Quantum dwell-correlation times in the scattering of two nonrelativistic particles

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In a previous paper [G. E. Hahne, J. Phys. A **36**, 7149 (2003)] the author studied a nontraditional boundary value problem associated with Schrödinger's partial differential equation for the wave function of a structureless particle moving in four-dimensional spacetime: in this boundary value problem, instead of the conventional specification of initial wave-function values on a time=constant surface, suitable time-dependent boundary and normal-derivative values are given on a three-dimensional space-time surface surrounding a slablike region of interaction in four-dimensional spacetime. The particle's time coordinate plays a natural role as an operator and observable in the modified formalism. In the present paper, the formalism is extended to describe a system of two nonrelativistic particles—each with its own time coordinate—scattering from background potentials and from one another in four-dimensional spacetime. The two-body interaction is taken as a generic noninstantaneous action-at-a-distance, which depends independently on the space-time positions of the two particles. The dynamics is expressed in terms of an integral equation for the wave function, that is, a nonrelativistic version of the Bethe-Salpeter equation. An optical theorem is derived for the transition operator associated with scattering processes; when the theorem holds, the pointwise probability current density derivable from the wave function is conserved globally, that is, in a region covering the space-time domain of significant interparticle interaction. A general formula for the expected dwell-correlation time for the two particles in the space-time region in terms of the scattering matrices is worked out.

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I. INTRODUCTION

As we are normally guided by well-established dynamical insights, and as the family of time=constant hyperplanes is a Galilean invariant in four-dimensional spacetime, it is customary in nonrelativistic quantum mechanics to consider the quantum state of a system to be defined as one of a complete set of states on some time=constant surface, and to be among the same, or an equivalent, kinematical set of states on each time=constant surface. The dynamics is then manifested as a Schrödinger-type partial differential equation, which controls the time evolution of a wave function from a given input state on one time=constant surface to an output state on any other such surface, with simple, time-independent or time-harmonic, values prescribed on the spatial boundaries. The time coordinate then appears to be a mere parameter in the formalism and was from early times not regarded as having operator or observable status ([1], p. 140, footnote; [2], p. 63, footnote), despite the impending conflict with the relativistic idea that space and time should be treated jointly and similarly ([3], p. 188; [4], p. 354). Over the years, there have been several, and often disparate, attempts to elaborate quantum mechanics so that various physical types of time (dwell times, arrival times, delay times, uncertainty in times of events, decay times, etc.) could be defined plausibly as quantum-mechanical observables subject to experimental test and verification—see the Introductory chapter in [5]. So far, apparently, no theoretical investigations have been attempted to describe quantum correlation times for two particles entering and exiting a scattering region, where the particles can interact with one another as well as with externally imposed fields in the scattering domain.

As a preliminary attempt to build such a superstructure upon quantum theory, it was suggested in [6] that single-particle quantum kinematics can be established differently, by defining a complete set of quantum states on a (say) z =constant plane, and an equivalent set of states on each of the family of z=constant planes, with the dynamics still being controlled by Schrödinger's equation: then z appears to be a mere parameter in the dynamical formalism, while the other three coordinates t, x, and y naturally take roles as operators or observables in each z=constant plane. Unlike waves evolving in the time direction, there can be reflection as well as transmission of waves moving in a spatial direction, so that waves representing motion in both directions must be described by the aforementioned complete set of quantum states. Another complication, which is commonly encountered in multichannel quantum scattering problems, is the presence of closed as well as open-channel states, with real exponential rise or fall of the wave function in (in this case) the z direction. As discussed in [6], under fairly general conditions quantum dynamics conserves the z component of the probability current overall, and a judicious choice of what constitute input states and what output states to a scattering process allows these states to have separately positive definite, and equal, norms; whence, the S matrix that maps input into output is unitary. In particular, if the particle is considered to be free everywhere and every when outside the slablike region contained between two z=constant planes, and if the interaction part of the Hamiltonian is suitably well behaved, it is physically unambiguous what constitute ingoing waves and what outgoing waves with respect to the slab, at least for open channels. A four-current density of the flow of time is then constructed straightforwardly from the probability flow current, such that the four-divergence of the time

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current represents the density of the creation of time at each point; an application of the divergence theorem then permits the calculation of the creation of time in a space-time box in terms of the time current's boundary values. In turn, the dwell time for the particle in a scattering region can then be expressed in terms of the S-matrix mapping input into output. With a few exceptions, we shall not review or cite herein the extensive literature on the problem of quantum time for a single particle, as that subject was discussed in [6], and the new edition of the book [5] covers the subject in considerable detail.

