite Δt, a different approximation of the full entanglement condition (47) can be given, which can be obtained using techniques and procedures analogous to the ones discussed in the previous sections. Neglecting again the Hamiltonian contribution, one finds

$$(1-R_1)(1+R_2) < \frac{1}{4}\left[\frac{\sin(\omega_{12}\Delta t/2)}{\omega_{12}\Delta t/2}\right]^2 \times \left[\left(\frac{\omega_1}{\omega_2}\frac{R_2}{R_1}\right)^{1/2} S_1 + \left(\frac{\omega_2}{\omega_1}\frac{R_1}{R_2}\right)^{1/2} S_2\right]^2, \quad (52)$$

where

$$R_\alpha = \frac{1-e^{-\beta\omega_\alpha}}{1+e^{-\beta\omega_\alpha}}, \quad S_\alpha = \frac{\sin(\omega_\alpha \ell)}{\omega_\alpha \ell}, \quad \alpha = 1, 2. \quad (53)$$

It is a further generalization of the condition discussed in [26] in the case of identical atoms, to which it reduces for $\omega_1 = \omega_2$ and Δt infinite. Although valid only for large (but finite) Δt, it can always be satisfied with suitably chosen β and ℓ. In particular, Eq. (52) is always true in the zero-temperature case (i.e., in the limit $\beta \to \infty$). In other words, a bath made of quantum fields in the vacuum state is always able to generate entanglement, for any finite spatial separation of the two atoms.

All the above considerations are based on the condition (47) for entanglement enhancement, which when satisfied assures that quantum correlations among the two atoms are generated as soon as $t > 0$. However, it is unable to determine the fate of these quantum correlations as time increases and, in particular, in the asymptotically long time regime. On general grounds, one expects that the effects of decoherence and dissipation that counteract entanglement production be dominant at large times, so that no entanglement is left in the end. There are however instances in which the entanglement generated at the beginning of the evolution persists also for asymptotically long times [5,27,28]. In order to fully clarify this situation, a complete study and classification of the set of the equilibrium states of the refined master equation (10)–(13) is necessary.[11] Work on this topic is presently in progress and will be reported elsewhere.

APPENDIX

We indicate here how to compute the integrals that appear in the expressions of the Kossakowski matrix $C_{ij}^{(\alpha\beta)}$ [Eqs. (30) and (31)] and in the effective Hamiltonian interaction term $H_{\rm eff}^{(12)}$ [Eqs. (41) and (42)]. In the high-temperature case, the explicit evaluation of Eq. (30) involves the computation of integrals of the following two types:

$$I_1 = \int_{-\infty}^{+\infty} dx\, \sin(cx)\frac{\sin(x-a)}{x-a}\frac{\sin(x-b)}{x-b}, \quad (A1)$$

$$I_2 = \int_{-\infty}^{+\infty} dx\, \frac{\sin(cx)}{x}\frac{\sin(x-a)}{x-a}\frac{\sin(x-b)}{x-b}, \quad (A2)$$

with a, b, c positive constants. By decomposing the products of the trigonometric functions in terms of linear combinations of sines and cosines, one can split, for example, I_1 into the

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[10] Because of the Riemann-Lebesgue lemma, the ergodic average, on which the so-called "rotating wave approximation" is based, is strictly justified only in the limit $\lambda \to 0$ and $\Delta t \to \infty$.

[11] To date, only partial results of the classification the equilibrium states of completely positive quantum dynamical semigroups have been obtained [2,40].

sum of three simpler integrals:

$$I_1 = I_0 + I(c) - I(-c), \tag{A3}$$

with

$$\begin{aligned} I_0 &= \frac{\cos(a-b)}{2}\int_{-\infty}^{+\infty} dx\, \frac{\sin(cx)}{(x-a)(x-b)},\\ I(c) &= \frac{1}{4}\int_{-\infty}^{+\infty} dx\, \frac{\sin[(2-c)x-a-b]}{(x-a)(x-b)}. \end{aligned} \tag{A4}$$

