Hidden symmetry and nonlinear paraxial atom optics

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A hidden symmetry of the nonlinear wave equation is exploited to analyze the propagation of paraxial and uniform atom-laser beams in time-independent and quadratic transverse potentials with cylindrical symmetry. The quality factor and the paraxial *ABCD* formalism are generalized to account exactly for mean-field interaction effects in such beams. Using an approach based on moments, these theoretical tools provide a simple yet exact picture of the interacting beam profile evolution. Guided atom laser experiments are discussed. This treatment addresses simultaneously optical and atomic beams in a unified manner, exploiting the formal analogy between nonlinear optics, nonlinear paraxial atom optics, and the physics of two-dimensional Bose-Einstein condensates.

I. INTRODUCTION

The realization of quasicontinuous atom lasers, building upon a great theoretical [1] and experimental [2–5] effort, is a topic of considerable excitement: beyond the challenge of producing a fully coherent atomic beam, such devices appear indeed as promising tools for various applications, ranging from precision experiments using atom interferometry [6] to the investigation of basic physical phenomena such as resonant nonlinear [7] quantum transport [8]. While the first atom lasers involved a matter-wave beam outcoupled into free space [2] and experiencing a free fall, recent experiments have demonstrated the possibility to obtain a guided atomic beam subject to an acceleration several orders of magnitude smaller than gravity [4,5]. In such beams, the atomic flux and longitudinal velocity are well monitored, and the beam linear density, i.e., the number of atoms per unit length, is constant on the propagation axis with a good approximation. Interaction effects on the propagation of the extracted atomic beam are usually considered as negligible outside the source condensate and as such are discarded from the analysis. Using a description of the atomic beam based on moments [9], it is in fact possible to account exactly for mean-field interaction effects in such guided atom lasers, provided that the atoms propagate at a constant velocity and in suitable transverse potentials. The purpose of this paper is to expose this method, exploiting commutation relations reminiscent of a hidden symmetry of the two-dimensional (2D) Schrödinger equation [10,11]. I explored the connection of previous results of nonlinear optics [12–14] to this symmetry and I extended them to nonlinear atom optics, giving a compact proof valid simultaneously for both optical and atomic beams.

The similarity between the equations of propagation for light and atomic waves [15,16], enabling one to reproduce with matter waves many nonlinear-optical phenomena [17], has already been used to adapt successful theoretical methods from optics to atom optics [18]. Concerning the linear propagation of matter waves, a relevant example is the *ABCD* matrix formalism [19] introduced by Bordé for atomic clouds [20]. This approach computes very efficiently

the propagation of dilute atomic wave packets in time-dependent quadratic potentials. Among other examples, let us mention the introduction of a paraxial wave equation and of a paraxial *ABCD* formalism for atomic beams [3,21], the characterization of a multiple-mode atomic beam with a quality factor [21–23], and the treatments of atomic interactions with a lensing term [3,24]. Although the optical approach of guided atom-laser propagation is not exclusive—other treatments rely on hydrodynamic equations [25]—, it provides a reliable and tractable characterization of such beams.

In this spirit, I considered two general results of nonlinear optics concerning the transverse profile of paraxial light beams. In a uniform medium, the second-order moment of the transverse intensity distribution follows a parabolic law, even in the presence of a Kerr effect [12]. In a graded-index medium, the transverse width oscillates with a period independent from the strength of the nonlinearity [13]. These results, which attest a universal behavior of the beamwidth evolution, are indeed intimately connected to a hidden symmetry of the paraxial equation [10,11,26]. They give a simple and yet exact expression of the transverse size of light beams propagating in a nonlinear medium and have had several applications in nonlinear optics [14,27]. As suggested in Refs. [21,22], these properties could be also relevant for nonlinear atom optics.

Here, I used them to develop a simple *ABCD* matrix formalism suitable to address the propagation of interacting atom-laser beams satisfying the following assumptions: the beam is stationary, paraxial, of constant linear density along the propagation axis, and it propagates in a transverse potential, which is time independent, quadratic, and of cylindrical symmetry. As a preliminary step, we will analyze the free expansion of such interacting atom-laser beams and we discuss experimental results [5]. The presented treatment avoids the limitations of previous nonlinear *ABCD* methods developed for matter waves [28,29]. In the nonparaxial *ABCD* approach [28], the interaction term is considered as a perturbation, assumption which becomes invalid for dense clouds or for long propagation times. The paraxial *ABCD* method used so far for interacting atomic beams [28,29], inspired from Ref. [30], requires the absence of external potential

and, most critically, assumes a Gaussian-shape approximation of the beam likely to break down after a finite propagation time. The advantage of the proposed treatment is that it addresses interacting atomic beams in a fully nonperturbative manner. As in previous nonlinear paraxial *ABCD* approaches [28,29], this method applies only to guided atomic beams experiencing no significant longitudinal potential and propagating at a constant longitudinal velocity. This hypothesis is realistic for the guided atom lasers reported in the Refs. [4,5].

II. EQUATIONS OF PROPAGATION FOR PARAXIAL LIGHT AND ATOMIC WAVES IN A NONLINEAR MEDIUM

In this section, we show that a paraxial light wave in a transparent and isotropic nonlinear medium and an interacting paraxial atomic beam of constant velocity follow equations of propagation, which are formally equivalent and take the form of a 2D Schrödinger equation. This similarity between the propagation of light and atomic waves had been pointed out, in the linear regime, by Bordé [15]. A formal analogy between mean-field atomic interactions and the Kerr effect is also discussed in Ref. [16].

