Quantum system under the actions of two counteracting baths: A model for the attenuation-amplification interplay

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We analyze the dynamical behavior of a quantum system under the actions of two counteracting baths: the inevitable energy draining reservoir and, in opposition, exciting the system, an engineered Glauber's amplifier. We follow the system dynamics towards equilibrium to map its distinctive behavior arising from the interplay of attenuation and amplification. Such a mapping, with the corresponding parameter regimes, is achieved by calculating the evolution of both the excitation and the Glauber-Sudarshan P function. Techniques to compute the decoherence and the fidelity of quantum states under the action of both counteracting baths, based on the Wigner function rather than the density matrix, are also presented. They enable us to analyze the similarity of the evolved state vector of the system with respect to the original one, for all regimes of parameters. Applications of this attenuation-amplification interplay are discussed.

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I. INTRODUCTION

Following developments from the early 1980s, focusing on measurement and other fundamental quantum problems, the subject of open quantum systems has recently been extended to encompass collective damping and diffusion mechanisms in nonideal networks. This broader perspective has largely been influenced by quantum information theory and its search for mechanisms to bypass decoherence in a realistic open quantum processor. Whereas collective diffusion is still at the beginning [1], a large number of results on collective damping, within different physical systems, have been derived [2–5], including the potentially useful emergence of relaxation- and decoherence-free subspaces [6]. In particular, efforts have been directed at the problem of high efficiency state transfer through a network of nonideal quantum channels [7,8].

In the present work, we revisit the problem of a single dissipative quantum system from a different and extended perspective, analyzing the interplay between the two quantum processes of dissipation and amplification. Whereas classical amplification is regularly used to achieve effective interactions, especially within atomic optics [9,10], its quantum counterpart has attracted less attention, since being suggested by Glauber [11]. Certainly, the difficulties associated with the practical implementation of a quantum multimodal amplifier discouraged further investigations on this subject. Glauber himself, however, has employed his amplifier model to analyze compelling problems, such as the feasibility of a laser gain tube [11] and the short-time behavior of superfluorescence [12]. The laser gain tube, suggested by Herbert [13], would allow exotic phenomena such as superluminal signaling and, as noted by Glauber [11], the amplification of a quantum system to its classical counterpart, followed by a set of nondemolitive (classical) measurements, and then the reverse attenuation process, to take the system back to its quantum domain with all known properties.

In this article, we examine specifically the competition between attenuators and amplifiers, which can be useful in the simulation of many physical phenomena, for example, super-radiant pulses. Thus, we analyze a system coupled on one hand to a reservoir (or attenuator) and on the other to an amplifier, as depicted in Fig. 1, addressing mainly the mapping of the behavior of the systems under different regimes of parameters.

Glaubers attenuator and amplifier

In a discussion of quantum-mechanical attenuators and amplifiers, Glauber [11] presented an original treatment of the former, based on the Glauber-Sudarshan P representation, focusing on a harmonic oscillator (HO) coupled to a reservoir modeled by quite a large number of oscillators. The dispersive Gaussian form of the quasiprobability density P is first derived and then, setting the HO initially in a pure coherent state α_0, the evolution of $P(\alpha,t|\alpha_0,0)$ is presented, in the phase space coordinates $\mathrm{Re}(\alpha)\times\mathrm{Im}(\alpha)$, exhibiting all the main features of a system damped by a thermal attenuator. Starting from a delta function associated with the pure coherent state α_0, the function P evolves discontinuously to a Gaussian whose increasing dispersion rate is controlled by the reservoir temperature. With the increasing dispersion, the mean value of the amplitude of the attenuated state traces an exponential spiral on $\mathrm{Re}(\alpha)\times\mathrm{Im}(\alpha)$, descending to the vacuum, where the initial delta function is discontinuously recovered.

On searching for a quantum amplifier, i.e., "a device that amplifies signal at the quantum level," Glauber's peculiar solution was to consider exactly the Hamiltonian describing

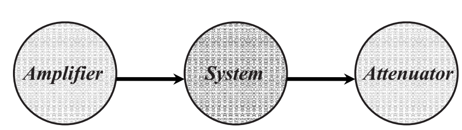

FIG. 1. Sketch of a quantum system under the action of an amplifier and an attenuator (reservoir).

the damped HO, but now considering an "inverted" HO, with negative kinetic and potential energies. The creation operator associated with such an oscillator creates de-excitations or negative energy quanta, whereas the annihilation operator conversely raises the energy of the inverted HO. Although the amplifier, like the reservoir, is modeled by a large set of oscillators, the interaction Hamiltonian takes the form of the energy-conserving counter-rotating terms rather than the usual rotating ones. Now, starting again from a coherent state α_0, the mean value of the amplitude of the amplified state describes an ascending exponential spiral on $\mathrm{Re}(\alpha)\times\mathrm{Im}(\alpha)$, with function P discontinuously evolving to a Gaussian whose dispersion rate, being non-null even at $T=0$ K, is greater than that of the attenuator model.

When considering a system under the action of both an attenuator and an amplifier, we must expect behavior of the P function to vary, depending on the relations between the coupling strengths on both sides of the system in Fig. 1. We can also assume these couplings to be either constants or variables. A detailed map of the system behavior is presented here for the case where both couplings of the system are constants.

