Inverse scattering transform for the Yajima-Oikawa equations with nonvanishing boundary conditions

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Application of the Yajima-Oikawa (YO) equations describing the long wave–short wave resonance interaction to study of vector solitons of mixed bright-dark types in quasi-one-dimensional spinor Bose-Einstein condensates is considered. The inverse scattering transform for the YO equations with the mixed vanishing and nonvanishing boundary values is constructed. Examples of management of the high-frequency waves by changing initial and boundary values associated with the low-frequency waves are presented.

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I. INTRODUCTION

The past decade has witnessed a tremendous explosion of interest in the experimental and theoretical studies of Bose-Einstein condensates (BECs). The recent development of techniques for trapping of ultracold atomic gases has opened new directions in the studies of BECs (see, e.g., Ref. [1]). One of the major achievements in this direction was the experimental creation of spinor BECs [1,2]. This gave rise to the observation of various phenomena that are not present in single-component BECs, including formation of spin domains [3] and spin textures [4].

A spinor condensate formed by atoms with spin F is described by a macroscopic wave function with $(2F+1)$ components. Accordingly, a number of theoretical works have been dealing with multicomponent vector solitons in $F=1$ spinor BECs. Bright [5–7] and dark [8,9] solitons have been predicted in this context. Mixed vector soliton solutions, composed of bright and dark components, of the respective system of coupled Gross-Pitaevskii equations (GPEs) have been reported in Ref. [10]. Compound solitons of the mixed type may be of particular interest, as they would provide for the possibility of all-matter-wave waveguiding, with the dark soliton component building an effective conduit for the bright component, similar to the all-optical waveguiding proposed in nonlinear optics [11].

In the physically relevant case of ^{87}Rb and ^{23}Na atoms with $F=1$, which are known to form spinor condensates of ferromagnetic and polar types, respectively, the system includes a naturally occurring small parameter δ, namely, the ratio of the strengths of the spin-dependent and spin-independent interatomic interactions [12,13]. Exploiting the smallness of δ, the authors of Ref. [10] developed a multiscale-expansion method to asymptotically reduce the nonintegrable GPE system to the Yajima-Oikawa (YO) system [14,15]. The latter one was integrated by using the inverse scattering transform method (ISTM) [16,17] in Refs. [14,15,18]. The YO equations were originally derived to describe the interaction of Langmuir and sound waves in plasma [19] and were used in studies of magnetic chains [20]. These equations were derived in the context of optics and used for study the nonlinear dynamics of the DNA chain [21] and binary BECs [22]. Borrowing exact soliton solutions from the YO system, the authors of Ref. [10] predicted two types of vector-soliton complexes in the spinor condensate, *viz*., dark-dark-bright and bright-bright-dark ones for the $m_F=+1,-1,0$ spin components. Numerical simulations of the underlying GPE system showed that these solitary pulses (including ones with moderate, rather than small amplitudes) emulate solitons in integrable systems quite well [10]. The solitons propagate undistorted for a long time and undergo quasielastic collisions [10]. In Ref. [10], the continuous-wave solutions as for high-frequency waves and as for low-frequency waves were considered. Stability of the continuous low-frequency waves in a case of polar BEC was demonstrated.

While the ISTM for the YO equations was developed many years ago, with vanishing boundary conditions, the basic formulation of ISTM has not been developed for the YO equations with nonvanishing boundary conditions. We should also remark that direct methods have been applied to YO equations as a way to derive explicit bright and dark soliton solutions (see, for instance, Refs. [23,24]).

At sufficiently low temperatures and in the framework of the mean-field approach, the spinor BEC with $F=1$ is described by the vector order parameter, $\mathbf{\Psi}(\mathbf{r},T)=[\Psi_{-1}(\mathbf{r},T),\Psi_0(\mathbf{r},T),\Psi_{+1}(\mathbf{r},T)]^T$, with the components corresponding to the three values of the vertical spin projection, $m_F=-1,0,+1$. Here, T and $\mathbf{r}=(X,Y,Z)$ are the time and space variables, respectively. Supposing that the condensate is confined in a highly anisotropic trap with frequencies $\omega_X \ll \omega_\perp$, one may assume the wave functions approximately separable, $\Psi_{0,\pm1}\approx\psi_{0,\pm1}(X)\psi_\perp(Y,Z)$, where the transverse wave function $\psi_\perp(Y,Z)$ is the ground state of the respective harmonic oscillator. Then, averaging of the underlying system of the coupled three-dimensional (3D) GPEs in transverse plane (Y,Z) leads to the following system of coupled one-dimensional (1D) equations for the longitudinal components of the wave functions (see Ref. [10] and references therein):

$$i\hbar\partial_T\psi_{\pm1}=\hat{H}_{\rm si}\psi_{\pm1}+c_2^{(1\rm D)}(|\psi_{\pm1}|^2+|\psi_0|^2-|\psi_{\mp1}|^2)\psi_{\pm1}+c_2^{(1\rm D)}\psi_0^2\psi_{\mp1}^*, \quad (1)$$

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$$i\hbar\partial_T\psi_0 = \hat{H}_{si}\psi_0 + c_2^{(1D)}(|\psi_{-1}|^2 + |\psi_{+1}|^2)\psi_0 + 2c_2^{(1D)}\psi_{-1}\psi_0^*\psi_{+1}, \tag{2}$$

where the asterisk denotes the complex conjugate and $\hat{H}_{si} \equiv -(\hbar^2/2m)\partial_X^2 + (1/2)m\omega_X^2 X^2 + c_0^{(1D)} n_{tot}$ is the spin-independent part of the Hamiltonian, with $n_{tot} = |\psi_{-1}|^2 + |\psi_0|^2 + |\psi_{+1}|^2$ being the total density (m is the atomic mass). The nonlinearity coefficients have an effectively 1D form, namely, $c_0^{(1D)} = c_0/2\pi a_\perp^2$ and $c_2^{(1D)} = c_2/2\pi a_\perp^2$, where $a_\perp = \sqrt{\hbar/m\omega_\perp}$ is the transverse harmonic-oscillator length, which defines the size of the transverse ground state. Finally, coupling constants c_0 and c_2 which account, respectively, for spin-independent and spin-dependent collisions between identical spin-1 bosons are given by (in the mean-field approximation)

$$c_0 = \frac{4\pi\hbar^2(a_0 + 2a_2)}{3m}, \quad c_2 = \frac{4\pi\hbar^2(a_2 - a_0)}{3m}, \tag{3}$$

where a_0 and a_2 are the s-wave scattering lengths in the symmetric channels with total spin of the colliding atoms $F=0$ and $F=2$, respectively. Note that the $F=1$ spinor condensate may be either ferromagnetic (such as the ^{87}Rb), characterized by $c_2 < 0$, or polar (such as the ^{23}Na), with $c_2 > 0$.

Measuring time, length, and density in units of $\hbar/c_0^{(1D)}n_0$, $\hbar/\sqrt{mc_0^{(1D)}n_0}$, and n_0, respectively (where n_0 is the peak density), the authors of Ref. [10] cast Eqs. (1) and (2) in the dimensionless form

$$i\partial_T\psi_{\pm 1} = H_{si}\psi_{\pm 1} + \delta(|\psi_{\pm 1}|^2 + |\psi_0|^2 - |\psi_{\mp 1}|^2)\psi_{\pm 1} + \delta\psi_0^2\psi_{\mp 1}^*, \tag{4}$$

$$i\partial_T\psi_0 = H_{si}\psi_0 + \delta(|\psi_{-1}|^2 + |\psi_{+1}|^2)\psi_0 + 2\delta\psi_{-1}\psi_0^*\psi_{+1}, \tag{5}$$

where $H_{si} \equiv -(1/2)\partial_X^2 + (1/2)\Omega_{tr}^2 X^2 + n_{tot}$, the normalized harmonic trap strength is given by

$$\Omega_{tr} = \frac{\omega_x}{\omega_\perp}\frac{3}{2(a_0 + 2a_2)n_0}, \tag{6}$$

and the small physical parameter is defined by

$$\delta \equiv \frac{c_2^{(1D)}}{c_0^{(1D)}} = \frac{a_2 - a_0}{a_0 + 2a_2}. \tag{7}$$

In the cases of ^{87}Rb and ^{23}Na atoms with $F=1$, $\delta = -4.66 \times 10^{-3}$ [12], and $\delta = +3.14 \times 10^{-2}$ [13], respectively, i.e., in either case, δ plays the role of a small parameter of Eqs. (4) and (5).