A. Equation of propagation for optical waves

We consider a paraxial and monochromatic light wave propagating in a nonabsorptive and nonlinear medium of graded index. Specifically, its linear relative permittivity is assumed to have a quadratic position dependence of the form $\epsilon_r(x,y,z) = \epsilon_r^0[1 - \beta^2(z)r^2]$. This yields a medium index $n(r,z) = n_0[1 - \beta^2(z)r^2]^{1/2}$ quadratic in the vicinity of the propagation axis, where $r \ll \beta^{-1}(z)$. The index is greater on the propagation axis so that the medium acts as a converging lens and traps rays around this axis. The electric field is assumed monochromatic written in the form

$$\mathbf{E}(x,y,z,t) = \psi_\perp(x,y,z)e^{i(kz-\omega t)}\mathbf{a},$$

and it satisfies the paraxial approximation, i.e., $|\partial\psi_\perp/\partial z| \ll k|\psi_\perp|$. Using this approximation in Maxwell's equation, one obtains the nonlinear paraxial equation for optical waves [31]

$$2ik\frac{\partial\psi_\perp}{\partial z} = \Delta_T\psi_\perp + \gamma|\psi_\perp|^2\psi_\perp - k^2\beta^2r^2\psi_\perp. \quad (1)$$

The wave vector k satisfies the dispersion relation $k^2 = \epsilon_r^0\omega^2/c^2$, and the nonlinear coefficient $\gamma = \chi_e^{(3)}\omega^2/c^2$ is proportional to the nonlinear susceptibility $\chi_e^{(3)}$ at the frequency ω. Δ_T is the transverse Laplacian acting on the coordinates x, y. This equation appears as a 2D nonlinear Schrödinger equation, in which the coordinate z along the propagation axis plays the role usually devoted to the time in quantum mechanics [15].

B. Equation of propagation for atomic waves

We give a reminder on the propagation equation for a paraxial and monochromatic atom laser evolving in the sta-

tionary regime within an external potential [3,21]. The propagation equation is derived in the general case, but the results presented later apply only in the absence of a longitudinal potential.

Let us consider a wave function ψ, solution of the time-independent Schrödinger equation

$$\hat{H}\psi(\mathbf{r}) = E\psi(\mathbf{r}).$$

The Hamiltonian $\hat{H}$ accounts for interactions treated in the mean-field approximation and for an external potential U, sum of a transverse cylindrical potential $U_\perp(r,z) = 1/2m\omega_\perp^2(z)r^2$, and of a longitudinal one $U_\parallel(z)$. As in Ref. [32], the wave function is factorized in a 2D +one dimensional (1D) decomposition as

$$\psi(x,y,z) = \psi_\perp(x,y,z)\psi_\parallel(z).$$

The transverse wave function is normalized to unity $\forall z \int dxdy|\psi_\perp|^2(x,y,z) = 1$. The longitudinal wave function $\psi_\parallel$ verifies the 1D time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_\parallel}{\partial z^2} + U_\parallel\psi_\parallel = E\psi_\parallel,$$

which can be solved with the WKB method as

$$\psi_\parallel(z) = \sqrt{\frac{m\mathcal{F}}{p(z)}}\exp\left[\frac{i}{\hbar}\int_{z_0}^z dup(u)\right].$$

$\mathcal{F}$ is the matter-wave flux through any transverse plane, $p(z) = \sqrt{2m[E-U_\parallel(z)]}$ is the classical momentum along z, and the integral in the exponential argument starts at an arbitrary coordinate z_0 of the propagation axis corresponding to a point within the considered propagation zone.

For a beam propagating in the classically allowed region, where $U_\parallel(z) < E$, such WKB approach is indeed valid only if the wave vector $k(z) = p(z)/\hbar$ satisfies the condition $|dk/dz|/k \ll k$, otherwise, quantum reflections may occur. Equivalently, the longitudinal potential $U_\parallel$ should vary smoothly enough to verify $|dU_\parallel/dz| \ll \sqrt{8m}(E-U_\parallel)^{3/2}/\hbar$.

In a paraxial beam, the average momenta $\langle p_{\perp x,y,z}\rangle$ of the transverse wave function $\psi_\perp$ must satisfy $|\langle p_{\perp x,y,z}\rangle| \ll p(z)$. The second derivative of the wave function $\psi_\perp$ on the coordinate z is then negligible, and the wave function $\psi_\perp$ satisfies

$$\left[i\hbar\frac{p(z)}{m}\frac{\partial}{\partial z} + \frac{\hbar^2}{2m}\Delta_T - \frac{4\pi\hbar^2 a_s}{m}|\psi_\parallel|^2|\psi_\perp|^2 - U_\perp\right]\psi_\perp = 0,$$

$$(2)$$

with a_s as the s-wave atomic scattering length.

At this point, we note a significant difference between the nonlinear-optical and atom-optical propagations. In typical experimental conditions in optics, the refraction index and the longitudinal wave vector of the beam are constant along the propagation axis with a good approximation. In contrast, atom-laser experiments can involve a longitudinal potential, which may change significantly the longitudinal atomic momentum during the propagation. This induces a difference between the optical propagation (1) and the atomic propagation (2): in the latter, the coefficients of the first-order derivative and of the nonlinear term become z dependent in the

presence of a longitudinal potential. One can eliminate this dependence in the first-order derivative term by performing the following variable change, replacing the longitudinal coordinate z by the parameter:

$$\tau(z) = \int_{z_0}^{z} dz \frac{m}{p(z)},$$

defined as the time needed classically to propagate from the axis point of coordinate z_0 to the point of coordinate z. Equation (2) takes then the form

$$2ik\frac{\partial \psi_\perp}{\partial \tau} = -\Delta_T \psi_\perp + 8\pi a_s n_{1D}(\tau)|\psi_\perp|^2\psi_\perp + k^2\omega_\perp^2 r^2\psi_\perp. \tag{3}$$

We noted $k=m/\hbar$ and introduced the linear atomic density $n_{1D}=|\psi_\parallel|^2$. For a dilute atomic beam, this equation is formally equivalent to the optical Eq. (1).