The paper in hand undertakes to establish, first, a quantum-dynamical formalism for scattering of two interacting structureless particles, each with its own time coordinate; hence, the wave function for the system will be a complex-valued function of eight variables—the space-time positions of particles 1 and 2—as in $\Psi_{12}(t_1,x_1,y_1,z_1,t_2,x_2,y_2,z_2)$. Instead of a partial differential equation the equation of motion will take the form of an integral equation, which is a nonrelativistic form of the Bethe-Salpeter equation (BSE). The formalism can account for noninstantaneous interaction at a distance between the particles. [When the interactions are instantaneous, these dynamics can be reduced to a boundary value problem for a partial differential equation of Schrödinger type for a wave function $\bar{\Psi}_{12}(t,x_1,y_1,z_1,x_2,y_2,z_2)$, where t is a common time coordinate for the two particles.] The integral equation formalism can handle problems of both temporal and spatial evolutions of wave functions: the same Green's functions, transition operator, and integral equation will generate scattering states of either type, depending on the type of input states.

The dynamics is then elaborated to define a current density for the flow of probability associated with a wave function: the pointwise current density will have 16 components and amounts to the tensor product of the single-particle current densities, as in [see Eq. (22a) below for details]

$$j^{\mu_1\mu_2}(t_1,x_1,y_1,z_1,t_2,x_2,y_2,z_2),$$

$$\text{with } \mu_1=0,1,2,3, \quad \mu_2=0,1,2,3, \tag{1}$$

where 0,1,2,3 correspond to t,x,y,z components, respectively, so that (ξ_1) can stand for $(\xi_1^0,\xi_1^1,\xi_1^2,\xi_1^3)=(t_1,x_1,y_1,z_1)$, etc. The double divergence of the current density (summation convention on μ's from 0 to 3 applies) is

$$\frac{\partial^2 j^{\mu_1\mu_2}}{\partial\xi_1^{\mu_1}\,\partial\xi_2^{\mu_2}}(\xi_1,\xi_2). \tag{2}$$

This double divergence is interpreted as the pointwise density of the creation of probability in the eight-dimensional spacetime. The equations of motion do not, unlike the usual case of single-time systems, permit the definition of a probability current density such that the above double divergence is everywhere zero. An application of the (double) divergence theorem permits the global integrated creation of probability in the eight-dimensional interior region to be expressed as an integral of the normal component of the tensor product current over the six-dimensional product of the three-dimensional boundary surfaces for the individual particles. Given the validity of an optical theorem for the transition operator, then whatever the input state, the complete scattering process does not create or destroy probability globally. We remark that the proof of the optical theorem entails the use of partially time-reversed Green's functions for intermediate particle states in some of the terms in a perturbation theory expansion for the transition operator [see Eqs. (33a) and (33b) and Eqs. (34a)–(34d)]; these terms disappear when the interparticle interaction is instantaneous.

The final task begins with defining from the wave function a function representing the 16-component density of flow of the product time operator t_1t_2; this is just the product $j^{\mu_1\mu_2}(\xi_1,\xi_2)t_1t_2$. The double divergence of this current represents the local density of creation of product time at the point (ξ_1,ξ_2). The divergence theorem reduces the computation of the total amount of product time generated in the interior region to an integral over the region's surface. In turn, the integrand in the surface integral will be expressed in terms of the S matrices associated with the scattering of either one or of both particles. The resulting formula achieves the paper's title objective, that is, expressing the correlation time, which is the expected value of t_1t_2, for the two particles dwelling in the scattering region in terms of the region taken, the input wave function, and the input-to-output map.