II. HARMONIC OSCILLATOR COUPLED WITH TWO COUNTERACTING MULTIMODAL SYSTEMS

We take our system to be a quantum HO, as in Glauber's treatment [11], under the action of two quantum systems, an attenuator and an amplifier. We also assume, for completeness, that our system is subjected to a classical amplification field. The Hamiltonian governing the system evolution is thus given by $H=H_0+H_I$, where

$$H_0=\hbar\omega a^\dagger a+\hbar\sum_{m=1}^{2}\sum_k \omega_{mk}b_{mk}^\dagger b_{mk}, \tag{1}$$

encompasses the HO of frequency ω, where $a^\dagger$ (a) is the associated creation (annihilation) operator, as well as the attenuator ($m=1$) and the amplifier ($m=2$), where $b_{mk}^\dagger$ (b_{mk}) is the creation (annihilation) operator associated with their kth mode ω_{mk}. Finally, the interaction term

$$H_I=\hbar\sum_k[\lambda_{1k}(ab_{1k}^\dagger+a^\dagger b_{1k})+\lambda_{2k}(e^{i\nu t}ab_{2k}+e^{-i\nu t}a^\dagger b_{2k}^\dagger)], \tag{2}$$

describes the couplings of the system with the attenuator and the amplifier modes, of strengths λ_{1k} and λ_{2k}, given by the rotating and the counter-rotating terms, respectively. We stress that we have not resorted to an inverted harmonic oscillator to account for the amplification dynamics as in Glauber's original model [11]. After all, our damped HO must be the same as our amplified one. To circumvent this problem, retaining the noninverted HO even for the amplification process, we merely introduce the phase factors $e^{\pm i\nu t}$ into the system-amplifier coupling. As a matter of fact, under an appropriate choice of the frequency ν, Glauber's original amplifier Hamiltonian

$$H_{\mathrm{Glauber}}=-\hbar\omega a^\dagger a+\hbar\sum_k[\omega_k b_k^\dagger b_k+\lambda_k(ab_k+a^\dagger b_k^\dagger)]$$

turns out to be exactly the same as the one given in our version: $\hbar\omega a^\dagger a+\hbar\Sigma_k[\omega_k b_k^\dagger b_k+\lambda_k(e^{i\nu t}ab_k+e^{-i\nu t}a^\dagger b_k^\dagger)]$.

The convenient transformation

$$U(t)=\exp\left[-\frac{i\nu t}{2}\left(a^\dagger a+\sum_{m=1}^{2}\sum_k b_{mk}^\dagger b_{mk}\right)\right],$$

leads to the Hamiltonian $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_I$, with time-independent interaction terms, given by

$$\mathcal{H}_0=\hbar\varpi a^\dagger a+\hbar\sum_{m=1}^{2}\sum_k \varpi_{mk}b_{mk}^\dagger b_{mk}$$

$$\mathcal{H}_I=\hbar\sum_k[\lambda_{1k}(ab_{1k}^\dagger+a^\dagger b_{1k})+\lambda_{2k}(ab_{2k}+a^\dagger b_{2k}^\dagger)],$$

where $\varpi=\omega-\nu/2$ and $\varpi_{mk}=\omega_{mk}-\nu/2$.

Next, we adopt a set of commonly used approximations with which the master equation of the system described by the Hamiltonian $\mathcal{H}$ can be derived. We firstly assume that both couplings of the HO, with the attenuator and the amplifier, are weak enough for us to perform a second-order perturbation approximation on these parameters, and then trace out the bath degrees of freedom. We also assume Markovian attenuator and amplifier, to be able to factorize the density operator of the global system as $\rho_S(t)\otimes\rho_{\mathrm{att}}(0)\otimes\rho_{\mathrm{amp}}(0)$. With these assumptions, we obtain the reduced density operator of the HO, given by

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t}=-\frac{1}{\hbar^2}\int_0^t \mathrm{d}t'\ \mathrm{Tr}_R[\mathcal{H}_I(t),[\mathcal{H}_I(t'),\rho_S(t)\otimes\rho_{\mathrm{att}}(0)\otimes\rho_{\mathrm{amp}}(0)]]. \tag{3}$$

Since for thermal attenuator and amplifier we obtain $\langle b_{mk}b_{m'k'}\rangle=\langle b_{mk}^\dagger b_{m'k'}^\dagger\rangle=0$, for $m,m'=1,2$ or vice-versa, we have only to solve the integrals [appearing in Eq. (3)] related to correlation functions of the form $\int_0^t dt'\langle\Gamma_m^\dagger(t)\Gamma_m(t')\rangle$, where $\Gamma_m(t)=\Sigma_k\lambda_{mk}e^{-i\Omega_{mk}t}b_{mk}$, with $\Omega_{mk}=\varpi_{mk}+(-1)^m\varpi$. We further assume the frequencies of the attenuator and the amplifier to be closely spaced, to allow a continuum summation, $\sigma_m(\omega_m)$ being the density of states of both multimodal systems. The average excitation of the kth mode associated with either the attenuator or the amplifier is defined by the relation $\langle b_m^\dagger(\omega_m)b_{m'}(\omega_{m'})\rangle=\delta_{mm'}N_m(\omega_m)\delta(\omega_m-\omega_{m'})$, where $N_m(\omega_m)$ is the thermal average photon number. Finally, making the usual assumption that $\lambda_m(\omega_m)$, $\sigma_m(\omega_m)$, and $N_m(\omega_m)$ are slowly varying functions, we present some convenient variable substitutions. First, the transformations $\varepsilon=\omega_1-\omega$ and $\tau=t-t'$ lead to the simplified form for the attenuator correlation

$$\int_0^t d\tau \langle \Gamma_1^\dagger(t)\Gamma_1(t-\tau)\rangle = \int_0^t d\tau \int_{-\omega}^{\infty} \frac{d\varepsilon}{2\pi}\sigma_1(\varepsilon+\omega)\lambda_1^2(\varepsilon+\omega)N_1(\varepsilon+\omega)e^{-i\varepsilon\tau} = \frac{1}{2}\sigma_1(\omega)\lambda_1^2(\omega)N_1(\omega),$$

whereas the transformations $\varepsilon=\omega_2+\omega-\nu$ and $\tau=t-t'$ entails the amplifier correlation

$$\int_0^t d\tau \langle \Gamma_2^\dagger(t)\Gamma_2(t-\tau)\rangle = \int_0^t d\tau \int_{\omega-\nu}^{\infty} \frac{d\varepsilon}{2\pi}\sigma_2(\varepsilon-\omega+\nu)\lambda_2^2(\varepsilon-\omega+\nu) \times N_2(\varepsilon-\omega+\nu)e^{-i\varepsilon\tau} = \begin{cases} \frac{1}{2}\sigma_2(\nu-\omega)\lambda_2^2(\nu-\omega)N_2(\nu-\omega) & \text{for } \omega-\nu \ll 0 \\ \frac{1}{4}\sigma_2(0)\lambda_2^2(0)N_2(0) & \text{for } \omega-\nu \simeq 0 \\ 0 & \text{for } \omega-\nu \gg 0 \end{cases}.$$