For an interacting atomic beam, however, the coefficient of the nonlinear term in Eq. (3) still depends on the propagation parameter τ. In the WKB treatment, where the linear density reads as $n_{1D}(\tau)=m\mathcal{F}/p(\tau)$, this coefficient is a constant only if no significant longitudinal potential is applied to the atomic beam, which propagates then with a constant longitudinal velocity. The nonlinear-optical Eq. (1) and atom-optical Eq. (3) are then fully equivalent.

C. Quantum-mechanical interpretation of the nonlinear wave equation

Both equations of propagation can be written in the generic form

$$2ik\frac{\partial \psi_\perp}{\partial u} = -\epsilon\Delta_T\psi_\perp + \gamma|\psi_\perp|^2\psi_\perp + \epsilon k^2\alpha^2(u)r^2\psi_\perp, \tag{4}$$

with $u=z$, $k=\sqrt{\epsilon_r^0}\omega/c$, $\epsilon=-1$, $\gamma=\chi_e^{(3)}k^2/\epsilon_r^0$, and $\alpha=\beta$ for the optical equation, and $u=\tau$, $k=m/\hbar$, $\epsilon=1$, $\gamma=8\pi n_{1D}a_s$, and $\alpha=\omega_\perp$ for the atomic equation.

One must assume that no longitudinal potential is driving the atoms in order to obtain a nonlinear coefficient γ independent of the coordinate τ in the atomic equation. As stated previously, we make this assumption for the subsequent developments: otherwise, the equations of motion for the beam moments are not solved by the *ABCD* matrix approach presented here. However, the coefficient α of the quadratic effective potential may depend on the coordinate u in both the optical and atomic equations.

If one makes the correspondence $2k\rightarrow\hbar$ and $u\rightarrow t$, Eq. (4) appears as the Schrödinger equation[1] of a 2D wave function [15]. The corresponding Hamiltonian

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[1]To treat atomic beams only, one could indeed obtain directly a 2D Schrödinger equation from Eq. (2) instead of Eq. (4), without having to do the correspondence $2k\rightarrow\hbar$. The great advantage of starting from Eq. (4) is that one obtains a treatment addressing simultaneously optical and atomic beams. This also shows the insight provided by a quantum-mechanical approach of the optical paraxial equation.

$$\hat{H} = \epsilon\hat{K} + \hat{V} + \epsilon\hat{U} \tag{5}$$

involves three terms

$$\hat{K} = \hat{p}^2/\hbar^2,$$

$$\hat{V} = \frac{1}{2}\gamma\delta^{(2)}(\mathbf{r}-\mathbf{r}'),$$

$$\hat{U} = k^2\alpha^2\hat{r}^2. \tag{6}$$

We have introduced the operator $\hat{\mathbf{p}}=-i\hbar(\nabla_x,\nabla_y)$, satisfying $[\hat{x},\hat{p}_x]=i\hbar$, associated with the transverse momentum of the atoms or photons inside the considered beam. The average values of these operators are defined with the normalized transverse wave function $\psi_\perp$, following usual conventions in quantum mechanics. For instance, the second-order moment $\langle\hat{r}^2\rangle$ reads as

$$\langle\hat{r}^2\rangle(u) = \int dx\ dy\psi_\perp^*(x,y,u)(x^2+y^2)\psi_\perp(x,y,u).$$

Let us note that because of the correspondence $2k\rightarrow\hbar$, the Hamiltonian (5) does not have the dimension of an energy. Accordingly, we shall refer to the three operators $\hat{K}$, $\hat{V}$, $\hat{U}$ as "effective potentials." $\hat{K}$ is proportional to the transverse kinetic energy, while $\hat{V}$ and $\hat{U}$ play, respectively, the roles of an effective 2D-contact potential and of an external potential. The proposed analysis of the propagation essentially relies on the commutation relations between these three operators.

III. FREE NONLINEAR PROPAGATION OF OPTICAL AND ATOMIC BEAMS

We treat here the simpler case for which the paraxial Eq. (4) contains no effective quadratic potential. This corresponds to the propagation of an optical beam in a Kerr medium of a uniform linear index or to that of an interacting atom laser in the absence of external potential. We use the hidden symmetry of the paraxial equation to derive the equations of motion for the moment $\langle\hat{r}^2\rangle$ associated with such beams. We obtain a simple expansion law, which we apply to discuss experimental results [5] on guided atom lasers.

A. Effective Hamiltonian scale invariance and propagation of the second-order moment

The Hamiltonian (5) associated with the free propagation equation reduces to

$$\hat{H}_0 = \epsilon\hat{K} + \hat{V}. \tag{7}$$

The key point is to investigate how this operator transforms in a space dilatation $\mathbf{r}\rightarrow\lambda\mathbf{r}$ [11]

$$\mathbf{r}\rightarrow\lambda\mathbf{r}, \quad \psi(\mathbf{r})\rightarrow\lambda^{d/2}\psi(\mathbf{r}/\lambda), \quad \hat{H}_0\rightarrow-\frac{\epsilon}{\lambda^2}\Delta_T+V(\lambda\mathbf{r}).$$

To ensure the scale invariance of the Hamiltonian $\hat{H}_0$, the interaction potential should transform as $V(\lambda\mathbf{r})=\lambda^{-2}V(\mathbf{r})$.

