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Perfect imaging with positive refraction in three dimensions

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Maxwell's fish eye has been known to be a perfect lens within the validity range of ray optics since 1854. Solving Maxwell's equations, we show that the fish-eye lens in three dimensions has unlimited resolution for electromagnetic waves.

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Perfect imaging with positive-index materials has been discussed since 1854 [1], but only for light rays and not for waves [2,3]. In fact, the defining property of such ideal optical instruments [3] is the perfect focusing of rays: for each point in a region of space, all emitted rays meet in a corresponding point in the image space. Yet it is the wave nature of light that sets the resolution of optical instruments. Only very recently [4], one of us proved that the archetype of all ideal optical instruments [2], Maxwell's fish eye [1], perfectly images light waves in two dimensions. Here we show that the fish eye in three dimensions has infinite resolution as well, provided its medium is impedance-matched. Absorption does not appear to significantly reduce the image quality, in contrast to imaging with negative refraction [5,6]. However, fish-eye lenses contain both the source and the image inside the medium, and impedance-matched devices are still difficult to make in practice. On the other hand, our case supports the idea [4] that perfect imaging is not necessarily caused by the amplification of evanescent waves [5] but rather by the geometry of light [7–9]. Such conceptual insights may be vital for the future direction of technical developments for perfect imaging.

According to Fermat's principle [3,7], light rays follow extremal optical paths with a path length measured by the refractive index n, i.e., geodesics. Optical materials thus establish virtual spatial geometries for rays [7,10]. Whether these geometries are also valid for waves depends on the type of wave propagation. In two-dimensional (2D) structures, typical of integrated optics, the polarization of light decides whether the structures are perceived as geometries: for materials with purely electrical response TE polarization [11] is required [4]. In three dimensions, the material must be impedance-matched for establishing a virtual geometry for electromagnetic fields [7], with equal electric permittivity ε and magnetic permeability μ,

$$\varepsilon = \mu = n. \tag{1}$$

This crucial condition has not been considered in the previous treatment of Maxwell's fish eye with Maxwell's equations [12,13]. Scalar waves [14] should not obey the Helmholtz or Schrödinger equation either, for perfect imaging in three dimensions, but rather a wave equation we consider here as well, Eq. (19). In the previous wave theories of Maxwell's fish eye [12–15], perfect imaging was impossible. If, however, the medium or polarization is chosen such that the geometry of light is not restricted to rays but extends to waves, waves may be as perfectly imaged as rays [4].

It has been found [2] that Maxwell's fish eye establishes a non-Euclidean geometry [16], the three-dimensional (3D) surface of the four-dimensional (4D) hypersphere, by the refractive-index profile

$$n = \frac{2}{1+r^2}, \tag{2}$$

where r denotes the distance from the center measured in the characteristic length scale of the device. As the index profile (2) is radially symmetric, the trajectory of a light ray lies in a plane, due to the conservation of angular momentum. So, for light rays, the propagation in three dimensions is the same as in two dimensions where Maxwell's fish eye corresponds to the surface of a 3D sphere. Here all the rays emitted from one point travel along the great circles, meeting again at the antipodal point, which proves that Maxwell's fish eye fits the definition of an ideal optical instrument [3]. In physical space, the antipodal image of a point $\mathbf{r}_0$ was found [3] to appear at

$$\mathbf{r}_0' = -\frac{\mathbf{r}_0}{r_0^2}. \tag{3}$$

In this paper, we show that electromagnetic waves are perfectly imaged at $\mathbf{r}_0'$, apart from a phase delay of

$$\varphi = \pi k, \tag{4}$$

where k denotes the wave number in units of the inverse length scale of the fish eye. As the phase (4) is uniform, objects are not only faithfully but also coherently imaged. Furthermore, the phase (4) is linear in wave number and so in frequency if the refractive index is not frequency dependent. In this case, the time delay between source and image is uniform as well: a flash of light emitted at the source arrives as a flash at the image after a time delay of π in our units. Note that the phase delay is different in two dimensions [4]: $\pi\nu$ with $\nu^2 = k(k+1)$, which seems to be related to the fact that Huygens' principle is not valid in two dimensions; there, elementary light waves do not propagate as sudden flashes (Green functions in the time domain are not delta functions). In the following, we prove perfect imaging in three dimensions and obtain the quantitative results (3) and (4) by analyzing the electromagnetic Green function.