The remainder of the paper is divided into four sections, which treat of the above several topics as follows. In Sec. II, we shall propose a preliminary version of the Bethe-Salpeter equation that governs the quantum dynamics of two particles, each with its own associated time. The proposed integral equation is based on an inference from the classical variational principle for the dynamics of two particles, each with its own time, having a given (in general, noninstantaneous) mutual interaction at a distance. The classical variational principle is modeled on the relativistically invariant variational principle of Schwarzschild, Tetrode, and Fokker for two interacting charged particles (see, e.g., [7] for a review, and [8] for a more recent work) but the nonrelativistic interaction proposed herein has no symmetries in general. In Sec. III, we propose a transition operator formalism for the two-particle problem, then infer an optical theorem that—if satisfied by the transition operator—guarantees overall probability conservation, and derive a correction to the classically derived interaction Hamiltonian such that—with the correction included—the optical theorem is satisfied by the derived transition operator. In Sec. IV, we derive a formula for the dwell-correlation time for the two particles to remain in the zone of interaction. Section V contains a discussion of the results obtained and of some possible tasks for future investigations along these lines.

There are four appendixes, all of which deal with one-particle systems. Appendix A gives a tutorial on one-particle formal scattering theory in spacetime (as distinguished from the more usual space-energy domain), for the evolution of the wave function both with time t as a parameter and with the spatial coordinate z as a parameter. This material is intended to supersede the scattering theory of [6]; but for the detailed computation of single-particle dwell (sojourn) times, crossing (arrival) times, or delay times, we refer the reader to [6]. Appendix B exhibits a classical single-particle variational principle that can be used to obtain either the Lagrangian or the Hamiltonian version of classical dynamics, and which will serve as a model for the two-particle two-time variational principle that is proposed in Eq. (3) below. Appendix C considers the relation of predicted theoretical currents of time with possible observations of times in an ensemble of single-particle scattering experiments where the dynamics is Schrödinger's equation. A reader of the technical details of this paper should begin with a study of Appendixes A–C. Appendix D gives a brief presentation of the computation of single-particle dwell times by the method of flux-flux correlations (FFCs); this material need not be studied in the preparation for the treatment of the two-particle problem.

II. BETHE-SALPETER EQUATION: FIRST APPROXIMATION

In this section we shall infer a preliminary version of an integral equation of Bethe-Salpeter type to express the quantum dynamics of two nonrelativistic particles, each with its own time coordinate. Let the particles' positions, etc., be labeled by subscripts 1 and 2. The particles' masses are m_1, m_2; electrical charges are e_1, e_2; and e_{12} is a coupling constant. We shall incorporate one-particle interactions with background electromagnetic vector fields, $e_1 A_1^{\mu_1}(\xi_1)$ and $e_2 A_2^{\mu_2}(\xi_2)$, and with a two-particle interaction in the form of a double-vector density $e_{12} A_{12}^{\mu_1\mu_2}(\xi_1,\xi_2)$ that can be noninstantaneous—in order to lighten the mathematical load, these fields will all be presumed to be of compact support in all coordinates.

We choose *ad hoc* a classical action functional modeled in Eq. (B6),

$$\begin{aligned}\mathcal{A}_{12} = \int_{\tau_{1a}}^{\tau_{1b}} d\tau_1 \Biggl\{ p_1^0 \frac{d\xi_1^0}{d\tau_1} + p_1^{j_1}\frac{d\xi_1^{j_1}}{d\tau_1} - u_1^0\Biggl[p_1^0 + \frac{1}{2m_1}\Biggl(p_1^{j_1} \\ - \frac{e_1}{c}A_1^{j_1}(\xi_1)\Biggr)\Biggl(p_1^{j_1} - \frac{e_1}{c}A_1^{j_1}(\xi_1)\Biggr) + e_1 A_1^0(\xi_1)\Biggr]\Biggr\} \\ + \int_{\tau_{2a}}^{\tau_{2b}} d\tau_2 \Biggl\{ p_2^0 \frac{d\xi_2^0}{d\tau_2} + p_2^{j_2}\frac{d\xi_2^{j_2}}{d\tau_2} - u_2^0\Biggl[p_2^0 + \frac{1}{2m_2}\Biggl(p_2^{j_2} \\ - \frac{e_2}{c}A_2^{j_2}(\xi_2)\Biggr)\Biggl(p_2^{j_2} - \frac{e_2}{c}A_2^{j_2}(\xi_2)\Biggr) + e_2 A_2^0(\xi_2)\Biggr]\Biggr\} \\ + \int_{\tau_{1a}}^{\tau_{1b}} d\tau_1 u_1^0 \int_{\tau_{2a}}^{\tau_{2b}} d\tau_2 u_2^0 \Biggl[- e_{12}A_{12}^{00}(\xi_1,\xi_2) + \frac{e_{12}}{m_1 c}\Biggl(p_1^{j_1} \\ - \frac{e_1}{c}A_1^{j_1}\Biggr)A_{12}^{j_1 0}(\xi_1,\xi_2) + \frac{e_{12}}{m_2 c}\Biggl(p_2^{j_2} - \frac{e_2}{c}A_2^{j_2}\Biggr)A_{12}^{0 j_2}(\xi_1,\xi_2) \\ - \frac{e_{12}}{m_1 m_2 c^2}\Biggl(p_1^{j_1} - \frac{e_1}{c}A_1^{j_1}\Biggr)\Biggl(p_2^{j_2} - \frac{e_2}{c}A_2^{j_2}\Biggr)A_{12}^{j_1 j_2}(\xi_1,\xi_2)\Biggr].\end{aligned} \tag{3}$$