This is verified by the potential $V(\mathbf{r}) = 1/r^2$ and also by the 2D-contact potential $V(\mathbf{r}) = \frac{1}{2}\gamma\delta^{(2)}(\mathbf{r})$ [10]. The scale invariance of $\hat{H}_0$ can be expressed equivalently in terms of commutation with the generator $\hat{Q}$ of dilatations as

$$[\hat{Q}, \hat{H}_0] = 2i\hat{H}_0. \tag{8}$$

This generator reads as $\hat{Q} = \frac{1}{2\hbar}(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}} + \hat{\mathbf{r}}\cdot\hat{\mathbf{p}})$, it operates at each point $\mathbf{r}$ a translation proportional to the vector $\mathbf{r}$. The evolution of the second-order moment (identified to the width $w^2 = \langle\hat{r}^2\rangle$ for a light beam) is given by Ehrenfest's theorem, applied with the Hamiltonian (5)

$$\frac{d\langle\hat{r}^2\rangle}{du} = \frac{2\epsilon}{k}\langle\hat{Q}\rangle, \quad \frac{d^2\langle\hat{r}^2\rangle}{du^2} = \frac{\epsilon}{ik^2}\langle[\hat{Q}, \hat{H}_0]\rangle. \tag{9}$$

The parabolic evolution of $\langle\hat{r}^2\rangle$ is indeed a direct consequence of the Hamiltonian scale invariance (8): the quantity $d^2\langle\hat{r}^2\rangle/du^2$ is proportional to the average effective energy $\langle\hat{H}_0\rangle$ of the transverse wave function

$$\frac{d^2\langle\hat{r}^2\rangle}{du^2} = \frac{2\epsilon}{k^2}\langle\hat{H}_0\rangle,$$

which is a constant of motion. This result holds only for a 2D transverse space, the only dimensionality giving the desired scale transformation of the contact potential. In this case, using Eqs. (8) and (9), one obtains readily the free expansion law valid for optical and atomic beams

$$\langle\hat{r}^2\rangle(u) = \frac{\epsilon}{k^2}\langle\hat{H}_0\rangle u^2 + \frac{2\epsilon}{k}\langle\hat{Q}\rangle_0 u + \langle\hat{r}^2\rangle_0. \tag{10}$$

We noted $\langle\ \rangle_0$ the average value of the considered operator at the initial time of the propagation, namely, for the coordinate $u = 0$.

B. Application to the time-of-flight analysis of an atom-laser beam

The previous result can be extended easily to account for the presence of a linear gravitational potential in the propagation of an atomic beam. By considering the free-fall frame or alternatively by applying Ehrenfest's theorem as in Eq. (9) with an additional potential $V_g = 2k^2 gy$ (with O_y the vertical axis), one retrieves a parabolic law of Eq. (10) for the transverse width $w^2(\tau) = \langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2 + \langle\hat{y}^2\rangle - \langle\hat{y}\rangle^2$. We consider a wave function corresponding to a centered atomic beam, verifying initially $\langle\hat{\mathbf{r}}\rangle_0 = 0$ and $\langle\hat{Q}\rangle_0 = 0$. $\epsilon = 1$ and $k = m/\hbar$ in the atomic equation, so the free atomic Hamiltonian is $\hat{H}_0 = \hat{K} + \hat{V}$. The width evolution reads as

$$w^2(\tau) = \frac{2}{k^2}(\langle\hat{K}\rangle_0 + \langle\hat{V}\rangle_0)\tau^2 + w^2(0). \tag{11}$$

The quantity $\langle\hat{K}\rangle_0 = m^2/\hbar^2\langle\Delta v^2\rangle_0$ reflects the contribution of the transverse kinetic energy to the beam divergence, while the quantity $\langle\hat{V}\rangle_0$ reflects that of mean-field interactions. This second contribution is usually discarded in the time-of-flight analysis of atom-laser beams.

The expansion law (11) can be readily applied to investigate interactions effects on the transverse dynamics of a guided atom laser. Assuming that one knows sufficiently the initial wave-function profile as to estimate $\langle\hat{V}\rangle_0$, this equation gives the transverse velocity dispersion $\langle\Delta v^2\rangle_0$ as a function of the width expansion $w(\tau)$ measured experimentally

$$\langle\Delta v^2\rangle_0 = \frac{1}{2\tau^2}[w^2(\tau) - w^2(0) - \hbar^2/m^2\langle\hat{V}\rangle_0].$$

Interaction effects have been neglected in the analysis of the expansion given in [4,5], in which the expansion is simply attributed to the initial transverse kinetic energy. This leads to a different estimate of the transverse velocity dispersion $\langle\Delta v^2\rangle_{free}$ expressed as a function of the beamwidth

$$\langle\Delta v^2\rangle_{free} = \frac{1}{2\tau^2}[w^2(\tau) - w^2(0)].$$

Discarding interaction effects in the beam expansion thus yields the following error on the transverse velocity:

$$\frac{\langle\Delta v^2\rangle_{free} - \langle\Delta v^2\rangle_0}{\langle\Delta v^2\rangle_0} = \frac{\langle\hat{V}\rangle_0}{\langle\hat{K}\rangle_0}. \tag{12}$$