By first changing the integration variable to $y = (2-c)x - a - b$ in $I(c)$, with $c \leqslant 2$, and then reducing the denominators in the partial fractions in both integrands, one can express I_0 and $I(c)$ as combinations of the following integral (see , e.g., [41]):

$$\int_{-\infty}^{+\infty} dx\, \frac{\sin(\alpha x)}{x+z} = \pi\cos(\alpha z), \quad \alpha > 0. \tag{A5}$$

Explicitly, one finds

$$I_0 = -\pi\frac{\cos(a-b)}{a-b}\sin\left[\frac{(a+b)c}{2}\right]\sin\left[\frac{(a-b)c}{2}\right], \tag{A6}$$

$$I(c) = \frac{\pi}{4(a-b)}\{\cos[a(c-1)+b] - \cos[a+b(c-1)]\}, \tag{A7}$$

so that, recalling (A3), one finally obtains

$$I_1 = \pi\sin\left[\frac{(a+b)c}{2}\right]\frac{\sin[(a-b)(1-c/2)]}{(a-b)}. \tag{A8}$$

This result holds for $c \leqslant 2$. When $c > 2$, one is forced to use a different integration variable in the expression of $I(c)$ in Eq. (A4), $y' = (c-2)x + a + b$ and as a result end up with a vanishing value for I_1. However, as a function of the parameter c, the integral I_1 is continuous, since the expression in Eq. (A8) also vanishes at the boundary point $c = 2$.

From the result (A8), one further obtains

$$\begin{aligned} \lim_{c\to 0}\left(\frac{I_1}{c}\right) &= \int_{-\infty}^{+\infty} dx\, x\, \frac{\sin(x-a)}{x-a}\,\frac{\sin(x-b)}{x-b}\\ &= \pi\left(\frac{a+b}{2}\right)\frac{\sin(a-b)}{a-b}. \end{aligned} \tag{A9}$$

The integral I_2 in Eq. (A2) can be evaluated using similar manipulations. When $c \leqslant 2$, one explicitly finds

$$\begin{aligned} I_2 = \frac{\pi}{a-b}\Bigg\{&\frac{\sin(ac/2)}{a}\sin[a(1-c/2)-b]\\ &+ \frac{\sin(bc/2)}{b}\sin[a-b(1-c/2)]\Bigg\}, \end{aligned} \tag{A10}$$

whereas for $c > 2$, a simpler expression holds:

$$I_2 = \pi\frac{\sin a}{a}\frac{\sin b}{b}. \tag{A11}$$

Here again we see that I_2 is a continuous function of c, since the expression in Eq. (A10) reduces to the expression in Eq. (A11) at the boundary value $c = 2$. Further, from the expression in Eq. (A10), one easily obtains the following limiting results:

$$\begin{aligned} \lim_{c\to 0}\left(\frac{I_2}{c}\right) &= \int_{-\infty}^{+\infty} dx\, \frac{\sin(x-a)}{x-a}\,\frac{\sin(x-b)}{x-b}\\ &= \pi\frac{\sin(a-b)}{a-b}, \end{aligned} \tag{A12}$$

and, similarly,

$$\lim_{a,b\to 0} I_2 = \int_{-\infty}^{+\infty} dx\, \frac{\sin(cx)}{x}\left(\frac{\sin x}{x}\right)^2 = \pi c\left(1 - \frac{c}{4}\right). \tag{A13}$$

The integrals appearing in the evaluation of the Hamiltonian contribution $H_{\mathrm{eff}}^{(12)}$ can instead all be reduced to expressions of the form

$$J = \int_{-\infty}^{+\infty} dx\, \cos(cx)\frac{\sin(x-a)}{x-a}\,\frac{\sin(x-b)}{x-b}. \tag{A14}$$

With the help of manipulations similar to those used above, J can be reduced to combinations of the following integral [41]:

$$\int_{-\infty}^{+\infty} dx\, \frac{\cos(\alpha x)}{x+z} = \pi\sin(\alpha z) \quad \alpha > 0. \tag{A15}$$

When $c \leqslant 2$, the integral in Eq. (A14) can be cast in the following form:

$$J = \pi\cos\left[\frac{(a+b)c}{2}\right]\frac{\sin\left[(a-b)(1-c/2)\right]}{(a-b)}, \tag{A16}$$

whereas it vanishes for $c > 2$. In the limit of vanishing a and b, it reduces to $J = \pi(1 - c/2)$.

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