In physical terms, the four types of two-particle contributions represent charge-charge (A_{12}^{00}), current-charge ($A_{12}^{j_1 0}$), charge-current ($A_{12}^{0 j_2}$), and current-current ($A_{12}^{j_1 j_2}$) interactions, to first order in A_{12}'s. These terms can be thought of as comprising retarded interactions with the source point at the earlier of the two times or as advanced interactions with the source point at the later of the two times. Note that Eq. (3) is gauge invariant with respect to separate gauge transformations $\Lambda_1(\xi_1)$ on $A_1^{\mu_1}(\xi_1), p_1^{\mu_1}(\tau_1)$ and $\Lambda_2(\xi_2)$ on $A_2^{\mu_2}(\xi_2), p_2^{\mu_2}(\tau_2)$ [cf. Eq. (A43)] but is not invariant with respect to gauge transformations by a $\Lambda_{12}(\xi_1,\xi_2)$ of $A^{\mu_1\mu_2}(\xi_1,\xi_2)$.

It is possible to start with an action functional of type (B1), with the two-particle contributions being of the form

$$\begin{aligned}\int_{\tau_{1a}}^{\tau_{1b}} d\tau_1 \int_{\tau_{2a}}^{\tau_{2b}} d\tau_2 e_{12}[- u_1^0 u_2^0 A_{12}^{00}(\xi_1,\xi_2) + u_1^{j_1} u_2^0 A_{12}^{j_1 0}(\xi_1,\xi_2) \\ + u_1^0 u_2^{j_2} A_{12}^{0 j_2}(\xi_1,\xi_2) - u_1^{j_1} u_2^{j_2} A_{12}^{j_1 j_2}(\xi_1,\xi_2)].\end{aligned} \tag{4}$$

Note that this variational principle is invariant, modulo the constraints, with respect to transformations by a gauge function $\Lambda_{12}(\xi_1,\xi_2)$ of the $A_{12}^{\mu_1\mu_2}(\xi_1,\xi_2)$; insofar as Eq. (4) would lead to a complicated canonical formalism that would distract from our purpose here, we shall not utilize it. In this case, the procedure corresponding to Eq. (B5) for eliminating $u_1^{j_1}/u_1^0$ and $u_2^{j_2}/u_2^0$ leads to Eq. (3) only to first order in the components of A_{12}. We shall in Sec. III analyze the necessity of introducing a different kind of higher-order terms in A_{12} at the quantum level in order to secure unitarity of the S matrix.

We want to establish a Bethe-Salpeter equation for the system wave function $\Psi_{12}^{(2)}(\xi_1,\xi_2)$ along the following lines:

$$\begin{aligned}\Psi_{12}^{(2+)}(\xi_1,\xi_2) \approx \Psi_{12}^{(1+)}(\xi_1,\xi_2) + i\hbar \int d^4\xi_1' \int d^4\xi_2' \\ \times G_1^{(1+)}(\xi_1;\xi_1')G_2^{(1+)}(\xi_2;\xi_2')\Delta H_{12}(\xi_1',\xi_2') \\ \times \Psi_{12}^{(2+)}(\xi_1',\xi_2').\end{aligned} \tag{5}$$