For the quasimonomode beam reported in [5], one can estimate the ratio $\langle\hat{V}\rangle_0/\langle\hat{K}\rangle_0$ by modeling the initial transverse wave function with a Gaussian profile verifying $\langle\mathbf{r}\rangle_0 = \langle\mathbf{p}\rangle_0 = 0$.[2] Equation (12) shows then that the value $\langle\Delta v^2\rangle_{free}$ overestimates the correct square velocity dispersion $\langle\Delta v^2\rangle_0$ as $\langle\Delta v^2\rangle_{free} = (1 + 2n_{1D}a_s)\langle\Delta v^2\rangle_0$, thus of roughly 18% in [5] with the reported linear density. In fact, interaction effects may affect significantly the flight of the atomic dloud, and can only be safely neglected in the limit $n_1 a_s \ll 1$ [33].

IV. *ABCD* MATRIX ANALYSIS OF THE NONLINEAR PROPAGATION OF OPTICAL AND ATOMIC BEAMS IN THE PRESENCE OF A QUADRATIC POTENTIAL

We now address the nonlinear propagation described by the full Eq. (4) and obtain an exact *ABCD* matrix analysis of the beam evolution. As seen previously, the additional effective potential contained in this equation reflects either a position-dependent linear permittivity of the form $\epsilon_r(x, y, z) = \epsilon_r^0[1 - \beta(z)^2 r^2]$ for an optical beam or a time-independent quadratic potential $U_\perp(r, z) = 1/2m\omega_\perp^2(z)r^2$ for an atomic beam. We remind that although they are assumed time independent, the coefficient β and the trapping frequency $\omega_\perp$ may depend on the longitudinal coordinate z or parameter τ. The presented treatment can thus address atom-laser experiments focusing the beam thanks to a time-independent quadratic transverse potential, which sharpens along the propagation axis.

We proceed as follows. Using commutation relations between the three operators $\hat{K}$, $\hat{V}$, $\hat{U}$ defined in Eqs. (6), we

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[2]To evaluate this error for a multiple-mode atomic beam [4], one needs to know the mode decomposition of the transverse wave profile in order to estimate $\langle\hat{V}\rangle_0$.

derive the equations of motion for the second-order moment. This approach is inspired from the treatment of Ghosh [26], but indeed these equations had been derived several times before [12–14,34]. We then introduce a nonlinear quality factor for matter-wave beams, the invariance of which is related to the hidden symmetry of the wave equation. This parameter enables the definition of a nonlinear complex radius of curvature q "renormalized" by the mean-field interactions. We show that this radius of curvature can be propagated through $2 \times 2 ABCD$ matrices identical to those used for linear optical and atomic beams. This conclusion agrees with the results obtained by Porras *et al.* [14] in nonlinear optics. To ease the comparison between the linear and nonlinear regimes, an introduction to the linear $ABCD$ matrix formalism for optical and atomic beams is given in the Appendix.

A. Solutions for the propagation in a transverse quadratic potential

In the presence of a quadratic potential $\hat{U}$, the Hamiltonian (5) $\hat{H} = \epsilon \hat{K} + \hat{V} + \epsilon \hat{U}$ is no longer scale invariant, but the operators $\hat{H}_0 = \epsilon \hat{K} + \hat{V}$, $\hat{U}$, $\hat{Q}$ verify nonetheless the special set of commutation relations [11]

$$[\hat{Q}, \hat{H}_0] = 2i\hat{H}_0 \quad [\hat{Q}, \epsilon\hat{U}] = -2i\epsilon\hat{U} \quad [\epsilon\hat{U}, \hat{H}_0] = 4i\alpha^2 k^2 \hat{Q}.$$
$$(13)$$

Thanks to this set of equations, signature of a hidden symmetry, the Hamiltonian can be embedded into an SO(2,1) algebra [11]. These relations yield, through Ehrenfest's theorem applied with $\hat{H} = \hat{H}_0 + \epsilon \hat{U}$, a closed set of coupled equations for the derivatives of the second-order moment

$$\frac{d\langle \hat{r}^2 \rangle}{du} = \frac{2\epsilon}{k} \langle \hat{Q} \rangle,$$

$$\frac{d^2\langle \hat{r}^2 \rangle}{du^2} = \frac{2\epsilon}{k^2}(\langle \hat{H}_0 \rangle - \epsilon\langle \hat{U} \rangle),$$

$$\frac{d^3\langle \hat{r}^2 \rangle}{du^3} = -4\alpha^2 \frac{d\langle \hat{r}^2 \rangle}{du} - 2\frac{d\alpha^2}{du}\langle \hat{r}^2 \rangle.$$
$$(14)$$

Equations (14) show that for a constant coefficient $\alpha(u) = \alpha_0$ [or uniform trap frequency $\omega_\perp(z) = \omega_0$], the second-order moment oscillates with a frequency independent from the strength of the nonlinearity. This property of nonlinear optics, established in [13], is formally identical to the universal low-energy mode found for 2D condensates [11]. Exact solutions may be found with a periodic quadratic coefficient $\alpha(u)$ [or periodically modulated frequency $\omega_\perp(\tau)$ for atomic beams] by noticing that the width obeys a Hill's equation [26,34].