Generally, Eqs. (4) and (5) give rise to spin-mixing states. However, there also exist nonspin mixing, or spin-polarized states, which are stable stationary solutions of Eqs. (4) and (5). Consider the spatially homogeneous system ($\Omega_{tr}=0$) and focus on such solutions having at least one component equal to zero, the remaining ones being continuous waves. The corresponding exact stationary solutions are

$$\psi_{-1} = \psi_{+1} = \sqrt{\frac{\mu}{2}}\exp(-i\mu t), \quad \psi_0 = 0 \tag{8}$$

and

$$\psi_{-1} = \psi_{+1} = 0, \quad \psi_0 = \sqrt{\mu}\exp(-i\mu t). \tag{9}$$

Starting from Eqs. (4) and (5) and using a multidimensional analysis, the authors of Ref. [10] derived the YO equations in the context of BECs in the following form:

$$\partial_{\tilde{t}} n = -\partial_{\tilde{x}}|F|^2, \tag{10}$$

$$i\partial_{\tilde{t}} F + \frac{1}{2}\partial_{\tilde{x}}^2 F = nF. \tag{11}$$

Where $\tilde{t} = \delta T$, $\tilde{x} = \sqrt{\delta}(X - \sqrt{\mu}T)$, $\sqrt{\mu}$ is the speed of sound, F is the high-frequency wave amplitude corresponding to the spin projection $m_F=0$, and n is the amplitude of the low-frequency wave corresponding to the spin projections $m_F = +1, -1$ [10].

Let us choose the following shifted coordinates:

$$F = F(x,t), \quad n = n(x,t),$$

$$t = \tilde{t}, \quad x = \tilde{x} + \tilde{t}, \tag{12}$$

and rewrite the system (10) and (11) as follows:

$$i\frac{\partial F}{\partial t} + i\frac{\partial F}{\partial x} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = nF, \tag{13}$$

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial x} = -\frac{\partial |F|^2}{\partial x}. \tag{14}$$

The purpose of this study is to solve the YO Eqs. (13) and (14) with a mixture of vanishing and nonzero boundary values by using the ISTM based on solution of the Riemann-Hilbert problem (RHP). The present paper will consider a case of the nonzero boundary values for the low-frequency wave $n(x,t)$.

The paper is organized as follows. In Sec. II, we develop the direct-scattering problem. Section II A presents the transformed zero curvature presentation of the YO system with nonvanishing boundary conditions. In Sec. II B, an adjoint scattering problem, which provides two additional analytic solutions of the original scattering problem, is introduced. The symmetry and analytical properties of the Jost functions are described in Secs. II C and II D, respectively. Solution of the inverse problem is presented in Sec. III. In Sec. IV, we present examples of management of the high-frequency wave by changing initial and boundary values associated with the low-frequency wave. The respective RHP is formulated and solved in the Appendix.

II. DIRECT PROBLEM

A. Linear systems

The zero curvature presentation of systems (13) and (14) is derived by Yajima and Oikawa [14,15] in the following form:

$$\frac{\partial \mathbf{\Theta}}{\partial x} = (i\mathbf{J} + \mathbf{Q})\mathbf{\Theta}, \tag{15}$$

where

$$\mathbf{J}_0 = \lambda \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

$$\mathbf{Q} = \frac{i}{2\lambda} \begin{pmatrix} n & 2i\lambda \bar{F} & n \\ iF & 0 & iF \\ -n & -2i\lambda \bar{F} & -n \end{pmatrix}, \tag{17}$$

and

$$\frac{\partial \mathbf{\Theta}}{\partial t} = \mathbf{V}\mathbf{\Theta} \coloneqq \begin{pmatrix} i(2\lambda^2/3 + 2\lambda) & 0 & 0 \\ 0 & -4i\lambda^2/3 & 0 \\ 0 & 0 & i(2\lambda^2/3 - 2\lambda) \end{pmatrix} \mathbf{\Theta} \tag{18}$$

$$-\frac{i}{2\lambda} \begin{pmatrix} n + \frac{|F|^2}{2} & 2i\lambda\left(\frac{i}{2}\partial_x \bar{F} + \bar{F} + \lambda \bar{F}\right) & n + \frac{|F|^2}{2} \\ iF + \frac{1}{2}\partial_x F + i\lambda F & 0 & iF + \frac{1}{2}\partial_x F - i\lambda F \\ -n - \frac{|F|^2}{2} & -2i\lambda\left(\frac{i}{2}\partial_x \bar{F} + \bar{F} - \lambda \bar{F}\right) & -n - \frac{|F|^2}{2} \end{pmatrix} \mathbf{\Theta}, \tag{19}$$

where $\mathbf{\Theta}(x,t;\lambda)$ is the 3×3 matrix-valued eigenfunction, λ is the spectral parameter, and the overbar signifies complex conjugation.

Throughout this work, we consider potentials with the same space- and time-independent amplitudes at both space infinities

$$F(x,t) = 0, \quad n(x,t) = 2q, \quad x \to \pm\infty, \tag{20}$$

where $q = \pm r$ and $r = |q|$. Except for Sec. IV where a particular case of time-dependent parameter q is considered.

Let us rewrite system (15)–(17) as follows:

$$\frac{\partial \mathbf{\Theta}}{\partial x} = (i\mathbf{J_0} + \mathbf{Q_0})\mathbf{\Theta} + (\mathbf{Q} - \mathbf{Q_0})\mathbf{\Theta}, \tag{21}$$

where

$$\mathbf{Q}_0 = \mathbf{Q}(x \to \pm\infty) = \frac{i}{\lambda} \begin{pmatrix} q & 0 & q \\ 0 & 0 & 0 \\ -q & 0 & -q \end{pmatrix}. \tag{22}$$

Define the function $\mathbf{\Phi}$ as

$$\mathbf{\Phi} = \mathbf{D}\mathbf{\Theta} \coloneqq \begin{pmatrix} 1 & 0 & \frac{q}{\lambda(2\lambda + \Lambda) - q} \\ 0 & 1 & 0 \\ \frac{q}{\lambda(2\lambda + \Lambda) - q} & 0 & 1 \end{pmatrix} \mathbf{\Theta}, \tag{23}$$

where $\Lambda = 2\sqrt{\lambda^2 - q}$.

Let us add to the right-hand side (RHS) of system (21) the term $i\lambda \mathbf{I}\mathbf{\Phi}$, where $\mathbf{I}$ is the unite matrix. Then, introducing the spectral parameter ζ such that

$$\Lambda = \left(\zeta - \frac{q}{\zeta}\right), \quad \lambda = \frac{1}{2}\left(\zeta + \frac{q}{\zeta}\right), \tag{24}$$

we rewrite system (21) in the form

$$\frac{\partial \mathbf{\Phi}(x;\zeta)}{\partial x} = [i\mathbf{\Omega}(\zeta) + \mathbf{R}(\zeta)]\mathbf{\Phi}(x;\zeta), \tag{25}$$

where

$$\mathbf{\Omega}(\zeta) = \begin{pmatrix} -\Lambda(\zeta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda(\zeta) \end{pmatrix}, \tag{26}$$

$$\mathbf{R}(\zeta) = \begin{pmatrix} \frac{i\zeta(n-2q)}{\zeta^2 - q} & -\frac{\zeta^2 \bar{F}}{\zeta^2 - q} & \frac{i\zeta(n-2q)}{\zeta^2 - q} \\ -\frac{F}{\zeta} & 0 & -\frac{F}{\zeta} \\ -\frac{i\zeta(n-2q)}{\zeta^2 - q} & \frac{\zeta^2 \bar{F}}{\zeta^2 - q} & -\frac{i\zeta(n-2q)}{\zeta^2 - q} \end{pmatrix}. \tag{27}$$

We define for real $\zeta \neq 0, \pm\sqrt{q}$ Jost functions $\mathbf{\Phi}_{\pm}(x;\zeta) = \{\phi_{\pm}^{(1)}, \phi_{\pm}^{(2)}, \phi_{\pm}^{(3)}\}$ which are the solution of system (25) with the asymptotics

$$\mathbf{\Phi}_{\pm}(x;\zeta) \to \mathbf{J}(x;\zeta) = \begin{pmatrix} e^{-i\Lambda x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\Lambda x} \end{pmatrix}, \quad x \to \pm\infty. \tag{28}$$

Two such solutions of system (25) ϕ_i and ψ_i are related by the scattering matrix $\mathbf{S} \coloneqq \{s_{jk}\}$,

$$\phi_{-}^{(j)}(x;\zeta) = \sum_{k=1}^{3} s_{jk}(\zeta)\phi_{+}^{(k)}(x;\zeta) \tag{29}$$

or

$$\mathbf{\Phi}_{-}(x;\zeta) = \mathbf{\Phi}_{+}(x;\zeta)\mathbf{S}^{\mathbf{T}}, \tag{30}$$

where $\mathbf{S}^T \coloneqq \{s_{kj}\}$.