Therefore, the choice $\nu=2\omega$ provides the amplifier correlation with exactly the form $\int_0^t d\tau\langle\Gamma_2^\dagger(t)\Gamma_2(t-\tau)\rangle = \frac{1}{2}\sigma_2(\omega)\lambda_2^2(\omega)N_2(\omega)$ derived from Glauber's original amplifier Hamiltonian.

Returning to the Schrödinger picture, we finally obtain the master equation

$$\frac{d\rho(t)}{dt} = i\omega[\rho(t),a^\dagger a] + \frac{\Gamma_1(N_1+1)+\Gamma_2 N_2}{2}[2a\rho(t)a^\dagger - \rho(t)a^\dagger a - a^\dagger a\rho(t)] + \frac{\Gamma_1 N_1+\Gamma_2(N_2+1)}{2}[2a^\dagger\rho(t)a - \rho(t)aa^\dagger - aa^\dagger\rho(t)], \tag{4}$$

where the attenuation and amplification rates are given by $\Gamma_m=\gamma_m(\omega)=\sigma_m(\omega)\lambda_m^2(\omega)$. We call attention to the fact that the superoperator describing the action of the amplifier follows directly from the well-known dissipative Liouvillian operator, by exchanging a with $a^\dagger$.

III. QUASIPROBABILITY DISTRIBUTION FUNCTIONS

From the master equation (4), we derive the evolution equation for the normal ordered characteristic function

$$\frac{d\chi(\eta,\eta^*,t)}{dt} = \Phi\left[\frac{1}{2}\left(\eta\frac{\partial}{\partial\eta}+\eta^*\frac{\partial}{\partial\eta^*}\right)+\mathfrak{N}|\eta|^2\right]\chi(\eta,\eta^*,t),$$

where we have defined the *effective transference rate* $\Phi=\Gamma_1-\Gamma_2$ and, through the quantity $\Lambda=\Gamma_1 N_1+\Gamma_2(N_2+1)$, the *effective thermal average photon number* $\mathfrak{N}=\Lambda/\Phi$. Assuming that the HO is prepared in the superposition of coherent states $|\Psi\rangle=\sum_{m=1}^M c_m|\beta_m\rangle$, the above equation has the solution

$$\chi(\eta,\eta^*,t) = \sum_{m,n=1}^{M} c_m c_n^* \langle\beta_n|\beta_m\rangle \times \exp[(\eta\beta_n^* e^{-i\omega t} - \eta^*\beta_m e^{i\omega t}) \times e^{-\Phi t/2} - \mathfrak{N}|\eta|^2(1-e^{-\Phi t})], \tag{5}$$

which will be further employed, together with the Glauber-Sudarshan P function, to map the dissipation x amplification competition. From a two-dimensional Fourier transform of solution (5), we obtain the P function as

$$P(\alpha,\alpha^*,t) = \frac{1}{\pi^2}\int d^2\eta \exp(\alpha\eta^* - \alpha^*\eta)\chi(\eta,\eta^*,t) = \frac{1}{\pi\mathcal{D}(t)}\sum_{m,n=1}^{M} c_m c_n^*\langle\beta_n|\beta_m\rangle \times \exp\left(-\frac{(\alpha-\beta_m e^{i\omega t-\Phi t/2})(\alpha-\beta_n e^{i\omega t-\Phi t/2})^*}{\mathcal{D}(t)}\right),$$

where the positive definite function $\mathcal{D}(t)=\mathfrak{N}(1-e^{-\Phi t})$ stands for the *effective dispersion rate*. We note that the diffusion $\mathcal{D}(t)$ becomes null, giving a delta-P function, only for the particular case where $N_1=\Gamma_2=0$, indicating that the presence of the amplifier forbids a coherent evolution even at absolute zero ($N_2=0$).

For the case considered by Glauber in his study of the amplifier, where the HO is prepared in a coherent state $|\Psi\rangle=|\beta\rangle$, the P function reduces to the simple form

$$P(\alpha,\alpha^*,t) = \frac{1}{\pi\mathcal{D}(t)}\exp\left(-\frac{|\alpha-\beta e^{i\omega t-\Phi t/2}|^2}{\mathcal{D}(t)}\right), \tag{6}$$

which enables us to derive both of the extreme cases analyzed by Glauber in Ref. [11]: (i) when the HO is coupled only to the attenuator ($\Gamma_2=0$), giving

$$P_{(\Gamma_2=0)}(\alpha,\alpha^*,t) = \frac{1}{\pi D(t)}\exp\left(-\frac{|\alpha-\beta e^{i\omega t-\Gamma_1 t/2}|^2}{D(t)}\right),$$

with the dispersion $D(t)=N_1(1-e^{-\Gamma_1 t})$, and (ii) when the HO is coupled only yo the amplifier ($\Gamma_1=0$), giving

$$P_{(\Gamma_1=0)}(\alpha,\alpha^*,t) = \frac{1}{\pi\tilde{D}(t)}\exp\left(-\frac{|\alpha-\beta e^{i\omega t+\Gamma_2 t/2}|^2}{\tilde{D}(t)}\right),$$

where $\tilde{D}(t)=(N_2+1)(e^{\Gamma_2 t}-1)$. As already noted by Glauber [11], when proposing his inverted HO model, the diffusion process associated with the amplifier is finite even at absolute zero, differently from that arising from the attenuator, which is null when $N_1=0$. As demonstrated below, this feature inevitably has negative consequences for the phase coherence of superposed states of the HO under the action of an amplifier.