The superscripts 0,1,2 in parentheses indicate the number-of-particle interactions accounted for in the wave and Green's-function determinations. The factor $i\hbar$ before the integral on the right-hand side (rhs) guarantees that the product Green's function reduces to the conventional two-particle single-time Green's function when the times are made equal. This follows straightforwardly for the free-particle Green's functions from Eq. (A16), from which we can infer

$$\begin{aligned}i\hbar G_1^{(0\pm)}(t,\mathbf{r}_1;t',\mathbf{r}_1')G_2^{(0\pm)}(t,\mathbf{r}_2;t',\mathbf{r}_2') \\ = \pm \bar{G}_{12}^{(0\pm)}(t,\mathbf{r}_1,\mathbf{r}_2;t',\mathbf{r}_1',\mathbf{r}_2'),\end{aligned} \tag{6}$$

where $\bar{G}_{12}^{(0\pm)}$ is a free-particle Green's function associated with the Schrödinger operator

$$i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m_1}\boldsymbol{\nabla}_1\cdot\boldsymbol{\nabla}_1 + \frac{\hbar^2}{2m_2}\boldsymbol{\nabla}_2\cdot\boldsymbol{\nabla}_2. \tag{7}$$

A corresponding result holds for the Green's functions with one-body interactions included. If the two-body interactions are instantaneous, that is,

$$\Delta H_{12}(t_1,\mathbf{r}_1,t_2,\mathbf{r}_2) = \overline{\Delta H}_{12}(t_1,\mathbf{r}_1,\mathbf{r}_2)\,\delta^1(t_1-t_2), \tag{8}$$

then we have for the single-time wave functions

$$\bar{\Psi}^{(2+)}_{12}(t,\mathbf{r}_1,\mathbf{r}_2)=\bar{\Psi}^{(1+)}_{12}(t,\mathbf{r}_1,\mathbf{r}_2)+\int dt'\int d^3r'_1\int d^3r'_2 \times\bar{G}^{(1+)}_{12}(t,\mathbf{r}_1,\mathbf{r}_2;t',\mathbf{r}'_1,\mathbf{r}'_2)\overline{\Delta H}_{12}(t',\mathbf{r}'_1,\mathbf{r}'_2)\times\bar{\Psi}^{(2+)}_{12}(t',\mathbf{r}'_1,\mathbf{r}'_2). \quad (9)$$

In what follows, we shall secure a formalism that reduces to Eq. (9) when the two-body interactions satisfy Eq. (8). In this case, the classical-to-quantum construction is familiar, as the entire action principle (3) can be made to depend on a single common time coordinate for both particles.

The action functional of Eq. (3) is stationary for variations of $u_1^0(\tau_1)$, whereupon

$$-p_1^0-H_1^{(1)}=\int_{\tau_{2a}}^{\tau_{2b}}d\tau_2u_2^0\Delta H_{12}(\xi_1(\tau_1),\xi_2(\tau_2)), \quad (10)$$

where ΔH_{12}, symmetrized in the preparation for quantum mechanics, is

$$\begin{aligned}\Delta H_{12}(\xi_1,\xi_2)=e_{12}A_{12}^{00}-\frac{e_{12}}{2m_1c}\Bigg[\left(p_1^{j_1}-\frac{e_1}{c}A_1^{j_1}\right)A_{12}^{j_10}\\+A_{12}^{j_10}\left(p_1^{j_1}-\frac{e_1}{c}A_1^{j_1}\right)\Bigg]-\frac{e_{12}}{2m_2c}\\\times\left[\left(p_2^{j_2}-\frac{e_2}{c}A_2^{j_2}\right)A_{12}^{0j_2}+A_{12}^{0j_2}\left(p_2^{j_2}-\frac{e_2}{c}A_2^{j_2}\right)\right]\\+\frac{e_{12}}{4m_1m_2c^2}\Bigg[\left(p_1^{j_1}-\frac{e_1}{c}A_1^{j_1}\right)\left(p_2^{j_2}-\frac{e_2}{c}A_2^{j_2}\right)A_{12}^{j_1j_2}\\+\left(p_1^{j_1}-\frac{e_1}{c}A_1^{j_1}\right)A_{12}^{j_1j_2}\left(p_2^{j_2}-\frac{e_2}{c}A_2^{j_2}\right)\\+\left(p_2^{j_2}-\frac{e_2}{c}A_2^{j_2}\right)A_{12}^{j_1j_2}\left(p_1^{j_1}-\frac{e_1}{c}A_1^{j_1}\right)\\+A_{12}^{j_1j_2}\left(p_1^{j_1}-\frac{e_1}{c}A_1^{j_1}\right)\left(p_2^{j_2}-\frac{e_2}{c}A_2^{j_2}\right)\Bigg].\end{aligned} \quad (11)$$