Inverting relation (30), we obtain

$$\mathbf{\Phi}_{+}(x;\zeta) = \mathbf{\Phi}_{-}(x;\zeta)[\mathbf{S}^T]^{-1}. \tag{31}$$

Introduce the eigenfunctions $\mathbf{\Psi}_{\pm}(x;\zeta)=\{\psi_{\pm}^{(1)},\psi_{\pm}^{(2)},\psi_{\pm}^{(3)}\}$,

$$\mathbf{\Psi}_{\pm}(x;\zeta) = \mathbf{J}(x;\zeta)\mathbf{\Phi}_{\pm}(x;\zeta)\mathbf{J}^{-1}(x;\zeta), \tag{32}$$

and the scattering matrix $\mathbf{T}(x;\zeta)$,

$$\mathbf{T}(x;\zeta) = \mathbf{J}(x;\zeta)\mathbf{S}^T(x;\zeta)\mathbf{J}^{-1}(x;\zeta). \tag{33}$$

From Eqs. (30), (32), and (33) we obtain

$$\mathbf{\Psi}_{\pm}(x;\zeta) \to \mathbf{I}, \quad |x| \to \infty, \tag{34}$$

$$\det \mathbf{T}(x;\zeta) = \det \mathbf{S} = 1, \tag{35}$$

$$\det \mathbf{\Psi}_{\pm}(x;\zeta) = 1. \tag{36}$$

As a result, instead of Eqs. (30) and (31), we have

$$\mathbf{\Psi}_{-}(x;\zeta) = \mathbf{\Psi}_{+}(x;\zeta)\mathbf{T}, \quad \mathbf{\Psi}_{+}(x;\zeta) = \mathbf{\Psi}_{-}(x;\zeta)\mathbf{T}^{-1}. \tag{37}$$

B. Adjoint problem and auxiliary eigenfunctions

In order to formulate and solve the inverse scattering problem, one needs to construct two independent sets of analytic eigenfunctions. However, unlike some classical integrable equations, such as the nonlinear Schrödinger equations, which is solved by using a 2×2 spectral problem [16], analytical properties of solutions of the spectral problem (25) are not self-consistent for this purpose. For instance, the function $\phi^{(2)}$ is analytic neither on the upper nor on the lower sheet of the complex plane ζ.

Method of solving the 3×3 spectral problems was first developed by Zakharov and Manakov [25]. Their approach introduced for investigating the three-wave interaction is used here in order to obtain a representation of the nonanalytic eigenfunctions in terms of analytic eigenfunctions and scattering data. The key idea is to consider the "adjoint" eigenvalue problem. Following to [25] (see also [26] and [27]), in addition to system (25), we consider the adjoint eigenvalue problem

$$\frac{\partial \tilde{\mathbf{\Phi}}}{\partial x} = -[i\mathbf{\Omega}(\zeta) + \mathbf{R}^T(\zeta)]\tilde{\mathbf{\Phi}}. \tag{38}$$

Define for a real ζ, $\zeta \neq 0, \pm\sqrt{q}$, the Jost functions $\tilde{\mathbf{\Phi}}_{\pm}(x,t;\zeta)=\{\tilde{\phi}_{\pm}^{(1)},\tilde{\phi}_{\pm}^{(2)},\tilde{\phi}_{\pm}^{(3)}\}$ solutions for system (38) with the asymptotics

$$\tilde{\mathbf{\Phi}}_{\pm}(x;\zeta) \to \mathbf{J}^{-1}(x;\zeta), \quad x \to \pm\infty \tag{39}$$

and the Jost functions $\tilde{\mathbf{\Psi}}_{\pm}(x;\zeta)=\{\tilde{\psi}_{\pm}^{(1)},\tilde{\psi}_{\pm}^{(2)},\tilde{\psi}_{\pm}^{(3)}\} = \tilde{\mathbf{\Phi}}_{\pm}(x;\zeta)\mathbf{J}(x;\zeta)$ with the asymptotics

$$\tilde{\mathbf{\Psi}}_{\pm}(x;\zeta) \to \mathbf{I}, \quad x \to \pm\infty. \tag{40}$$

One may find that the following relations hold:

$$\tilde{\mathbf{\Phi}}_{-}(x;\zeta) = \tilde{\mathbf{\Phi}}_{+}(x;\zeta)\tilde{\mathbf{S}}^T(\zeta), \tag{41}$$

$$\tilde{\mathbf{\Psi}}_{-}(x;\zeta) = \tilde{\mathbf{\Psi}}_{+}(x;\zeta)\tilde{\mathbf{T}}(\zeta), \tag{42}$$

$$\tilde{\mathbf{\Psi}}_{+}(x;\zeta) = \tilde{\mathbf{\Psi}}_{-}(x;\zeta)\tilde{\mathbf{T}}^{-1}(\zeta), \tag{43}$$

$$\tilde{\mathbf{T}}(\zeta) = \mathbf{J}^{-1}(x;\zeta)\tilde{\mathbf{S}}^T\mathbf{J}(x;\zeta). \tag{44}$$

Solutions of original (25) and adjoint equations (38) satisfy to the conditions

$$\tilde{\mathbf{\Psi}}_{\pm}^T(x;\zeta)\mathbf{\Psi}_{\pm}(x;\zeta) = \tilde{\mathbf{\Phi}}_{\pm}^T(x;\zeta)\mathbf{\Phi}_{\pm}(x;\zeta) = \mathbf{I}. \tag{45}$$

Comparing Eqs. (37) with Eqs. (42) and (43) and by means of Eq. (45), we obtain

$$\tilde{\mathbf{T}}_{\pm}^T(\zeta)\mathbf{T}_{\pm}(\zeta) = \mathbf{I}, \tag{46}$$

$$\mathbf{T}(\zeta) = \tilde{\mathbf{\Psi}}_{+}^T(x;\zeta)\mathbf{\Psi}_{-}(x;\zeta), \tag{47}$$

$$\tilde{\mathbf{T}}(\zeta) = \mathbf{\Psi}_{+}^T(x;\zeta)\tilde{\mathbf{\Psi}}_{-}(x;\zeta), \tag{48}$$

for two arbitrary vector functions $u^{(j)}=\{u_1^{(j)},u_2^{(j)},u_3^{(j)}\}^T$ and $v^{(m)}=\{v_1^{(m)},v_2^{(m)},v_3^{(m)}\}^T$ denote the standard vector product as

$$u^{(j)}v^{(m)} = \{u_2^{(j)}v_3^{(m)} - u_3^{(j)}v_2^{(m)}, u_3^{(j)}v_1^{(m)} - u_1^{(j)}v_3^{(m)}, u_1^{(j)}v_2^{(m)} - u_2^{(j)}v_1^{(m)}\}^T.$$

To construct complete set of the eigenfunctions following Kaup [26], we introduce the auxiliary vector functions in the form

$$\chi_{+}(x;\zeta) = \tilde{\phi}_{+}^{(3)}\tilde{\phi}_{-}^{(1)}, \quad \chi_{-}(x;\zeta) = \tilde{\phi}_{-}^{(3)}\tilde{\phi}_{+}^{(1)}, \tag{49}$$

$$\tilde{\chi}_{+}(x;\zeta) = \phi_{-}^{(3)}\phi_{+}^{(1)}, \quad \tilde{\chi}_{-}(x;\zeta) = \phi_{+}^{(3)}\phi_{-}^{(1)}. \tag{50}$$

One can directly verify that Eqs. (49) and (50) are solutions to the spectral problems (25) and (38), respectively.

C. Symmetries

Let us define the automorphisms $(\hat{G})_k(\zeta)$, which act on the set fundamental solutions $\{\mathbf{\Phi}(x,\zeta)\}$ of the system (25)

$$\hat{G}_k(\zeta)\mathbf{\Phi}(x,\zeta) \coloneqq \mathbf{D}_k\mathbf{\Phi}(x,g_k(\zeta))\mathbf{D}_k^{-1} \in \{\mathbf{\Phi}(x,\zeta)\}. \tag{51}$$

Where $g_k(\zeta)$ is a fractional-linear transform of ζ and $\mathbf{D}_k(\zeta)$ is a matrix operator, which are determined by the properties of symmetry of Eqs. (25).

The symmetry transform

$$g_1(\zeta) = -\frac{q}{\zeta} \tag{52}$$

does not change analytical properties of eigenfunctions and corresponds to the following automorphism:

$$\mathbf{\Phi}(x;g_1(\zeta)) = \mathbf{D}_1(\zeta)\mathbf{\Phi}(x;\zeta)\mathbf{D}_1^{-1}(\zeta), \tag{53}$$

where

$$\mathbf{D}_1(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\zeta^2}{q} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{54}$$

Respective transform of the matrix $\mathbf{S}$ is

$$\mathbf{S}(g_1(\zeta)) = \mathbf{D}_1^{-1}(\zeta)\mathbf{S}(\zeta)\mathbf{D}_1(\zeta). \tag{55}$$

The parity transform

$$g_2(\zeta) = -\zeta \tag{56}$$

corresponds to the symmetries

$$\mathbf{\Phi}(x;-\zeta) = \mathbf{D}_2\mathbf{\Phi}(x;\zeta)\mathbf{D}_2^{-1} \tag{57}$$

and

$$\mathbf{S}(-\zeta) = \mathbf{D}_2^{-1}(\zeta)\mathbf{S}(\zeta)\mathbf{D}_2(\zeta), \tag{58}$$

where

$$\mathbf{D}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{59}$$

The third automorphism is

$$\mathbf{\Phi}(g_3(\zeta)) = \mathbf{D}_3(\zeta)\mathbf{\Phi}(\zeta)\mathbf{D}_3^{-1}(\zeta), \tag{60}$$

where $g_3(\zeta)=g_2(g_1(\zeta))=q/\zeta$ and $\mathbf{D}_3(\zeta)=\mathbf{D}_2\mathbf{D}_1(\zeta)$.