B. Nonlinear quality factor and radius of curvature

The proposed $ABCD$ law applies to the propagation a parameter q, which can be interpreted as a complex radius of curvature generalized for the nonlinear propagation. This pa-

rameter has been introduced in previous developments of nonlinear optics [13,14], and here we extend its definition to atom optics. This definition involves an invariant of propagation, obtained from the set of commutation relations [Eq. (13)] behind the hidden symmetry, which reads as

$$M_I^4 = \epsilon \langle \hat{r}^2 \rangle \langle \hat{H}_0 \rangle - \langle \hat{Q} \rangle^2.$$
$$(15)$$

For the optical propagation in which $\epsilon = -1$, one retrieves the nonlinear-optics quality factor [14,35]. The connection of this optics invariant with the hidden symmetry of the paraxial equation has never been pointed out to my knowledge. For the atomic propagation in which $\epsilon = 1$, the parameter M_I constitutes the generalization to paraxial interacting beams of the quality factor introduced by Riou *et al.* [22] for dilute atomic beams. The term $\langle \hat{H}_0 \rangle$ includes in M_I the additional divergence resulting from interaction effects, which were not accounted for in the quality factor defined in Ref. [22]. The interacting quality factor of a cylindrical and centered ($\langle \mathbf{r} \rangle = 0$) atomic beam in the fundamental Gaussian mode reads as $M_I^4 = 1 + 2n_{1D}a_s$. In the limit of dilute atomic waves, one retrieves $M_I = 1$, so the interacting quality factor coincides with the parameter defined in [22]. It is also useful to introduce the radius of curvature R [13,14]

$$\frac{1}{R} = \frac{1}{2w^2}\frac{dw^2}{du}.$$
$$(16)$$

With these parameters at hand, the generalized complex radius of curvature reads as

$$\frac{1}{q} = \frac{1}{R} + \frac{iM_I^2}{kw^2}.$$
$$(17)$$

C. Demonstration of the $ABCD$ law for the nonlinear propagation

The previous assumptions regarding the uniform linear atomic density and cylindrical symmetry of the effective transverse potential still apply. Unless specified otherwise, we also assume that the beam is centered ($\langle \hat{\mathbf{r}} \rangle = 0$) and that the potential contains no linear term.[3]

Considering the radii of curvature $q_1 = q(0)$ and $q_2 = q(u)$ at different stages of the propagation, we seek to establish an $ABCD$ law of the form

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}.$$
$$(18)$$

It is sufficient that the two following relations be satisfied [14]:

$$w_2^2 = w_1^2 \left[A + \frac{B}{R_1}\right]^2 + \frac{M_I^4 B^2}{k^2 w_1^2}$$
$$(19)$$

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[3]In the presence of a quadratic potential, a linear gravitational potential can nonetheless be easily accounted for with a change of coordinates $x' = x$, $y' = y + g/\omega_\perp^2$ (sag), and by considering the evolution of the moment $\langle \hat{r}'^2 \rangle$ instead of $\langle \hat{r}^2 \rangle$.

$$\frac{w_2^2}{R_2} = w_1^2 \left[A + \frac{B}{R_1} \right] \left[C + \frac{D}{R_1} \right] + \frac{M_I^4 BD}{k^2 w_1^2}. \qquad (20)$$

The two functions $F_1(u)$ and $F_2(u)$ defined, respectively, by the right-hand side of Eqs. (19) and (20) obey the same first-order differential system as the functions w^2 and w^2/R—obtained by considering the set of Eqs. (14)–(16),

$$\frac{dF_1}{du} = 2F_2, \quad \frac{dF_2}{du} F_1 = \frac{M_I^4}{k^2} + F_2^2 - \alpha^2(u) F_1^2, \qquad (21)$$

if and only if the *ABCD* matrix satisfies the differential equation

$$\frac{d}{du} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha^2(u) & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad (22)$$

as well as the condition $(AD - BC)(u) = 1$ for any parameter u. Setting the initial condition $[A, B, C, D](0) = [1, 0, 0, 1]$ guarantees then that $F_1(u)$ and $F_2(u)$ coincide with w^2 and w^2/R at all times, thereby, ensuring the *ABCD* law (18). The nonlinear *ABCD* matrix obeys Eq. (22), which is also the differential equation satisfied by the *ABCD* matrix describing the linear propagation of dilute atomic beams [see Eq. (A11)]. This shows that in spite of the mean-field interactions, the usual interaction-free paraxial *ABCD* matrices [21] apply rigorously to propagate the parameter q. In particular, one retrieves the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

for the free propagation[4] and the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos(\alpha u) & \alpha^{-1} \sin(\alpha u) \\ -\alpha \sin(\alpha u) & \cos(\alpha u) \end{pmatrix}$$

for the propagation in a quadratic index or cylindrical potential described by Eq. (4). These matrices are identical to those given in Eqs. (A2) and (A3) for the linear optical propagation or in Eqs. (A12) and (A13) for the linear atom-optical propagation. It is a rather remarkable fact that the linear *ABCD* laws persist for an interacting beam: the inclusion of the invariant M_I in the definition of the radius of curvature q captures the nonlinear effects on the propagation. This is indeed a consequence of the special set of commutation relations resulting from the hidden symmetry.

Setting $u = \tau$ and $\alpha = \omega_\perp$ in these matrices, one obtains the *ABCD* law for the propagation of an interacting atomic beam. An analogous property for the propagation of the parameter q had been established in a more restrictive framework: it applied only to free-propagating paraxial Gaussian atomic beams treated with a Gaussian-shape approximation [28,29], which implies a perturbative treatment of the interaction term. The result obtained here is much more general:

it is nonperturbative, it applies to atomic beams propagating in transverse potentials, and it does not require any approximation once the assumptions of paraxiality, uniform linear density of the atomic beam (and thus absence of a longitudinal potential), and cylindrical symmetry of the transverse potential are satisfied.