Next, we present the Wigner distribution $W(\zeta,\zeta^*,t)$, which is seen later to be indispensable for estimating the decoherence and the fidelity of the HO states under both diffusive processes. For the particular case where the HO is prepared in a "Schrödinger cat"-like state $|\Psi_{SC}\rangle = \mathcal{N}(|\beta\rangle + |-\beta\rangle)$ ($\mathcal{N}$ being a normalization factor), employing again the characteristic function (5), we obtain the relation

$$W(\zeta,\zeta^*,t) = \frac{1}{\pi^2}\int \exp(\eta^*\zeta - \eta\zeta^*)\chi(\eta,\eta^*,t)e^{-|\eta|^2/2}d^2\eta$$
$$= \frac{2\mathcal{N}^2}{\pi[2\mathcal{D}(t)+1]}\sum_{m,n=0}^{1}\langle\beta_n|\beta_m\rangle$$
$$\times\exp\left(-\frac{2(\zeta-\beta_m e^{i\omega t-\Phi t/2})(\zeta-\beta_n e^{i\omega t-\Phi t/2})^*}{2\mathcal{D}(t)+1}\right). \tag{7}$$

In the above expression, we observe the characteristic residual width of the Wigner distribution, which remains even when the effective diffusion is made null, by switching off the amplifier and assuming the attenuator to be at absolute zero.

IV. MAPPING THE DISSIPATION X AMPLIFICATION INTERPLAY

A. Excitation

From the solution for $\chi(\eta,\eta^*,t)$ in Eq. (5), we find that the mean excitation of the HO evolves as

$$\langle a^\dagger a\rangle(t) = \sum_{m,n=0}^{M} c_m c_n^*\langle\beta_n|\beta_m\rangle[\beta_m(\beta_n)^* e^{-\Phi t} + \mathfrak{N}(1-e^{-\Phi t})], \tag{8}$$

showing that the dissipation-amplification dynamics is essentially governed by both parameters Φ and $\mathfrak{N}$, i.e., the attenuation and amplification rates, apart from the temperatures of the two counteracting systems coupled to the HO. To simplify our analysis of the dissipation x amplification interplay, we assume, without loss of generality, the coherent initial state $|\Psi\rangle = |\beta\rangle$, simplifying Eq. (8) to $\langle a^\dagger a\rangle(t) = |\beta|^2 e^{-\Phi t} + \mathfrak{N}(1-e^{-\Phi t})$. For the general case in Eq. (8), the only difference is in the coefficients multiplying the attenuation and amplification functions given by $e^{-\Phi t}$ and $(1-e^{-\Phi t})$, respectively. With this assumption, in Fig. 2, we plot the excitation $\langle a^\dagger a\rangle(t)$ against $\Gamma_1 t$, fixing $|\beta|^2=1$ and $N_1=N_2=0.1$. When $\Gamma_2<\Gamma_1$ and the diffusion rises to the asymptotic value $\mathcal{D}=\mathfrak{N}$, we have three different behaviors for the excitation, which depend on the relation between the effective thermal average photon number $\mathfrak{N}$ and the intensity $|\beta|^2$ of the initial state.

(a) For (i) $\mathfrak{N}<|\beta|^2$, the action of the attenuator overcomes that of the amplifier and the excitation decreases exponentially to the asymptotic value $\mathfrak{N}=0.7$, obtained with $\Gamma_2=1/3$ (in units of Γ_1), as indicated by the thick solid line in Fig. 2.

(b) For (ii) $\mathfrak{N}=|\beta|^2$, achieved by fixing $\Gamma_2=(1-N_1)/(2+N_2)=3/7$, the attenuator and the amplifier play equal roles, leaving the excitation unaffected as indicated by the solid line.

(c) For (iii) $\mathfrak{N}>|\beta|^2$, the amplifier overcomes the attenuator (in spite of the inequality $\Gamma_2<\Gamma_1$), and the excitation increases exponentially to the asymptotic value $\mathfrak{N}=1.3$, obtained when $\Gamma_2=1/2$, as indicated by the dashed-dotted line.

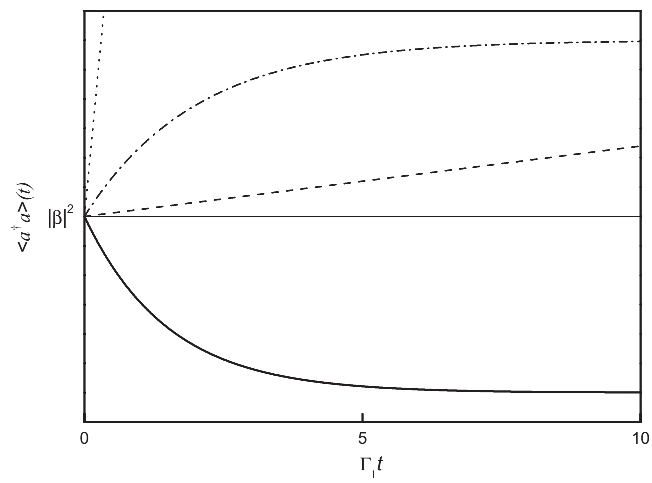

FIG. 2. Plot of the excitation $\langle a^\dagger a\rangle(t)$ against $\Gamma_1 t$, for the cases where $\Gamma_2<\Gamma_1$, $\Gamma_2=\Gamma_1$, and $\Gamma_2>\Gamma_1$, fixing $|\beta|^2=1.0$ and $N_1=N_2=0.1$. The thick solid, solid, and dashed-dotted curves, for $\Gamma_2<\Gamma_1$, follow from distinct values of the ratio $\mathfrak{N}=\Lambda/\Phi$.

Two other regimes, defined by the relation between Γ_2 and Γ_1, are included in Fig. 2.