Using Eqs. (25) and (38), we reveal that $\overline{\mathbf{\Phi}(\bar{\zeta})}$ can be transformed to the solution for the adjoint equations as follows:

$$\overline{\mathbf{\Phi}(\bar{\zeta})} = \tilde{\mathbf{D}}^{-1}(\zeta)\tilde{\mathbf{\Phi}}(\zeta)\tilde{\mathbf{D}}(\zeta), \tag{61}$$

where

$$\tilde{\mathbf{D}}(\zeta) = \mathrm{diag}\{d_1,d_2,d_3\} = \mathrm{diag}\left\{-1,\frac{\zeta^3}{(\zeta^2-q)},1\right\}. \tag{62}$$

Substituting the asymptotics (34) and (40) in Eqs. (30) and (41) and using relation (61), we derive

$$\tilde{s}_{nm}(\zeta) = \frac{d_n}{d_m}\overline{s_{mn}(\bar{\zeta})}. \tag{63}$$

Thus, by means of relations (55), (58), and (63), we obtain

$$\tilde{s}_{21}(\bar{\zeta}) = -\overline{d_2(\zeta)s_{12}(\zeta)} = -\frac{\bar{\zeta}^3}{\bar{\zeta}^2-q}\overline{s_{12}(\zeta)}, \tag{64}$$

$$\tilde{s}_{23}(-\bar{\zeta}) = \overline{d_2(\zeta)s_{12}(\zeta)} = \frac{\bar{\zeta}^3}{\bar{\zeta}^2-q}\overline{s_{12}(\zeta)}, \tag{65}$$

$$\tilde{s}_{21}(-q/\bar{\zeta}) = \frac{q\bar{\zeta}}{\bar{\zeta}^2-q}\overline{s_{12}(\zeta)}, \tag{66}$$

$$\tilde{s}_{23}(q/\bar{\zeta}) = -\frac{q\bar{\zeta}}{\bar{\zeta}^2-q}\overline{s_{12}(\zeta)}. \tag{67}$$

Automorphisms of the adjoint functions that operate on a set of fundamental solutions $\{\tilde{\mathbf{\Phi}}(x;\zeta)\}$ of systems (38) can be obtained through the symmetry transforms obtained above and relation (61).

D. Analyticity

Using asymptotics (28) and (39) and relations (29) and (31), we obtain

$$s_{11}(\zeta) = \lim_{x\to\infty}\psi_{1-}^{(1)}(x;\zeta), \quad s_{33}(\zeta) = \lim_{x\to\infty}\psi_{3-}^{(3)}(x;\zeta), \tag{68}$$

$$\tilde{s}_{11}(\zeta) = \lim_{x\to-\infty}\psi_{1+}^{(1)}(x;\zeta), \quad \tilde{s}_{33}(\zeta) = \lim_{x\to-\infty}\psi_{3+}^{(3)}(x;\zeta). \tag{69}$$

Using standard methods (see, e.g., [17]), one can show that $\psi_+^{(1)}(x;\zeta)$, $\psi_-^{(3)}(x;\zeta)$, $\tilde{\psi}_-^{(1)}(x;\zeta)$, $\tilde{\psi}_+^{(3)}(x;\zeta)$, $s_{33}(\zeta)$, and $\tilde{s}_{11}(\zeta)$ are analytic functions of ζ on the lower half plane ($\mathrm{Im}\,\zeta<0$) while $\psi_-^{(1)}(x;\zeta)$, $\psi_+^{(3)}(x;\zeta)$, $\tilde{\psi}_+^{(1)}(x;\zeta)$, $\tilde{\psi}_-^{(3)}(x;\zeta)$, $s_{11}(\zeta)$, and $\tilde{s}_{33}(\zeta)$ are analytic functions of ζ on the upper half plane ($\mathrm{Im}\,\zeta>0$).

Asymptotics of the Jost function we derive from the spectral problem (25) are

$$\psi_{j+}^{(1)}(x;\zeta) = \delta_{j1} + O(|\zeta|^{-1}), \quad \mathrm{Im}\,\zeta<0, \ |\zeta|\to\infty, \tag{70}$$

$$\psi_{j-}^{(3)}(x;\zeta) = \delta_{j3} + O(|\zeta|^{-1}), \quad \mathrm{Im}\,\zeta<0, \ |\zeta|\to\infty, \tag{71}$$

$$\psi_{j-}^{(1)}(x;\zeta) = \delta_{j1} + O(|\zeta|^{-1}), \quad \mathrm{Im}\,\zeta>0, \ |\zeta|\to\infty, \tag{72}$$

$$\psi_{j+}^{(3)}(x;\zeta) = \delta_{j3} + O(|\zeta|^{-1}), \quad \mathrm{Im}\,\zeta>0, \ |\zeta|\to\infty, \tag{73}$$

where δ_{jk} is the delta function. If $q\neq0$, then the same asymptotics exist for $|\zeta|\to0$.

From Eqs. (68)–(73) we get

$$s_{11}(\zeta) = 1 + O(|\zeta|^{-1}), \ \tilde{s}_{33}(\zeta) = 1 + O(|\zeta|^{-1}), \quad \mathrm{Im}\,\zeta>0, \tag{74}$$

$$s_{33}(\zeta) = 1 + O(|\zeta|^{-1}), \ \tilde{s}_{11}(\zeta) = 1 + O(|\zeta|^{-1}), \quad \mathrm{Im}\,\zeta<0. \tag{75}$$

Starting from analytical properties of functions on the RHS of Eqs. (49) and (50), one can show that $\chi_-(x;\zeta)$, $\tilde{\chi}_-(x;\zeta)$ and $\chi_+(x;\zeta)$, $\tilde{\chi}_+(x;\zeta)$ are analytical in $\mathrm{Im}\,\zeta<0$ and $\mathrm{Im}\,\zeta>0$, respectively.

Asymptotics for $|\zeta|\to\infty$ of $\chi_\pm(x;\lambda)$, we find from Eqs. (70)–(75)

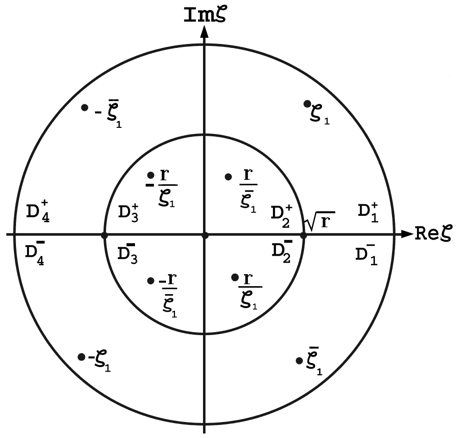

FIG. 1. The complex ζ plane. The inner circle has the radius $\sqrt{r}$ and the outer circle has an infinite radius. Domains $D_k^{\pm}$ are placed between the intervals lying on the axis and the quarters of cycles. Contour $\Gamma_k^+(\Gamma_k^-)$ runs along the boundary of domain $D_k^+(D_k^-)$ in the clockwise (counterclockwise) direction.

$$\chi_{\pm}(x;\zeta) = \pm \delta_{j2} + O(|\zeta|^{-1}), \quad \text{Im} \pm \zeta > 0. \tag{76}$$

Denote $r=|q|$ and define the following domains of the complex plane ζ:

$$D_1^+ = \{\text{Im}\,\zeta > 0 \cap \text{Re}\,\zeta > 0 \cap |\zeta|^2 > r\}, \tag{77}$$

$$D_2^+ = \{\text{Im}\,\zeta > 0 \cap \text{Re}\,\zeta > 0 \cap |\zeta|^2 < r\}, \tag{78}$$

$$D_3^+ = \{\text{Im}\,\zeta > 0 \cap \text{Re}\,\zeta < 0 \cap |\zeta|^2 < r\}, \tag{79}$$

$$D_4^+ = \{\text{Im}\,\zeta > 0 \cap \text{Re}\,\zeta < 0 \cap |\zeta|^2 > r\}, \tag{80}$$

$$D_1^- = \{\text{Im}\,\zeta < 0 \cap \text{Re}\,\zeta > 0 \cap |\zeta|^2 > r\}, \tag{81}$$

$$D_2^- = \{\text{Im}\,\zeta < 0 \cap \text{Re}\,\zeta > 0 \cap |\zeta|^2 < r\}, \tag{82}$$

$$D_3^- = \{\text{Im}\,\zeta < 0 \cap \text{Re}\,\zeta < 0 \cap |\zeta|^2 < r\}, \tag{83}$$

$$D_4^+ = \{\text{Im}\,\zeta < 0 \cap \text{Re}\,\zeta < 0 \cap |\zeta|^2 > r\}, \tag{84}$$

which are shown in Fig. 1.