Electric and magnetic fields. The Green function G describes the electric field of a stationary wave with wave number k emitted by an elementary dipole at position $\mathbf{r}_0$ that may point in all three directions. The Green function is a matrix also known, in electrical engineering, as a dyade or a bitensor with the first index referring to the electric-field strength at the spectator point $\mathbf{r}$ and the second describing the direction of the dipole source at $\mathbf{r}_0$. If not otherwise stated, we use Cartesian coordinates. As any source is a weighted and directed distribution of elementary point dipoles, it is sufficient to consider the Green function $G(\mathbf{r}, \mathbf{r}_0)$ for perfect imaging. The Green function obeys the wave equation

$$\nabla \times \frac{1}{n} \nabla \times G - n\, k^2 G = \iota \mathbb{1}, \tag{5}$$

with the infinitely localized current density ι. In a stationary regime, the wave emitted at the source must disappear somewhere. Usually the wave ultimately disappears at ∞; but in perfect imaging, the entire energy of the emitted wave is focused at the image and does not reach ∞ at all. In practice, the light will be absorbed at the image, for example, in a lithographic photoresist or a detector. In any case, to maintain a stationary regime we must supplement the source by a drain at image (3) with phase delay (4) such that [4]

$$\iota = \boldsymbol{\delta}(\mathbf{r} - \mathbf{r}_0) - e^{i\varphi} \boldsymbol{\delta}(\mathbf{r} - \mathbf{r}_0'). \tag{6}$$

The minus indicates that the drain is a source run in reverse. Finally, causality implies [4] that G is analytic in k on the upper-half complex plane and vanishing for $k \to \infty$ there. If G meets all these requirements, our case is settled: Maxwell's fish eye makes a perfect lens in three dimensions.

To deduce the Green function, it is advantageous to represent G in terms of its magnetic field, the tensor H, as

$$G = \frac{\nabla \times H - \iota \mathbb{1}}{n\, k^2}, \tag{7}$$

where we require H to obey the wave equation

$$\nabla \times \frac{1}{n} \nabla \times H - n\, k^2 H = \nabla \times \frac{\iota \mathbb{1}}{n}. \tag{8}$$

From Eqs. (7) and (8) follows Faraday's law of induction

$$\nabla \times G = n\, H \tag{9}$$

and subsequently the defining wave Eq. (5) of the Green function. Faraday's law also reveals that H describes the magnetic-field strength divided by $ic\mu_0 k$, where c denotes the speed of light and μ_0 the permeability of vacuum. Consider a special case of wave propagation first.

Source at origin. Imagine the source is placed at the center of the fish eye [and image (3) would be at ∞],

$$\mathbf{r}_0 = \mathbf{0}, \quad \iota = \boldsymbol{\delta}(\mathbf{r}). \tag{10}$$

Similar to the electromagnetic wave emitted by a point dipole in free space [17], we represent H by the ansatz

$$H = \nabla \times 2D(r)\, \mathbb{1}. \tag{11}$$

In free space, $2D$ describes the scalar Green function [17]. In Maxwell's fish eye, D turns out to be the scalar Green function, as we will show, where the factor of 2 represents the refractive index (2) at the source. We apply the expression

$$\nabla f(r) = \frac{\mathbf{r}}{r}\, \partial_r f \tag{12}$$

for the gradient of any radial function, where ∂_r abbreviates the derivative with respect to the radius, and obtain

$$H = \mathbf{r} \times \frac{2\, \partial_r D}{r}\, \mathbb{1}\,. \tag{13}$$

We calculate the curl of H by expanding the double vector product in $\nabla \times (\nabla \times 2D\mathbb{1})$, using the radial gradient (12) and the standard expression of the Euclidean Laplacian in spherical coordinates,