(a) For $\Gamma_2=\Gamma_1$, we have linear diffusion, $\mathcal{D}(t)=\Lambda t$, and a threshold between the bounded exponential behavior defined by $\Gamma_2<\Gamma_1$ and the unbounded exponential growth for $\Gamma_2>\Gamma_1$. The dashed line describes the linear evolution of the excitation given by $\langle a^\dagger a\rangle(t)=|\beta|^2+\Lambda t$.

(b) Finally, as already mentioned, for $\Gamma_2>\Gamma_1$, the amplifier completely overcomes the attenuator and the excitation undergoes an unbounded exponential growth, as does the diffusion rate, as indicated by the dotted line derived for $\Gamma_2=1.01$. We have chosen Γ_2 only slightly higher than Γ_1, owing to the strong increase in the excitation rate exhibited in this regime.

It is worth stressing again that the phenomenology embodied in the above analysis remains the same in the case of the general expression (8), the only changes arising from the choice of the effective thermal average photon number $\mathfrak{N}$ regarding the excitation of the initial state $|\Psi\rangle$. We also note that the phenomenology displayed by the curves in Fig. 2 also remains the same for a distinct choice of the temperatures of the attenuator and the amplifier, associated with the thermal average photon numbers N_1 and N_2, respectively; the only difference being the rate of change in these curves.

B. Variances

Computing the variances of the oscillator field from the characteristic function (5), we obtain, for both quadratures ($\ell=1,2$)

$$[\Delta X_\ell(t)]^2 = \frac{\hbar}{4}[1+\mathcal{D}(t)],$$

showing that the minimum uncertainty occurs in the absence of the diffusion process. Such a null diffusion occurs only in

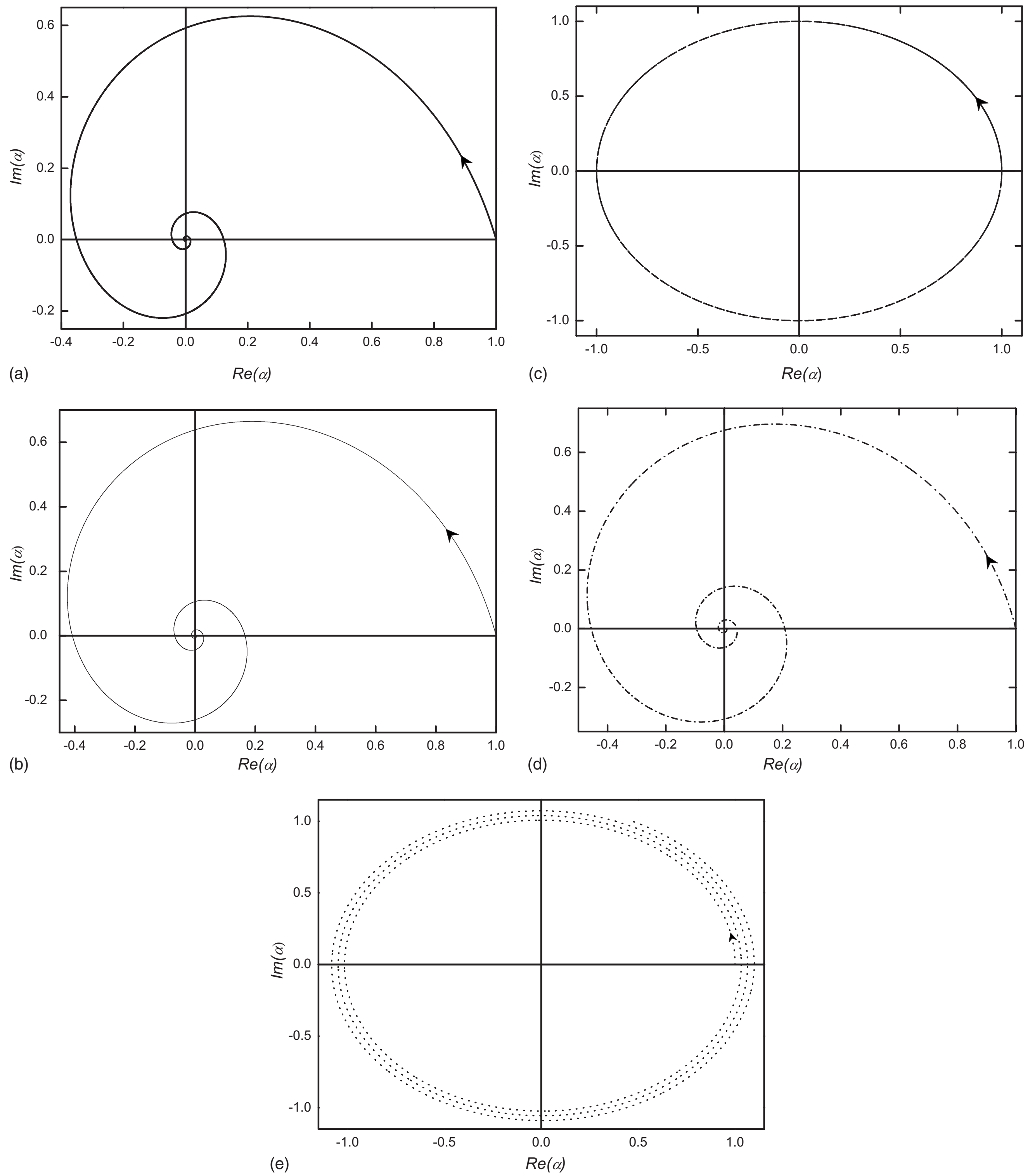

FIG. 3. Plot, on the plane $\text{Re}(\alpha)\times\text{Im}(\alpha)$, of the time evolution of the projection of the maximum value of P. The curves from (a) to (e) correspond exactly to those in Fig. 2, with the same fixed parameters, apart from the choice $\beta=1$ and the lengthening of the duration of the time evolution to $\Gamma_1 t=20$.

the case where $\mathfrak{N}=0$, implying that the amplifier must be turned off, such that $\mathcal{D}(t)=N_1$ and consequently, that an absolute zero attenuator must be assumed. Otherwise, the diffusion rate increases proportionally to the excitation $\langle a^{\dagger}a\rangle(t)$, forbidding, as concluded by Glauber [11], the feasibility of the *laser gain tube* proposed by Herbert [13] to clone superposition states for superluminal communication. In fact, as follows from the diffusion rate $\mathcal{D}(t)$, any amount of excitation provided by the amplifier device carries with it a corresponding amount of noise.