Then denote the following pairs of these domains:

$$D^+ = \begin{cases} D_1^+ \cup D_3^+, & r=q \\ D_1^+ \cup D_2^-, & r=-q, \end{cases} \tag{85}$$

$$\tilde{D}^+ = \begin{cases} D_3^+ \cup D_4^+ & r=q \\ D_3^- \cup D_4^+, & r=-q, \end{cases} \tag{86}$$

$$D^- = \begin{cases} D_2^- \cup D_4^-, & r=q \\ D_2^+ \cup D_4^-, & r=-q, \end{cases} \tag{87}$$

$$\tilde{D}^- = \begin{cases} D_1^- \cup D_3^-, & r=q \\ D_1^- \cup D_3^+, & r=-q. \end{cases} \tag{88}$$

Next we show that the eigenfunction $\phi_+^{(1)}(x;\zeta)$ is defined in D^+. For this purpose, we construct three linear combinations of the Jost function components

$$\varphi_1(x;\zeta) = [\phi_{+1}^{(1)}(x;\zeta) + \phi_{+3}^{(1)}(x;\zeta)], \tag{89}$$

$$\varphi_2(x;\zeta) = \phi_{+2}^{(1)}(x;\zeta), \tag{90}$$

$$\varphi_3(x;\zeta) = [\phi_{+1}^{(1)}(x;\zeta) - \phi_{+3}^{(1)}(x;\zeta)]. \tag{91}$$

Excluding the function φ_1 from system (25) and integrating the obtained equation, we find

$$\left[i\varphi_3 \frac{\partial \bar{\varphi}_3}{\partial x} - i\bar{\varphi}_3 \frac{\partial \varphi_3}{\partial x} + |\varphi_2|^2 \right] \Bigg|_{-\infty}^{\infty} (\zeta) = i(\bar{\Lambda}^2 - \Lambda^2) \int_{-\infty}^{\infty} |\varphi_2|^2 dx. \tag{92}$$

Substituting asymptotics of $\phi_+^{(1)}(x;\zeta)$ in Eq. (92), we derive the restriction to the spectral parameter $\text{Re}\,\Lambda > 0$. Thus, the eigenfunction $\phi_+^{(1)}(x;\zeta)$ is defined in domain D^+. Using the symmetry (57), one can show that $\phi_+^{(3)}(x;\zeta)$ is defined in domain D^-. Relations analogous to Eqs. (89)–(92) may be derived for the adjoint eigenfunctions as well. It may be shown, for instance, that $\phi_+^{(1)}(\phi_+^{(3)})$ is defined in $\tilde{D}^+(\tilde{D}^-)$.

E. Time dependence

In this section, we present time evolution of the scattering data we derive for $q(t)=\text{const}$ by using the transformed linear system (19)

$$\frac{\partial \mathbf{\Phi}}{\partial t} = \mathbf{D}^{-1}\mathbf{V}\mathbf{D}\mathbf{\Phi}. \tag{93}$$

In the limit $|x| \to \infty$, the matrix in the RHS of Eq. (93) takes the diagonal form

$$\mathbf{D}^{-1}\mathbf{V}\mathbf{D}|_{|\mathbf{x}|\to\infty} = \begin{pmatrix} \frac{2i\lambda^2}{3} + i\Lambda & 0 & 0 \\ 0 & -\frac{4i\lambda^2}{3} & 0 \\ 0 & 0 & \frac{2i\lambda^2}{3} - i\Lambda \end{pmatrix}. \tag{94}$$

Using the Eqs. (93) and (94) and relation of completeness (29), we find the following t dependence of the coefficients

$$s_{12}(t;\zeta) = s_{12}(0;\zeta)\exp[-it(2\lambda^2 + \Lambda)],$$
$$s_{13}(t;\zeta) = s_{13}(0;\zeta)e^{-2it\Lambda},$$

$$s_{21}(t;\zeta) = s_{21}(0;\zeta)\exp[it(2\lambda^2 + \Lambda)], \quad s_{31}(t;\zeta) = s_{31}(0;\zeta)e^{2it\Lambda},$$

$$s_{23}(t;\zeta) = s_{23}(0;\zeta)\exp[it(2\lambda^2 - \Lambda)],$$
$$s_{32}(t;\zeta) = s_{32}(0;\zeta)\exp[-it(2\lambda^2 - \Lambda)], \tag{95}$$

where λ and Λ are defined by Eq. (24). Diagonal elements $a_{ii}(\zeta)$ of matrix $\mathbf{S}$ and the spectral parameter ζ do not depend on t.

III. SOLUTION OF THE INVERSE PROBLEM

To solve the inverse problem, we need to find relations between $\phi^{(k)}(x,\zeta)$ and "potential" $F(x)$, $n(x)$. For this aim, we determine the asymptotic behavior of the eigenfunctions for large values of the scattering parameter. Using Eq. (A41), we derive as $|\zeta|\to\infty$,

$$\phi^{(1)}(x;\zeta)e^{i\Lambda x}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\sum_{n=1}^{N}\sum_{k=2,3}\begin{pmatrix}\zeta/\zeta_n\\1\\\zeta/\zeta_n\end{pmatrix}\frac{2\lambda_n C_k(t;\zeta_n)\exp(ix\Lambda_n)}{\zeta^2}\phi_k(x;\zeta_n)+O\begin{pmatrix}|\zeta|^{-2}\\|\zeta|^{-3}\\|\zeta|^{-2}\end{pmatrix}, \tag{96}$$

where $\lambda_n=\frac{1}{2}(\zeta_n+\frac{q}{\zeta_n})$, $\Lambda_n=\zeta_n-\frac{q}{\zeta_n}$, and

$$C_k(t;\zeta_n)=\frac{s_{1k}(t;\zeta_n)}{\partial_\zeta s_{11}(\zeta)|_{\zeta=\zeta_n}},\quad k=2,3,\quad n=1,2,\ldots,N. \tag{97}$$

On the other hand, by iteration we get from Eq. (25)

$$\phi_2^{(1)}(x;\zeta)e^{i\Lambda x}=-\frac{i}{2\zeta^2}F(x)+O\left(\frac{1}{|\zeta|^2}\right). \tag{98}$$

By comparing Eqs. (96) and (98), we obtain

$$F(x)=\sum_{n=1}^{N}\sum_{k=2,3}4i\lambda_n\exp(ix\Lambda_n)C_k(t;\zeta_n)\phi_k(x;\zeta_n). \tag{99}$$

Amplitude of the low-frequency wave $n(x)$ can be found by similar way as well as by using Eq. (14).

We consider here solutions associated with a set of nondegenerate poles in the corresponding integral equations. These integral equations, representing a solution to the RHP problem, are derived in the Appendix.

Let $s_{11}(\zeta)$ have N nondegenerate zeros ζ_n, $n=1,2,\ldots,N$, lying in domain D^+, i.e.,

$$s_{11}(\zeta_n)=0,\quad \zeta_n\in D^+. \tag{100}$$

Using the symmetry properties of scattering coefficients (see Sec. II C), we obtain

$$s_{33}(-\zeta_n)=\tilde{s}_{33}(-\bar{\zeta}_n)=\tilde{s}_{11}(\bar{\zeta}_n)=s_{33}(q/\zeta_n)=s_{11}(-q/\zeta_n)$$
$$=\tilde{s}_{33}(q/\bar{\zeta}_n)=\tilde{s}_{11}(-q/\bar{\zeta}_n)=0. \tag{101}$$

Calculating the residues in respective zeros on the RHS of Eq. (A41) and using the symmetries defined by the automorphism $\hat{G}_1$, we obtain the algebraic relation

$$\phi_1(x;\zeta)=e^{-i\Lambda x}\begin{pmatrix}1\\0\\0\end{pmatrix}-\sum_{n=1}^{N}\begin{pmatrix}\zeta/\zeta_n\\1\\\zeta/\zeta_n\end{pmatrix}[\alpha_n^{(2)}(\zeta)\phi_2(x;\zeta_n)+\alpha_n^{(3)}$$
$$\times(\zeta)\phi_3(x;\zeta_n)]\exp[ix(\Lambda_n-\Lambda)], \tag{102}$$

where

$$\alpha_n^{(k)}(\zeta)=\frac{(\zeta_n^2+q)C_k(t;\zeta_n)}{(\zeta_n-\zeta)(q+\zeta\zeta_n)},\quad k=2,3,\quad n=1,2,\ldots,N. \tag{103}$$

Using relations (65)–(67) and transforms $\hat{G}_2$ and $\hat{G}_3$, we derive from Eq. (A45)

$$\phi_2(x;\zeta)=\begin{pmatrix}0\\1\\0\end{pmatrix}+\sum_{n=1}^{N}\begin{pmatrix}\beta_n^{(1)}(\zeta)&0&\beta_n^{(3)}(\zeta)\\0&\beta_n^{(2)}(\zeta)&0\\\beta_n^{(3)}(\zeta)&0&\beta_n^{(1)}(\zeta)\end{pmatrix}\phi_1(x;\bar{\zeta}_n), \tag{104}$$

where

$$\beta_n^{(1)}(\zeta)=\frac{(q+\bar{\zeta}_n^2)[q(\bar{\zeta}_n-\zeta)+\zeta\bar{\zeta}_n^2]}{(\bar{\zeta}_n^2-q)(\zeta-\bar{\zeta}_n)(q+\bar{\zeta}_n\zeta)}\overline{C_2(t;\zeta_n)},\quad n=1,2,\ldots,N, \tag{105}$$

$$\beta_n^{(2)}(\zeta)=\frac{2\zeta^2\bar{\zeta}_1^2(q+\bar{\zeta}_n^2)}{(\bar{\zeta}_n^2-\zeta^2)(\bar{\zeta}_n^2\zeta^2-q^2)}\overline{C_2(t;\zeta_n)},\quad n=1,2,\ldots,N, \tag{106}$$

$$\beta_n^{(3)}(\zeta)=-\frac{q\bar{\zeta}_n(q-\bar{\zeta}_n^2)+\zeta(q^2+\bar{\zeta}_n^4)}{(\bar{\zeta}_n^2-q)(\zeta+\bar{\zeta}_n)(q-\bar{\zeta}_n\zeta)}\overline{C_2(t;\zeta_n)},$$
$$n=1,2,\ldots,N. \tag{107}$$