$$\begin{aligned} \nabla \times H &= \nabla \otimes \frac{\mathbf{r}}{r}\, \partial_r 2D - \mathbb{1} \left(\partial_r^2 + \frac{2}{r} \partial_r \right) 2D \\ &= \frac{\mathbf{r} \otimes \mathbf{r}}{r^2} \left(\partial_r^2 - \frac{1}{r} \partial_r \right) 2D - \mathbb{1} \left(\partial_r^2 + \frac{1}{r} \partial_r \right) 2D\,. \end{aligned} \tag{14}$$

Then we turn to $\nabla \times n^{-1} \nabla \times H$ that occurs in the wave Eq. (8) of the magnetic field. For the first term in expression (14), the only nonzero contribution to the curl originates from $\nabla \times \mathbf{a} \otimes \mathbf{r} = -\mathbf{a} \times \nabla \otimes \mathbf{r} = -\mathbf{a} \times \mathbb{1}$. For the curl of the second term divided by n, we apply the radial gradient (12) as in formula (13). Combining these terms, we obtain

$$\begin{aligned} \nabla \times \frac{1}{n} \nabla \times H &= \frac{2\mathbf{r}}{r^2} \times \mathbb{1} \left(\frac{2}{n\, r} - \partial_r \frac{1}{n} \partial_r\, r \right) \partial_r D \\ &= \frac{2\, n\, \mathbf{r}}{r} \times \mathbb{1} \partial_r \left(\frac{1}{r^2 n^3}\, \partial_r\, r^2\, n\, \partial_r D - D \right), \end{aligned} \tag{15}$$

where in the last step we used the explicit formula (2) of the fish-eye profile. All expressions in the wave equation (8), including the curl of the delta functions on the right-hand side, have the same matrix structure as the magnetic field (13). We only need to require

$$\frac{1}{r^2 n^3}\, \partial_r\, r^2\, n\, \partial_r\, D + (k^2 - 1)\, D = -\frac{\delta(\mathbf{r})}{4\, n} \tag{16}$$

for finding the Green function G in the special case (10). But before we write down the solution of the radially symmetric scalar wave equation (16), we cast it in a geometric form and transform it, for investigating perfect imaging of scalar waves in three dimensions.

Scalar waves. The virtual geometry of the "fish-eye world" is characterized by the line element

$$ds = n(r)\, dl\,. \tag{17}$$

In this geometry, the Laplacian [7] appears in spherical coordinates r, θ, and ϕ as

$$\begin{aligned} \sum_a \nabla_a \nabla^a &= \frac{1}{r^2 n^3}\, \partial_r\, r^2\, n\, \partial_r + \frac{1}{r^2}\, \partial_\theta^2 + \frac{1}{r^2 \sin^2 \theta}\, \partial_\phi^2 \\ &= \frac{1}{n^3} \nabla \cdot n \nabla \end{aligned} \tag{18}$$

in Cartesian coordinates. Consequently, we can write the radial wave equation (16) as

$$\left(\frac{1}{n^3} \nabla \cdot n \nabla + k^2 - 1 \right) D = -\frac{\iota}{n^3}\,. \tag{19}$$

This is the wave equation of the scalar Green function (for a conformally coupled scalar field on the hypersphere). The equation is invariant under coordinate transformations that preserve the geometry, the line element (17), because it is entirely composed of geometrical constructions: the left-hand side contains the Laplacian and $k^2 - 1$, a scalar, while the right-hand side consists of delta functions divided by the density n^3, the measure of length (17) cubed. Such coordinate transformations are the Möbius transformations

$$\mathbf{r}' = \frac{\mathbf{r}\big(1 + r_0^2\big) - \mathbf{r}_0(1 + 2\,\mathbf{r}\cdot\mathbf{r}_0 - r^2)}{1 + 2\,\mathbf{r}\cdot\mathbf{r}_0 + r^2 r_0^2}\,, \tag{20}$$