In a similar manner as in [13,14], this approach could also be used to discuss the self-trapping in future experimental beams involving atoms with attractive interactions such as fermions with Feshbach-tuned interactions. Self-trapping is expected at a critical density yielding $M_I = 0$, i.e. $n_{1D} = 1/(2|a_s|)$ for Gaussian beams, beyond which the beam collapses.

V. CONCLUSION

To summarize, I have used the symmetries of the Hamiltonian associated with the nonlinear paraxial wave equation—scale invariance in the free propagation, specific set of commutation relations in the presence of a quadratic external potential—to extract simple propagation laws for the width of a paraxial atomic beam of uniform density. These results can be applied to analyze the divergence of atom-laser beams in recent experiments [4,5] and evaluate their number of transverse modes. A quality factor [Eq. (15)] has been defined, valid for paraxial interacting atomic beams propagating in cylindrical potentials, thereby, generalizing the parameter introduced in [22] for dilute paraxial atom lasers. This parameter, together with a generalized radius of curvature, allows one to describe with linear *ABCD* matrices the propagation of a paraxial interacting atomic beam in time-dependent, cylindrical, and quadratic potentials. An interesting question is the robustness of the presented approach to a relaxation of the hypothesis of paraxiality and uniform density of the atomic beam. Recent theoretical work [28] on a matter-wave resonator [36], involving a free-falling atomic cloud bouncing on curved atomic mirrors, shows that the width oscillations of universal frequency persist with a good approximation in spite of a time-dependent linear density. This suggests that our treatment could be fruitfully applied to guided atom lasers presenting a slow variation in this parameter. Last, let us point out a strong connection between the physics of guided atom lasers and of 2D condensates, involving a similar equation. We have seen that theoretical tools for the latter can also be relevant for nonlinear atom optics. This also suggests that several effects specific of 2D condensates, such as the breathing modes, might be reproduced within the transverse profile of guided atom-laser beams.

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[4]For the free propagation, this result can be obtained directly by noting that Eq. (9) implies $(1/q)' = -1/q^2$ and thus simply $q(\tau) = q(0) + \tau$. By replacing $\langle \hat{r}^2 \rangle$ with w^2 in the definition (15) of the quality factor, this relation can be extended to the propagation of a noncentered beam ($\langle \hat{\mathbf{r}} \rangle \neq 0$) in a linear potential.

APPENDIX: THE *ABCD* MATRIX FORMALISM FOR LINEAR PARAXIAL OPTICS AND ATOM OPTICS

This appendix gives an introduction to the *ABCD* matrix formalism for optical and atomic beams experiencing a linear propagation.

1. *ABCD* formalism in linear paraxial optics

The *ABCD* matrix formalism, discussed in various textbooks [37], was originally introduced to analyze the light wave propagation in linear optical systems involving a constant or piecewise quadratic index of refraction [19]. We present briefly this approach for ray and Gaussian optics.

One associates to a light ray a two-component vector map, involving the ray distance to the axis $r(z)$ and the slope $r'(z) = dr(z)/dz$. In a linear optical system, the propagation of this vector is given by a linear input-output relation of the form

$$\begin{pmatrix} r(z_2) \\ r'(z_2) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r(z_1) \\ r'(z_1) \end{pmatrix} \tag{A1}$$

called *ABCD* law, in which the matrix coefficients depend on the propagation distance $\Delta z = z_2 - z_1$, and satisfy the relation $AD - BC = 1$. In a medium of uniform index, the *ABCD* matrix is simply

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \Delta z \\ 0 & 1 \end{pmatrix}. \tag{A2}$$

In a graded-index medium with $n(r,z) = n_0[1 - 1/2\beta^2(z)r^2]$, the ray propagation equation yields the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos(\beta\Delta z) & \beta^{-1}\sin(\beta\Delta z) \\ -\beta\sin(\beta\Delta z) & \cos(\beta\Delta z) \end{pmatrix}. \tag{A3}$$

The *ABCD* formalism is especially well suited to propagate Gaussian waves. Let us consider an input beam at the plane $z = z_1$ with a transverse profile

$$\psi_\perp(x,y,z_1) = \exp\left(-i\frac{kx^2}{2q_1}\right) \quad \text{with} \quad \frac{1}{q_1} = \frac{1}{R_1} - \frac{2i}{kw_1^2}. \tag{A4}$$

R_1 is the radius of curvature and w_1 the beamwidth in the direction O_x, q_1 is called the complex radius of curvature. The quantity $k = n_0\omega/c$ is the longitudinal wave vector of the optical wave. The beam is assumed to propagate between the planes $z = z_1$ and $z = z_2 = z_1 + \Delta z$ in a linear optical system characterized by an *ABCD* matrix. The beam propagator in this system is directly related to the *ABCD* matrix elements [37]

$$K(x,x',\Delta z) = \sqrt{\frac{ik}{2\pi B}}\exp\left[-\frac{ik}{2B}(Ax'^2 - 2x'x + Dx^2)\right]. \tag{A5}$$

The output beam at $z = z_2$ is given by the propagation integral

$$\psi_\perp(x,y,z_2) = \int_{-\infty}^{+\infty} dx' K(x',x,z_2 - z_1)\psi_\perp(x',y,z_1) \tag{A6}$$

and can be expressed as

$$\psi_\perp(x,y,z_2) = \sqrt{\frac{1}{A + B/q_1}}\exp\left(-i\frac{kx^2}{2q_2}\right),$$

with the complex radius of curvature q_2 given by the *ABCD* law rewritten here for convenience

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}. \tag{A7}$$

This treatment can be extended to the full Hermite-Gauss basis [37].