Finally, we obtain from Eq. (A42)

$$\phi_3(x;\zeta)=e^{ix\Lambda}\begin{pmatrix}0\\0\\1\end{pmatrix}+\sum_{n=1}^{N}\begin{pmatrix}0&0&\zeta/\zeta_n\\0&1&0\\\zeta/\zeta_n&0&0\end{pmatrix}[\gamma_n^{(2)}(\zeta)\phi_2(x;\zeta_n)$$
$$+\gamma_n^{(3)}(\zeta)\phi_3(x;\zeta_n)]\exp[ix(\Lambda-\Lambda_n)], \tag{108}$$

where

$$\gamma_n^{(k)}(\zeta)=\frac{(\zeta_n^2+q)C_k(t;\zeta_n)}{(\zeta_n+\zeta)(q-\zeta_n\zeta)},\quad k=2,3,\quad n=1,2,\ldots,N. \tag{109}$$

Since we require the symmetry conditions with respect to transforms $g_k(\zeta),k=1,2,3$, one can directly verify that Eqs.

(102), (104), and (108) preserve these symmetries due to the choice of zeros (101).

Solving the algebraic system (102)–(109) and using Eq. (99), we obtain for one pole $\zeta_1 \in D_1^+$,

$$F(x,t;\zeta_1) = \frac{4i\lambda_1 C_2(t;\zeta_n)}{\exp(-i\Lambda_1 x) + \gamma_1^{(3)}(t;\zeta_1)\exp(i\Lambda_1 x) + \beta_1^{(2)}(t;\zeta_1)\alpha_1^{(2)}(t;\bar{\zeta}_1)\exp(-i\bar{\Lambda}_1 x)}. \tag{110}$$

Denote $\zeta_1 = \nu + i\eta$ and $C_2(0;\zeta_1) = \exp(ix_1 + x_2)$, then we can rewrite solution (110) in the form

$$F(x,t) = \frac{4i[\nu(1+p) + i\eta(1-p)]\exp(iw_1 t + i[\nu(1-p)(x-t)] + ix_1)}{2b_1\cosh(\vartheta_+ - \ln b_1) + \gamma_0 \exp(2i[\nu(1-p)(x-t)] - \vartheta_-)}, \tag{111}$$

where

$$\vartheta_\pm = \eta(1+p)\{x - t[1 \pm \nu(1-p)]\} - x_2,$$

$$p = \frac{q}{\nu^2 + \eta^2}, \quad w_1 = \nu\eta(1-p^2),$$

$$b_1^2 = \frac{(\nu^2+\eta^2)(1+p^2) + 2p(\nu^2 - \eta^2)}{4\nu\eta^2(1-p^2)(1+p)},$$

$$\gamma_0 = \frac{q + \nu^2 - \eta^2 + 2i\nu\eta}{2(\nu + i\eta)(q - \nu^2 + \eta^2 - 2i\nu\eta)} C_3. \tag{112}$$

It may be proved that solution (111) is nonsingular. For the characteristic $\vartheta_+ = 0$, we find $\vartheta_- = 2t\eta\nu(1-p^2)$ and that soliton (111) does not disappear as $t \to \infty$ because $\eta\nu(1-p^2) > 0$ for any $\zeta_1 \in D^+$.

Let us $C_3(\zeta_1) = 0$. Then, the solitons have the conventional sech^2 shape for field $n(x,t)$ and sech shape for $F(x,t)$, which correspond to a gray soliton for components with spin projection $m_F = \pm 1$ and a bright soliton for $m_F = 0$. Solution for the low-frequency wave we derive from Eq. (14)

$$n(x,t;\zeta_1) = 2q + \frac{4[\nu^2(1+p)^2 + \eta^2(1-p)^2]|C_2|^2}{\nu(1-p)b_1^2\cos^2(\vartheta_+ - \ln b_1)}. \tag{113}$$

For a particular case $C_3 = 0$, $q \to 0$, but $\zeta_1 \to \text{const}$ soliton solutions (111) and (113) are reduced to the one-pole soliton solutions derived in Ref. [15].

IV. MANAGEMENT OF HIGH-FREQUENCY WAVE BY USING THE LOW-FREQUENCY WAVE

The authors of Ref. [10] tested the robustness of the derived vector soliton solutions in the case of a large normalized strength of the spin-dependent interaction, 1 order of magnitude larger than the value corresponding to the polar spinor condensate in sodium. They found that, although the solitons eventually get destroyed under such a strong perturbation, the lifetime of small- and moderate-amplitude solitons exceeds 300 ms, in physical units. Thus, the derived vector solitons of the YO equations have a good chance to be observed in experiments. From another side, these results demonstrated that lifetime of the solitons is large enough for observation of strong alteration of their parameters during evolution. In this section, we show that one can manage the parameters of a high-frequency wave by adjusting initial and boundary conditions associated with the low-frequency wave only. For this aim, we present two simple examples that support this statement and based on the ISTM technique developed in the present paper.

First, we show that the initial conditions of the low-frequency wave determine parameters of soliton (111). Proceed to the generation of solitons by simultaneous step change in the amplitude and the phase $n(x,0)$. Let $n(x,0) = 2q_0$ for $x < -L/2$, $n(x,0) = 0$ for $-L/2 < x < L/2$, and $n(x,0) = 2q_0\exp(i\phi_j)$ with $x > L/2$. Here, the phase jump ϕ_j is either 0 or π. Assume that $F(x,0) = 0$ for all x. Solving spectral problem (15) for these initial conditions, we find the equation, determining the eigenvalues λ_n,

$$\left(\frac{q_0}{\lambda_n} - 2\lambda_n\right)\tan\left[\left(\frac{q_0}{\lambda_n} - 2\lambda_n\right)L - \phi_j\right] = 2\sqrt{q_0 - \lambda_n^2}. \tag{114}$$

Consider the case of dark solitons with amplitude close to its maximum $\lambda_n = \sqrt{q_0} - \mu_n$, where $\mu_n \ll \sqrt{q_0}$. In this case, Eq. (114) has the approximate solution

$$\mu_n \approx \frac{\tan^2(\phi_j - \sqrt{q_0}L)}{8\sqrt{q_0}} \ll 1. \tag{115}$$

Generating a sequence of jumps of amplitude and phase of the low-frequency wave by the laser light [28], one can create a configuration of the high-frequency wave solitons.

Next, we show that fixing the time-depending boundary conditions for the low-frequency wave, one may control the trajectory and the amplitude of solitons (111). Let q depend on time variable t. Then the boundary conditions are defined by the following modification of system (93):

$$\frac{\partial \mathbf{\Phi}}{\partial t} = \mathbf{D}^{-1}(t)\mathbf{V}\mathbf{D}(\mathbf{t})\mathbf{\Phi} - \mathbf{D}^{-1}(\mathbf{t})\partial_t\mathbf{D}(\mathbf{t}), \tag{116}$$

where $\mathbf{D}$ now depends on t due to $q(t)$.

Consider a case of a dip or a spike occurring at the time being $t_0 > 0$, i.e., the function $q(t)$ has the following time dependence:

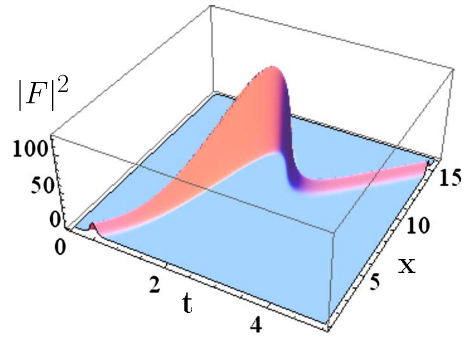

FIG. 2. (Color online) The intensity $|F|^2$ and of high-frequency wave soliton in the presence of spike in boundary conditions associated with the low-frequency wave in arbitrary units. $u=-v=6$, $m=t_0=2$, and $\zeta=-1+I$. The dimensionless variables x and t are related with the physical space and time variables by Eqs. (12).

$$q(t) = u - v\exp[-m(t-t_0)^2], \tag{117}$$

where u, v, and m are some real constants.