C. Glauber-Sudarshan P function

The $P(\alpha,\alpha^*,t)$ function in Eq. (6), associated with the HO prepared in the coherent state $|\beta\rangle$, provides a complementary scenario of the dissipation x amplification interplay. In fact, with the preparation of the coherent state $|\beta\rangle$, supposed to occur at $t=0$, we start from the delta-P function $\delta^{(2)}(\alpha-\beta)$. As time goes on, the Dirac delta evolves to a Gaussian distribution whose dispersion $\mathcal{D}(t)$ increases, even when the attenuator and the amplifier are both at absolute zero. In Fig. 3, we plot, on the plane $\mathrm{Re}(\alpha)\times\mathrm{Im}(\alpha)$, the time evolution of the projection of the maximum value of P. The curves from (a) to (e) correspond exactly to those in Fig. 2, with the same fixed parameters, apart from the choice $\beta=1$ and the lengthening of the duration of the evolution to $\Gamma_1 t=20$.

(a) For the case $\Gamma_2<\Gamma_1$, we observe in Figs. 3(a)–3(c) that the projected P function decreases on an exponential spiral to the origin, along the path defined by $\alpha(t)=\beta e^{i\omega t-\Phi t/2}$. Asymptotically, the P function reaches the value $(\pi\mathfrak{N})^{-1}e^{-|\alpha|^2/\mathfrak{N}}$, leading exactly to the excitations $\mathfrak{N}=0.7$, 1.0, and 1.3, corresponding to the choice $\Gamma_2=1/3$, 1, and 1/2, respectively. Evidently, a projected P function spiraling to the origin does not imply a decreasing excitation of the system.

(b) The threshold case $\Gamma_2=\Gamma_1$ results in the distribution $P=(\pi\Lambda t)^{-1}\exp(-|\alpha-\beta e^{i\omega t}|^2/\Lambda t)$, whose projection remains on a circle of radius β, as shown in Fig. 3(d). This circling diffusive distribution is associated with the linearly increasing excitation of the corresponding case in Fig. 2(d).

(c) Finally, for $\Gamma_2>\Gamma_1$ we obtain, as depicted in Fig. 3(e), a projected P distribution continuously increasing on an exponential spiral, together with the associated diffusion rate.

V. DECOHERENCE

From the Wigner function (7), obtained when the HO is prepared in the Schrödinger cat-like state $|\Psi_{SC}\rangle=\mathcal{N}(|\beta\rangle+|-\beta\rangle)$, we next compute the decoherence time by a technique presented in Ref. [1] for the particular case where the amplification process was absent ($\Gamma_2=0$). This technique is much more convenient whenever diffusion is present, overcoming the difficulties in estimating the decoherence time from the evolution of the density operator. In fact, when diffusion takes place together with the decay of the interference terms, we must consider the two separately, estimating both the diffusion time τ_{diff} and the decay time of the interference terms of the Wigner functions τ_{int}. The decoherence time then follows from the relation:

$$\frac{1}{\tau_D}=\frac{1}{\tau_{\mathrm{diff}}}+\frac{1}{\tau_{\mathrm{int}}}, \tag{9}$$

where the diffusion time, defined as

$$\frac{1}{\tau_{\mathrm{diff}}}=2\left.\frac{d}{dt}\mathcal{D}(t)\right|_{t=0}, \tag{10}$$

displays a tendency to a significant spread of the state. (We observe that in the absence of diffusion the term $\tau_{\mathrm{diff}}^{-1}$ becomes automatically null). To reach a definition of τ_{int}, we first decompose the Wigner function on its diagonal and off-diagonal elements $W(\zeta,\zeta^*,t)=\Sigma_{m,n=1}^{2}W_{m,n}(\zeta,\zeta^*,t)$, which are related to the components $\beta_m(t)=(-1)^{m-1}\beta e^{i\omega t-\Phi t/2}$ of the Schrödinger cat-like state, as

$$W_{m,n}(\zeta,\zeta^*,t)=\frac{2\mathcal{N}^2}{\pi[2\mathcal{D}(t)+1]}\langle\beta_n|\beta_m\rangle \times\exp\left\{\frac{-2[\zeta-\beta_m(t)][\zeta-\beta_n(t)]^*}{2\mathcal{D}(t)+1}\right\}. \tag{11}$$

These elements enable us to define the ratio

$$\Xi_{m,n}(t)\equiv\frac{W_{m,m}(\zeta,\zeta^*,t)W_{n,n}(\zeta,\zeta^*,t)}{W_{m,n}(\zeta,\zeta^*,t)W_{n,m}(\zeta,\zeta^*,t)},$$

which gives

$$\Xi_{1,2}(t)=\exp\left[4|\beta|^2\left(1-\frac{2e^{-\Phi t}}{2\mathcal{D}+1}\right)\right]$$

and offers a measure of the decay of interference, through the function

$$\wp_{1,2}(t)=\frac{\Xi_{1,2}(0)}{\Xi_{1,2}(t)}=\exp\left[-\frac{8|\beta|^2}{2\mathcal{D}+1}(2\mathcal{D}+1-e^{-\Phi t})\right]. \tag{12}$$

We note that the argument of the exponential is positive definite, since $2\mathcal{D}+1-e^{-\Phi t}=[2(\gamma_1N_1+\gamma_2N_2)+(\gamma_1+\gamma_2)](1-e^{-\Phi t})/\Phi$, showing that $\wp_{m,n}(t)$ equals unity at $t=0$, decaying as time goes on.