2. *ABCD* formalism in linear paraxial atom optics

Let us now apply the *ABCD* matrix technique to propagate dilute paraxial atomic waves [3,21]. The corresponding propagation equation has the form of Eq. (2) without the interaction term ($n_{1D} \simeq 0$)

$$i\hbar\frac{\partial\psi_\perp}{\partial\tau} = \left[-\frac{\hbar^2}{2m}\Delta_T + V_x(x,\tau)\right]\psi_\perp. \tag{A8}$$

The potential V_x is assumed to be quadratic of the form $V_x = m\omega_\perp^2 x^2/2$. The propagator K is given by the Van Vleck formula or by the *ABCD* formalism [15]

$$K(x,x',\tau) = \sqrt{\frac{im}{2\hbar\pi B}}\exp\left[\frac{im}{2\hbar B}(Ax'^2 - 2x'x + Dx^2)\right]. \tag{A9}$$

If one sets $k = m/\hbar$ as previously for atomic beams, this expression becomes formally identical to Eq. (A5) up to a sign change in the argument of the exponential. The corresponding *ABCD* matrix describes the one-dimensional motion of a classical particle moving in the potential V_x. Precisely, let us consider a classical particle initially at the position $x(0)$ and with a speed $v_x(0)$, moving in the potential V_x. At the instant $t = \tau$, this particle has a final position $x(\tau)$ and a speed $v_x(\tau)$. Since the equations of motion are linear, the final values are related to the initial ones by a linear map analogous to Eq. (A7)

$$\begin{pmatrix} x(\tau) \\ v_x(\tau) \end{pmatrix} = \begin{pmatrix} A(\tau) & B(\tau) \\ C(\tau) & D(\tau) \end{pmatrix} \begin{pmatrix} x(0) \\ v_x(0) \end{pmatrix}. \tag{A10}$$

The *ABCD* matrix involved in this relation is precisely the matrix used in the propagator of Eq. (A9). Let us verify this. One can check that the distribution defined by Eq. (A9) satisfies the Schrödinger equation if and only if the coefficients A, B, C, D satisfy the first-order differential equation

$$\frac{d}{d\tau}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_\perp^2 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{A11}$$

It is easy to show that the *ABCD* matrix defined in Eq. (A10) obeys the same equation. Besides, when the time τ tends toward 0, the *ABCD* matrix defined by Eq. (A10) tends toward identity. In this limit, the distribution defined with this matrix in Eq. (A9) yields a Dirac distribution, as expected for a propagator between equal instants. Equation (A9) and the *ABCD* matrix of Eq. (A10) thus define a distribution K,

which satisfies the Schrödinger equation as well as the correct initial condition. This shows that K is the propagator of the Schrödinger equation.

Using Eq. (A11), it is straightforward to establish the expression of the *ABCD* matrix for the free motion

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \qquad \text{(A12)}$$

and for the motion in a quadratic potential

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos(\omega_\perp \tau) & \omega_\perp^{-1} \sin(\omega_\perp \tau) \\ -\omega_\perp \sin(\omega_\perp \tau) & \cos(\omega_\perp \tau) \end{pmatrix}. \quad \text{(A13)}$$

The expression of the propagator can be readily applied to obtain, by analogy with Gaussian optics, an *ABCD* law for the propagation of Gaussian wave functions. Let us consider the complex radius of curvature decomposed as previously along its imaginary and complex part as

$$\frac{1}{q_1} = \frac{1}{R_1} + \frac{2i}{kw_1^2}.$$

We consider the initial transverse wave function $\psi_\perp$, initially of the form

$$\psi_\perp(x,y,\tau_1) = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{w_1}} \exp\left[i\frac{k}{2q_1}x^2\right]. \qquad \text{(A14)}$$

Note that this wave function is properly normalized to unity. Its evolution is given by the propagation integral

$$\psi_\perp(x,\tau_2) = \int dx' K(x,x',\tau)\psi_\perp(x',y',\tau_1),$$

with $\tau = \tau_2 - \tau_1$. Using the optical integral (A6), one obtains the wave function at the instant τ_2

$$\psi_\perp(x,y,\tau_2) = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{w_2}} \exp\left[i\frac{k}{2q_2}x^2\right],$$

where q_1 and q_2 satisfy the *ABCD* law of Eq. (A7).

This formalism can be readily extended to the propagation of wave functions in a separable quadratic transverse potential $V_\perp = V_x + V_y$. The corresponding propagator is simply the product of the one-dimensional propagators $K(x,x',y,y',\tau) = K_x(x,x',\tau)K_y(y,y',\tau)$ defined by Eq. (A9). A 2×2 *ABCD* matrix is associated with each direction. The propagation of a Gaussian wave function

$$\psi_\perp(x,y,\tau_1) = \sqrt{\frac{2}{\pi w_{1x} w_{1y}}} \exp\left[i\frac{k}{2q_{1x}}x^2\right] \exp\left[i\frac{k}{2q_{1y}}y^2\right]$$

is formally identical: the parameters q_x and q_y satisfy a relation of the form of Eq. (A7) with the *ABCD* matrices associated with their respective directions.

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