Assume for simplicity that the off-diagonal elements of the matrix $\{d'_{ij}\}=\mathbf{D}^{-1}(t)\partial_t\mathbf{D}(t)$ are small in comparison to the diagonal elements of this matrix, i.e.,

$$\epsilon = \frac{\zeta_1^2}{v - u\exp[-m(t-t_0)^2]} \ll 1, \tag{118}$$

where

$$\frac{d'_{11}}{d'_{13}} = \frac{d'_{33}}{d'_{31}} = \frac{d'_{11}}{d'_{31}} = \frac{1}{\epsilon}. \tag{119}$$

Note that in a general case, the off-diagonal terms give significant contribution to the radiation part of solution and may lead to soliton generation. However, even omitting the off-diagonal terms, we find that the time dependence of q leads to some nontrivial consequences, which may be revealed by using the ISTM.

Neglecting the off-diagonal elements of $\{d'_{ij}\}, i\neq j$, we found that the dependence of q on time (117) leads to the appearing of the following multiplier:

$$c_0(t) = \frac{[u - v\exp[-m(t-t_0)^2]]^2 - \zeta_1^4}{(u-v)^2 - \zeta_1^4} \tag{120}$$

to coefficient $C_2(t;\zeta_1)$.

In Fig. 2, the amplitude and the trajectory of soliton (111) in the presence of the spike (117) are shown. Without this spike, the trajectory of soliton is straightforward and the amplitude of soliton is constant.

These examples show that dependence of the soliton parameters on q allows one to control the high-frequency field (111) behavior by changing initial and boundary conditions associated with the low-frequency waves. In other words, controlling the initial and boundary conditions for the components with spin projection $m_F=\pm 1$, one can manage the propagation of bright-dark solitons in BEC. These are the examples of the all-matter-wave soliton guidance, with potential applications in the design of quantum switches and splitters.

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APPENDIX: SOLUTION OF THE RHP

Introduce the auxiliary functions

$$\theta_+(x;\zeta) = \tilde{\psi}_+^{(3)}\tilde{\psi}_-^{(1)}, \quad \theta_-(x;\zeta) = \tilde{\psi}_-^{(3)}\tilde{\psi}_+^{(1)}, \tag{A1}$$

$$\tilde{\theta}_+(x;\zeta) = \psi_-^{(3)}\psi_+^{(1)}, \quad \tilde{\theta}_-(x;\zeta) = \psi_+^{(3)}\psi_-^{(1)}. \tag{A2}$$

Then, we construct the complete set of eigenfunctions

$$\Theta_+ = \{\psi_+^{(1)}, \theta_+, \psi_-^{(3)}\}, \quad \Theta_- = \{\psi_-^{(1)}, \theta_-, \psi_+^{(3)}\}, \tag{A3}$$

$$\tilde{\Theta}_+ = \{\tilde{\psi}_-^{(1)}, \tilde{\theta}_+, \tilde{\psi}_+^{(3)}\}, \quad \tilde{\Theta}_- = \{\tilde{\psi}_+^{(1)}, \tilde{\theta}_-, \tilde{\psi}_-^{(3)}\}, \tag{A4}$$

satisfying the equalities

$$\partial_x\Theta_\pm = \mathbf{R}\Theta_\pm, \quad \partial_x\tilde{\Theta}_\pm = -\mathbf{R}^T\tilde{\Theta}_\pm. \tag{A5}$$

Then, we define the following matrix, depending on ζ and the time variable and x independent:

$$\tilde{\Theta}_+^T(x;\zeta)\Theta_+(x;\zeta) = \mathbf{H}_{++}(\zeta), \tag{A6}$$

$$\tilde{\Theta}_-^T(x;\zeta)\Theta_-(x;\zeta) = \mathbf{H}_{--}(\zeta), \tag{A7}$$

$$\tilde{\Theta}_+^T(x;\zeta)\Theta_-(x;\zeta) = \mathbf{H}_{+-}(\zeta), \tag{A8}$$

$$\tilde{\Theta}_-^T(x;\zeta)\Theta_+(x;\zeta) = \mathbf{H}_{-+}(\zeta). \tag{A9}$$

Let us introduce the matrix elements $t_j^k=t_{kj}$ and $\tilde{t}_j^k=t'_{jk}$, where $\mathbf{T}=\{t_{jk}\}$ and $[\mathbf{T}^{-1}]=\{t'_{jk}\}$. Then, the relations (37), (42), and (43) for the vector components of matrix functions $\mathbf{\Psi}_\pm(x;\zeta)$ and $\tilde{\mathbf{\Psi}}_\pm(x;\zeta)$ can be rewritten as

$$\psi_-^{(j)}(x;\zeta) = \sum_{k=1}^{3} t_j^k(\zeta)\psi_+^{(k)}(x;\zeta), \tag{A10}$$

$$\tilde{\psi}_-^{(j)}(x;\zeta) = \sum_{k=1}^{3} \tilde{t}_j^k(\zeta)\tilde{\psi}_+^{(k)}(x;\zeta), \tag{A11}$$

$$\psi_+^{(j)}(x;\zeta) = \sum_{k=1}^{3} \tilde{t}_k^j(\zeta)\psi_-^{(k)}(x;\zeta), \tag{A12}$$

$$\tilde{\psi}_+^{(j)}(x;\zeta) = \sum_{k=1}^{3} t_k^j(\zeta)\tilde{\psi}_-^{(k)}(x;\zeta). \tag{A13}$$

Taking into account the boundaries of functions and from Eqs. (A3) and (A4), we can derive the asymptotics

$$\Theta_+ \to \begin{pmatrix} \tilde{t}_1^1 & 0 & 0 \\ \tilde{t}_2^1 & t_3^3 & 0 \\ \tilde{t}_3^1 & -t_2^3 & 1 \end{pmatrix}, \quad x\to-\infty; \quad \begin{pmatrix} 1 & -\tilde{t}_1^2 & t_3^1 \\ 0 & \tilde{t}_1^1 & t_3^2 \\ 0 & 0 & t_3^3 \end{pmatrix}, \quad x\to\infty, \tag{A14}$$

$$\tilde{\Theta}_+ \to \begin{pmatrix} 1 & -\tilde{t}_2^1 & t_1^3 \\ 0 & \tilde{t}_1^1 & t_2^3 \\ 0 & 0 & t_3^3 \end{pmatrix}, \quad x \to -\infty; \quad \begin{pmatrix} \tilde{t}_1^1 & 0 & 0 \\ \tilde{t}_1^2 & t_3^3 & 0 \\ \tilde{t}_1^3 & -t_3^2 & 1 \end{pmatrix}, \quad x \to \infty, \tag{A15}$$

$$\Theta_- \to \begin{pmatrix} 1 & -t_2^1 & \tilde{t}_1^3 \\ 0 & t_1^1 & \tilde{t}_{23} \\ 0 & 0 & \tilde{t}_3^3 \end{pmatrix}, \quad x \to -\infty; \quad \begin{pmatrix} t_1^1 & 0 & 0 \\ t_1^2 & \tilde{t}_3^3 & 0 \\ t_1^3 & -\tilde{t}_3^2 & 1 \end{pmatrix}, \quad x \to \infty, \tag{A16}$$

$$\tilde{\Theta}_- \to \begin{pmatrix} t_1^1 & 0 & 0 \\ t_2^1 & \tilde{t}_3^3 & 0 \\ t_3^1 & -\tilde{t}_2^3 & 1 \end{pmatrix}, \quad x \to -\infty; \quad \begin{pmatrix} 1 & -t_1^2 & \tilde{t}_3^1 \\ 0 & t_1^1 & \tilde{t}_3^2 \\ 0 & 0 & \tilde{t}_3^3 \end{pmatrix}, \quad x \to \infty. \tag{A17}$$

By using these asymptotics and relations (A6)–(A9), we derive

$$\mathbf{H}_{++} = \mathrm{diag}\{\tilde{t}_1^1, \tilde{t}_1^1 t_3^3, t_3^3\}, \tag{A18}$$

$$\mathbf{H}_{+-} = \mathbf{I} - \mathbf{G}_{+-}, \tag{A19}$$

$$\mathbf{H}_{-+} = \mathbf{I} - \mathbf{G}_{-+}, \tag{A20}$$

$$\mathbf{H}_{--} = \mathrm{diag}\{t_1^1, t_1^1 \tilde{t}_3^3, \tilde{t}_3^3\}, \tag{A21}$$

where

$$\mathbf{G}_{+-} = \begin{pmatrix} 0 & t_2^1 & -\tilde{t}_1^3 \\ \tilde{t}_2^1 & t_3^1 \tilde{t}_3^1 & t_3^2 \\ -t_1^3 & \tilde{t}_3^2 & 0 \end{pmatrix}, \quad \mathbf{G}_{-+} = \begin{pmatrix} 0 & \tilde{t}_1^2 & -t_3^1 \\ t_1^2 & \tilde{t}_1^3 t_1^3 & \tilde{t}_2^3 \\ -\tilde{t}_3^1 & t_2^3 & 0 \end{pmatrix}. \tag{A22}$$

To solve the RHP, we must find the functions analytical in domains D_+ and D_- and possessing the following jump conditions on the boundary between these domains with asymptotics of the matrix functions

$$\Theta_\pm(x;\zeta), \quad \tilde{\Theta}_\pm(x;\zeta) \to \mathbf{I}, \quad |\zeta| \to \infty. \tag{A23}$$