From the above decay function, we define τ_{int} by generalizing the usual relation $\wp_{rs}(\tau_D)=e^{-4}$, used to compute the decoherence time in the case of reservoirs at finite temperatures [1], to

$$\wp_{1,2}(\tau_{\mathrm{int}})=\exp\left[-\frac{4}{2\mathcal{D}(\tau_{\mathrm{int}})+1}\right], \tag{13}$$

which corresponds to measuring the decay of the interference terms of the Wigner function by deducting their spread, common to all the diagonal and off-diagonal elements in Eq. (11); the decay of the off-diagonal elements is thus estimated with respect to the diagonals. This procedure amounts to an analysis of the decay of interference in a frame where the diagonal terms are frozen. From the relations (9), (10), (12), and (13), we thus obtain, for the superposition $|\Psi_{SC}\rangle$, the decoherence time

$$\tau_{\mathrm{int}}=\begin{cases}\dfrac{1}{|\Phi|}\ln\left(\dfrac{1}{1-\aleph}\right)\approx\dfrac{\aleph}{|\Phi|} & if\ \ \Phi\geq 0\\[2ex] \dfrac{1}{|\Phi|}\ln(1+\aleph)\approx\dfrac{\aleph}{|\Phi|} & if\ \ \Phi\leq 0\end{cases},$$

where

$$\aleph=\frac{|\Phi|}{2|\beta|^2[\Gamma_1(1+2N_1)+\Gamma_2(1+2N_2)]},$$

and consequently

$$\tau_D = \frac{1}{2|\beta|^2[\Gamma_1(1+2N_1)+\Gamma_2(1+2N_2)]+2[\Gamma_1 N_1+\Gamma_2(N_2+1)]}. \tag{14}$$

The distinct feature of the derived result (14) is the excess noise generated by the amplifier, the same source of noise responsible for non-null diffusion, even when both baths are at absolute zero. Analyzing Eq. (14) for the particular case where the amplifier is turned off ($\Gamma_2=0$), we obtain the well-known result $\tau_D=(2|\beta|^2\Gamma_1)^{-1}$ when the attenuator is at absolute zero ($N_1=0$), as well as the reasonable rate $\tau_D=[2|\beta|^2\Gamma_1(2N_1+1)+2N_1]^{-1}$ when the attenuator has a finite temperature.

VI. FIDELITY

Regarding the fidelity of a Schrödinger cat-like state $|\Psi_{SC}\rangle=\mathcal{N}(|\beta\rangle+|-\beta\rangle)$ prepared in the HO, we observe that it can be directly computed from the Wigner function (11), by the relation

$$\pi\int_{-\infty}^{\infty} d^2\zeta W(\zeta,\zeta^*,0)W(\zeta,\zeta^*,t)=\mathrm{Tr}[\rho_S(0)\rho_S(t)]. \tag{15}$$

The above equality can be proved from the relation between $W(\zeta,\zeta^*,t)$ and $P(\eta,\eta^*,t)$

$$W(\xi,\xi^*,t)=\frac{2}{\pi}\int d^2\eta P(\eta,\eta^*,t)\exp(-2|\eta-\xi|^2),$$

as follows:

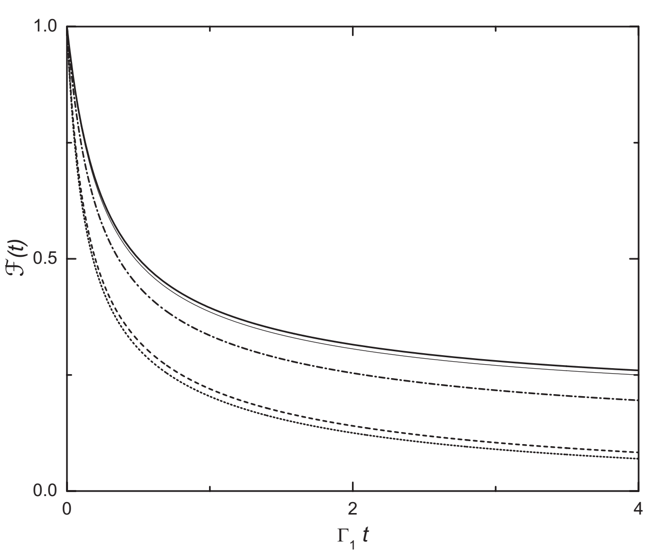

FIG. 4. Plot of the fidelity $\mathcal{F}(t)$ of the Schrödinger cat-like state $|\Psi_{SC}\rangle=\mathcal{N}(|\beta\rangle+|-\beta\rangle)$ under the attenuation-amplification interplay, for the cases $\Gamma_2<\Gamma_1$, $\Gamma_2=\Gamma_1$, and $\Gamma_2>\Gamma_1$, fixing $|\beta|^2=1.0$, $N_1=N_2=0.1$.

$$\pi\int_{-\infty}^{\infty} d^2\zeta W_S(\zeta,\zeta^*;t)W_S(\zeta,\zeta^*;0)$$
$$=\frac{4}{\pi}\int d^2\eta\int d^2\beta P_S(\eta,\eta^*,t)P_S(\beta,\beta^*,0)e^{-2(|\eta|^2+|\beta|^2)}$$
$$\times\int_{-\infty}^{\infty} d^2\zeta\exp[-4\zeta\zeta^*+2(\eta+\beta)^*\zeta+2(\eta+\beta)\zeta^*]$$
$$=\int d^2\eta\int d^2\beta P_S(\eta,\eta^*,t)P_S(\beta,\beta^*,0)\langle\beta|\eta\rangle\langle\eta|\beta\rangle$$
$$=\mathrm{Tr}\left\{\left[\int d^2\eta P_S(\eta,\eta^*,t)|\eta\rangle\langle\eta|\right]\times\left[\int d^2\beta P_S(\beta,\beta^*,0)|\beta\rangle\langle\beta|\right]\right\}=\mathrm{Tr}[\rho_S(t)\rho_S(0)].$$