because one verifies that

$$ds' = ds. \tag{21}$$

In two dimensions, the Möbius transformations correspond to the rotations on the surface of the 3D sphere in stereographic projection on the plane [18]. In three dimensions, the Möbius transformations (20) describe the rotations on the 3D surface of the 4D hypersphere, the virtual geometry of Maxwell's fish eye [2]. As the relationship (3) between source and image is also invariant under Möbius transformations, all scalar Green functions are simply Möbius-transformed solutions of the radial wave equation (16) with r replaced by the radius

$$r' = |\mathbf{r}'| = \frac{|\mathbf{r} - \mathbf{r}_0|}{\sqrt{1 + 2\,\mathbf{r}\cdot\mathbf{r}_0 + r^2 r_0^2}}\,. \tag{22}$$

A suitable solution is

$$D = \frac{1}{8\pi}\left(r' + \frac{1}{r'}\right)\exp(2ik\arctan r') = \frac{1}{8\pi}\left(r' + \frac{1}{r'}\right)\left(\frac{1 + \mathrm{i}\,r'}{1 - \mathrm{i}\,r'}\right)^k, \tag{23}$$

a curious variation of the free-space scalar Green function [17]. The Green function (23) has two singularities, one at $r' = 0$ and the other at $r' = \infty$; one singularity is at the source point $\mathbf{r}_0$ where expression (22) vanishes, and the other singularity is at the image (3) where the transformed radius (22) diverges. The phase $2k\arctan r'$ is finite, because the fish-eye world is closed, that is, the surface of the hypersphere is finite. At the image, we obtain the phase delay (4). The prefactor of the scalar Green function (23) was chosen to give delta functions on the right-hand side of the wave equation (19). The Green function (23) is analytic in k and decays exponentially on the upper-half complex k plane: D is causal. It is instructive to represent D in the time domain as

$$D(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} D\,\exp(-ikt)\,dk = \frac{\delta(2\arctan r' - t)}{8\pi(r' + 1/r')} = \frac{\delta(\zeta + \cos t)}{4\pi},\quad \zeta = \frac{r'^2 - 1}{r'^2 + 1}\,. \tag{24}$$

A light flash emitted at $\mathbf{r}_0$ and time $t = 0$ appears as a flash at the image (3) at time π in our units. All our requirements are met: scalar waves with the wave equation (19) and the fish-eye profile (2) support perfect imaging.

Green tensor. Having established the scalar Green function D, we follow a similar procedure to deduce the electromagnetic Green tensor G. First, we write down the Green function for the source at the origin in Möbius-primed coordinates. One verifies that formula (7) with expression (14) appears as

$$G = \frac{\nabla\times n(r')\nabla\otimes\nabla_0\,D(r')\times\overleftarrow{\nabla_0}}{n(r)\,n(r_0)\,k^2} - \frac{\iota\mathbb{1}}{n\,k^2} \tag{25}$$

evaluated at $r_0 = 0$. Here, ∇_0 denotes the gradient operator with respect to the source point $\mathbf{r}_0$, and the arrow indicates that all terms on the left of it are to be differentiated. Second, all we need to do to establish the Green function for an arbitrary source point $\mathbf{r}_0$ is to Möbius-boost the special case (25). We know that r' is Möbius invariant, but we also need to transform the bitensor components of G. For this, we write our result (25) in geometric terms for the fish-eye world with line element (17), which is most easily done in index notation. We use the permutation symbol $[abc]$ for the Cartesian curls [7] and express them by two Levi-Civita tensors ϵ^{abc} [7] in the fish-eye geometry, for spectator and source points separately. Finally, we lower the first index of each Levi-Civita tensor by the corresponding metric tensors $n(r)^2\mathbb{1}$ and $n(r_0)^2\mathbb{1}$, respectively. In formulas,

$$G_{ab} = \sum_{cdef}\frac{[acd]\,[bef]}{n(r)\,n(r_0)\,k^2}\,\frac{\partial^2 n(r')}{\partial x^c\,\partial x_0^e}\,\frac{\partial^2 D(r')}{\partial x^d\,\partial x_0^f} - \frac{\iota\delta_{ab}}{n\,k^2} = \sum_{cdef}\frac{\epsilon_a^{\ cd}\,\epsilon_b^{\ ef}}{k^2}\,\frac{\partial^2 n(r')}{\partial x^c\,\partial x_0^e}\,\frac{\partial^2 D(r')}{\partial x^d\,\partial x_0^f} - \frac{\iota\delta_{ab}}{n\,k^2}\,. \tag{26}$$