Using relations (A6) and (A7), rewrite Eqs. (A8) and (A9) as the RHPs

$$\Theta_+(x;\zeta) = \Theta_-(x;\zeta)\mathbf{H}_{--}^{-1}(\zeta)\mathbf{H}_{-+}(\zeta), \tag{A24}$$

$$\Theta_-(x;\zeta) = \Theta_+(x;\zeta)\mathbf{H}_{++}^{-1}(\zeta)\mathbf{H}_{+-}(\zeta), \tag{A25}$$

$$\tilde{\Theta}_+(x;\zeta) = \tilde{\Theta}_-(x;\zeta)\mathbf{H}_{--}^{-1}(\zeta)\mathbf{H}_{+-}^{T}(\zeta), \tag{A26}$$

$$\tilde{\Theta}_-(x;\zeta) = \tilde{\Theta}_+(x;\zeta)\mathbf{H}_{++}^{-1}(\zeta)\mathbf{H}_{-+}^{T}(\zeta). \tag{A27}$$

Equations (A24) and (A25) are related by the transform $\zeta \to -\zeta$. The same is true for Eqs. (A26) and (A27). These relations follow from the symmetries of functions (A3) and (A4),

$$\Theta_+(x,-\zeta) = \mathbf{D}_2^{-1}(\zeta)\Theta_-(x,\zeta)\mathbf{D}_2(\zeta), \tag{A28}$$

$$\tilde{\Theta}_+(x,-\zeta) = \mathbf{D}_2^{-1}(\zeta)\tilde{\Theta}_-(x,\zeta)\mathbf{D}_2(\zeta), \tag{A29}$$

and the symmetries of the matrices $\mathbf{H}_{\pm\pm}$ and $\mathbf{G}_{\pm\pm}$,

$$\mathbf{H}_{\pm\pm}(-\zeta) = \mathbf{D}_2^{-1}(\zeta)\mathbf{H}_{\mp\mp}(\zeta)\mathbf{D}_2(\zeta), \tag{A30}$$

$$\mathbf{G}_{+-}(-\zeta) = \mathbf{D}_2^{-1}(\zeta)\mathbf{G}_{-+}(\zeta)\mathbf{D}_2(\zeta). \tag{A31}$$

Using above results, we find that functions $\Theta_+(\zeta)$ and $\tilde{\Theta}_+(\zeta)$ are analytical in D^+ and $\tilde{D}^+$ and functions $\Theta_-(\zeta)$ and $\tilde{\Theta}_-(\zeta)$ are analytical in D^- and $\tilde{D}^-$, respectively.

Then we can define functions which are analytical in respective domains except a finite number of poles

$$\boldsymbol{\Xi}_{-+}(\zeta) = \begin{cases} \Theta_+(\zeta), & \zeta \in D^+ \\ \Theta_-(\zeta)\mathbf{H}_{--}^{-1}(\zeta), & \zeta \in D^-, \end{cases} \tag{A32}$$

$$\boldsymbol{\Xi}_{+-}(\zeta) = \begin{cases} \Theta_-(\zeta), & \zeta \in D^- \\ \Theta_+(\zeta)\mathbf{H}_{++}^{-1}(\zeta), & \zeta \in D^+, \end{cases} \tag{A33}$$

$$\tilde{\boldsymbol{\Xi}}_{-+}(\zeta) = \begin{cases} \tilde{\Theta}_+(\zeta), & \zeta \in \tilde{D}^+ \\ \tilde{\Theta}_-(\zeta)\mathbf{H}_{--}^{-1}(\zeta), & \zeta \in \tilde{D}^-, \end{cases} \tag{A34}$$

$$\tilde{\boldsymbol{\Xi}}_{+-}(\zeta) = \begin{cases} \tilde{\Theta}_-(\zeta), & \zeta \in \tilde{D}^- \\ \tilde{\Theta}_+(\zeta)\mathbf{H}_{++}^{-1}(\zeta), & \zeta \in \tilde{D}^+. \end{cases} \tag{A35}$$

Consider jump of the function (A32) on the boundary Γ between domains D^+ and D^-.

Due to Eq. (A32), the jump condition on the boundary $\Gamma = D^+ \cap D^-$,

$$\boldsymbol{\Xi}_{-+}(\zeta + i0) - \boldsymbol{\Xi}_{-+}(\zeta - i0) = \Theta_-(\zeta)\mathbf{H}_{--}^{-1}(\zeta)\mathbf{G}_{-+}(\zeta), \quad \zeta \in \Gamma, \tag{A36}$$

where $\zeta + i0 \in D^+$, $\zeta - i0 \in D^-$, and $\zeta \neq 0, \pm\sqrt{q}$. Or the same thing, by using Eqs. (A20), (A21), and (A24), we get

$$\Theta_+(\zeta) - \Theta_-(\zeta)\mathbf{H}_{--}^{-1}(\zeta) = \Theta_-(\zeta)\mathbf{H}_{--}^{-1}(\zeta)\mathbf{G}_{-+}(\zeta). \tag{A37}$$

The uniquely solutions of the RHP (A37) is

$$\boldsymbol{\Xi}_{-+}(\zeta) = \mathbf{I} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{\xi - \zeta}\Theta_-(\xi)\mathbf{H}_{--}^{-1}(\xi)\mathbf{G}_{-+}(\xi). \tag{A38}$$

The next connected by transform $\zeta \to -\zeta$,

$$\boldsymbol{\Xi}_{+-}(\zeta) = \mathbf{I} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{\xi - \zeta}\Theta_+(\xi)\mathbf{H}_{++}^{-1}(\xi)\mathbf{G}_{+-}(\xi). \tag{A39}$$

Due to Eq. (A34), the jump condition between domains $\tilde{D}^+$ and $\tilde{D}^-$ may be written as

$$\tilde{\boldsymbol{\Xi}}_{-+}(\zeta) = \mathbf{I} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\xi}{\xi - \zeta}\tilde{\Theta}_-(\xi)[\mathbf{H}_{--}^{-1}(\xi)]^T\mathbf{G}_{-+}^T(\xi). \tag{A40}$$

Let contours $\Gamma^{\pm}(\tilde{\Gamma}^{\pm})$ run over the boundary of domains $D^{\pm}(\tilde{D}^{\pm})$, respectively, in the clockwise directions. Using Eq. (A38), we get for $\zeta \in D^{+}$,

$$\psi_{+}^{(1)}(\zeta) = \delta_{1k} - \frac{1}{2\pi i}\int_{\Gamma^{+}} \frac{d\xi}{\xi-\zeta}\left[\frac{t_1^2}{t_1^1}\psi_{+}^{(2)} + \frac{t_1^3}{t_1^1}\psi_{+}^{(3)}\right]. \quad (A41)$$

Analogously, from Eqs. (A39), we get for $\zeta \in D^{-}$,

$$\psi_{+}^{(3)}(\zeta) = \delta_{3k} - \frac{1}{2\pi i}\int_{\Gamma^{-}} \frac{d\xi}{\xi-\zeta}\left[\frac{t_3^2}{t_3^3}\psi_{+}^{(2)} + \frac{t_3^1}{t_3^3}\psi_{+}^{(3)}\right]. \quad (A42)$$

Using Eqs. (A10)–(A13), (A1), (A2), and (A22), we derive

$$\psi_{+}^{(2)}(x;\zeta) = \frac{\theta_{+}(x;\zeta)}{\tilde{t}_1^1(\zeta)} + \frac{\tilde{t}_1^2(\zeta)}{\tilde{t}_1^1(\zeta)}\psi_{+}^{(1)}(x;\zeta), \quad (A43)$$

$$\psi_{+}^{(2)}(x;\zeta) = \frac{\theta_{-}(x;\zeta)}{\tilde{t}_3^3(\zeta)} + \frac{\tilde{t}_3^2(\zeta)}{\tilde{t}_3^3(\zeta)}\psi_{+}^{(3)}(x;\zeta). \quad (A44)$$

To reconstruct function $\psi_{+}^{(2)}$ normalized at $|\zeta|\to\infty$ with parts analytical in $\tilde{D}^{+}$ and $\tilde{D}^{-}$ defined by relations (A43) and (A44), we use the Cauchy formula

$$\psi_{+}^{(2)}(\zeta) = \delta_{2k} + \frac{1}{2\pi i}\left[\int_{\tilde{\Gamma}^{+}} \frac{d\xi}{\xi-\zeta}\frac{\tilde{t}_1^2}{\tilde{t}_1^1}\psi_{+}^{(1)}(\xi) + \int_{\tilde{\Gamma}^{-}} \frac{d\xi}{\xi-\zeta}\frac{\tilde{t}_3^2}{\tilde{t}_3^3}\psi_{+}^{(3)}(\xi)\right]. \quad (A45)$$

Taking into account x and t dependences of the coefficients t_k^i and $\tilde{t}_k^i$ following from Eqs. (33), (44), and (95), respectively, we derive the complete system of Eqs. (A41), (A45), and (A42). Note that the forms of these equations are the same as derived in [26]. Differences are in symmetries and domains of determinations.

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