To avoid the typical oscillations of the above overlap functions, we may either compute them with the restriction $\omega=0$, or consider the upper bound of the oscillations, i.e.,

$$\mathcal{F}(t)=\sup\left[\pi\int_{-\infty}^{\infty} d^2\zeta W(\zeta,\zeta^*,0)W(\zeta,\zeta^*,t)\right]. \tag{16}$$

The definition in Eq. (16) generated the curves in Fig. 4, where we fixed the parameters $N_1=N_2=0.1$ and $|\beta|^2=1.0$, so that $\mathfrak{N}=\langle\Psi_{SC}|a^\dagger a|\Psi_{SC}\rangle=(1-e^{-2|\beta|^2})|\beta|^2/(1+e^{-2|\beta|^2})=0.76$. We have considered, as in Fig. 2, the values $\mathfrak{N}=0.7$, obtained with $\Gamma_2=1/3$, for the case $\mathfrak{N}<|\beta|^2$ (thick solid line) and $\mathfrak{N}=1.3$, obtained with $\Gamma_2=1/2$, for the case $\mathfrak{N}>|\beta|^2$ (dashed-dotted line). The curve associated with the stationary excitation follows from the value $\mathfrak{N}=0.7$, now obtained with $\Gamma_2=0.36$ (solid line). Finally, we consider the cases where $\Gamma_2=1.0$ and 1.1, indicated by the dashed and dotted lines, respectively. As would have been expected, the larger the rate of the amplification process, the faster the decrease in fidelity. In fact, when $N_1=N_2$, the diffusion rate—and hence the noise injection from the amplification process into the system—becomes more pronounced with increasing excitation rate Γ_2/Γ_1.

VII. APPLICATIONS AND CONCLUDING REMARK

In this paper, we have analyzed the behavior of a harmonic oscillator (HO) under the action of two counteracting baths: a reservoir (or attenuator) and the Glauber amplifier. After deriving the master equation, we compute the characteristic, the Glauber-Sudarshan P function, and the Wigner distribution associated with a general state of the HO, to map the distinct classes of behavior of the time evolution of its excitation $\langle a^\dagger a\rangle(t)$. Basically, these classes are governed by the *effective transference rate* $\Phi=\Gamma_1-\Gamma_2$ and the *effective* *thermal average photon number* $\mathfrak{N}=\Lambda/\Phi$ associated with the attenuation-amplification interplay. A major feature of our competition problem is the noise injected by the amplifier, even when it is at absolute zero. This fact was pointed out by Glauber when introducing his amplification model. It has been attributed to the amplification of the zero-point fluctuations or the spontaneous emission of quanta from the amplifier into the system composed by the HO plus reservoir. Evidently, as we demonstrated, this inevitable dispersion arising from the amplification process plays a major part—besides the action of the reservoir—in the decoherence of a superposition state of the HO.

We have also presented a technique to compute the decoherence and the fidelity of the HO under the attenuation-amplification interplay, based on the Wigner distribution function instead of the reduced density matrix of the system. In fact, when diffusion is included in the analysis, it is more convenient to analyze the similarity between the evolved state vector and the original one by using the Wigner function of the system rather than its reduced density matrix.

A great number of physical systems are ruled by the attenuation-amplification interplay. The inevitable coupling of a quantum system to the environment imposes a non-null attenuation rate Γ_1 as an essential ingredient accounting for the decoherence process and the emergence of classical dynamics. Therefore, differently from the amplification rate Γ_2, Γ_1 is always a part of a realistic (nonideal) quantum evolution. Although not inevitable, amplification processes are familiar ingredients used to control coherent evolution. In atomic optics, the manipulation of electronic states is done with classical amplification fields or quantum radiation and vibrational excitations; conversely, the manipulation of the radiation and vibrational fields is achieved through their interaction with driven atoms [20,21]. However, such amplification processes are generally modeled by a single mode field with neglected linewidth. The above development thus furnishes a model for a more realistic multimodal amplification process, providing a quantitative account of the significant amount of noise—proportional to the linewidth of the amplification field—inevitably injected into the system.

As another application of the present attenuation-amplification interplay, we recall the von Neumann theory of quantum measurement [22]. In such a two-step theory, a unitary evolution is first established between the system to be measured and the apparatus, followed by a nonunitary evolution by which the quantum signal is amplified to the macroscopic regime, prompting the reduction in the state vector. It is quite reasonable to suppose that the attenuation-amplification interplay may provide a framework for modeling the nonunitary system-apparatus evolution, as formulated by von Neumann. In this connection, we note that the continuous photodetection theory has recently received a great deal of attention and significant developments have been made on this subject [23]. However, while the unitary first step of the process has been greatly advanced—leading to microscopic models of quantum jump superoperators with the inclusion of nonidealities—the nonunitary evolution is restricted to the dissipative dynamics, without accounting for the amplification process.

We finally mention that the problem of a harmonic oscillator under the action of one harmonic bath, the attenuator, had long been successfully treated through the functional integral approach [14,15], allowing significant applications in many different problems [16,17]. The quantum measurement problem had particularly benefited from the developments on dissipative quantum mechanics [18,19]. We also note that our problem, of a quantum system under the actions of two counteracting baths, could also be treated under the functional integral approach. On this regard, we note that phase factors similar to those presented in Eq. (2) to model our amplifier, must also be taken into account when pursuing the functional integral approach. Moreover, we observe that the well-known counter term taking place within the functional integral approach (the potential renormalization due to bath coupling), is associated, under our weak damping assumption, with a small shift in the frequency of the oscillator—the Lamb shift in the case where the system is a two-level atom. More specifically, under the weak damping assumption this frequency shift results to be a negligible Cauchy principal value integral [24].

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