Formula (26) describes a perfect bitensor in the fish-eye geometry that remains invariant after Möbius transformation. Consequently, we can simply drop the qualification $r_0 = 0$: our result (25) is valid for arbitrary source points. Note that the delta currents (6) in expression (25) conveniently cancel the delta functions arising in the derivatives of D; the Green function G is singular at source $\mathbf{r}_0$ and image $\mathbf{r}_0'$ but does not develop delta peaks. As the scalar Green function D is analytic and exponentially decaying on the upper-half complex k plane, so is the Green tensor G. Source, image (3), and phase delay (4) are the same as in the scalar case as well. In short, Maxwell's fish eye in three dimensions perfectly images electromagnetic waves.

Mirror. In practice, the fish-eye profile (2) poses a formidable challenge: it is infinitely extended across space and the refractive index $n < 1$ for $r > 1$; the speed of light exceeds c here and approaches infinity. Both problems are related: Maxwell's fish eye represents a finite virtual space, the 3D surface of the 4D hypersphere, stretched out to infinite physical space. Light can only reach infinity in a finite time with infinite speed. But, one can solve both problems in one stroke by placing a mirror at the unit sphere ($r = 1$) [4]. In this case, the device occupies a finite space, the interior of the unit sphere, and the refractive index ranges from 1 at the mirror to 2 in the center. The same trick has been applied in non-Euclidean cloaking [16]. The mirror creates the illusion that light propagates beyond the unit sphere, whereas in reality it is reflected at the mirror. After another reflection, the light focuses at the mirror image of the original focusing point (3). To show this, we try employing the inversion in the unit sphere as a mirror transformation of the spectator points,

$$\mathbf{r} \to \frac{\mathbf{r}}{r^2}, \tag{27}$$

or, in spherical coordinates, $r \to r^{-1}$. The mirror image of image (3) would appear at

$$\mathbf{r}_0'' = -\mathbf{r}_0. \tag{28}$$

The inversion (27) also preserves the line element (17) of Maxwell's fish eye, and hence the transformed scalar Green function remains a solution of the wave Eq. (19). We only need to transform the spectator indices of the Green tensor by the matrix P (but not the source indices). As reflection gives rise to a phase shift of π [17], we subtract the reflected field from the original one,

$$G' = G(r) - P\,G(r^{-1})\,. \tag{29}$$

In Cartesian coordinates,

$$P = r^2\,\mathbb{1} - 2\,\mathbf{r}\otimes\mathbf{r}, \tag{30}$$

but in spherical coordinates,

$$P^i_{\ j} = \mathrm{diag}(-r^2, 1, 1)\,. \tag{31}$$

Consequently, at the unit sphere, the tangential components of the Green function, representing the electric field, vanish, which is the defining property of a perfect mirror. The Green function (29) obeys not only the wave Eq. (5) but also the correct boundary conditions at the spherical mirror, which justifies formula (29).

Finally, we may also include absorption in our theory, although in a simplified model. Absorption is described by the imaginary part n'' of the refractive index. Suppose $n''(r)$ is proportional to the real part given by the fish-eye profile (2). This case is equivalent to having a complex wave number k in the definition (5) of the Green function. The singularities of the Green function (23) describe source and image, but they are not affected by the wave number that only reduces the amplitude: the imaging quality is resistant to absorption, at least in our simple model. In three dimensions, the fish-eye mirror images with a resolution no longer limited by the wave nature of light.

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