If we apply the quantum version of the left-hand side (lhs) of Eq. (10) to both sides of Eq. (5), we find that

$$\left(i\hbar\frac{\partial}{\partial\xi_1^0}-H_1^{(1)}\right)\Psi_{12}^{(2+)}(\xi_1,\xi_2)\approx i\hbar\int d^4\xi'_2G_2^{(1+)}(\xi_2,\xi'_2)\Delta H_{12}(\xi_1,\xi'_2)\Psi_{12}^{(2+)}(\xi_1,\xi'_2). \quad (12)$$

Comparing Eqs. (10) and (12), we infer that the classical-integral-to-quantum-operation passage is something like

$$\int_{\tau_{1a}}^{\tau_{2a}}d\tau_2u_2^0\to i\hbar\int d^4\xi'_2G_2^{(1+)}(\xi_2,\xi'_2), \quad (13)$$

and similarly for an integral over a dynamical path of particle 1. Note that both sides in Eq. (13) have the physical dimension of *time*. The use of a $i\hbar G^{(1+)}$ on the rhs of Eq. (13) may not be suitable in every case; $\pm i\hbar G^{(1-)}$ or $U^{(1)}$, or possibly another entity in this category, can be preferred on various grounds. I do not know of an *a priori* rule for making this choice for interior operations in, say, a higher-order perturbation; on physical grounds, however, the final Green's function that propagates the wave function into an exterior effectively interaction-free zone, where a classical intervention or measurement can occur, should be a causal Green's function.

The upshot of these considerations is that we are motivated to use the quantum version of Eq. (11) as the ΔH_{12} operator in Eq. (5). This operator is ‡ Hermitian. If the two-body interactions are instantaneous as in Eq. (8), and the one-particle interactions are zero, then Eq. (9) leads to a unitary S matrix for the time evolution of the single-time wave function by an argument along the lines of the one-particle case studied in Appendix A. For instantaneous two-particle interactions with nonzero single-particle interactions, the methods of the next section can be applied to show that Eq. (9) yields a unitary S matrix. In the next section we shall study how to rework Eq. (5) so that the two-body S matrix is unitary for noninstantaneous interactions.

III. BETHE-SALPETER EQUATION: OPTICAL THEOREM AND UNITARITY

In this section, we shall develop a transition operator formalism for the two-particle two-time case that was introduced in Sec. II. We shall work in the z-evolution environment, with the intended application being to the theory of expectation values of correlation times (see Sec. IV).

The physical domain of interest is the topological product of slabs,

$$\{(\xi_1)|\,\forall\,\xi_1^0,\xi_1^1,\xi_1^2,z_{1a}<\xi_1^3<z_{1b}\}\otimes\{(\xi_2)|\,\forall\,\xi_2^0,\xi_2^1,\xi_2^2,z_{2a}<\xi_2^3<z_{2b}\}. \quad (14)$$

This product slab is to contain the compact supports of all the one- and two-particle interaction fields. The input will be a sum of products of free-single-particle wave functions of type (A13a),

$$\Psi_{12}^{(0)}(\xi_1,\xi_2)=\int_{\text{open}}d^3w_1\sum_{\zeta_1}\int_{\text{open}}d^3w_2\sum_{\zeta_2}\chi_1(\xi_1;\mathbf{w}_1,\zeta_1)\times\chi_2(\xi_2;\mathbf{w}_2,\zeta_2)g_{12}(\mathbf{w}_1,\zeta_1,\mathbf{w}_2,\zeta_2), \quad (15)$$

where we assume no closed-channel input and will generally neglect closed-channel output. g_{12} is a complex-valued input probability-amplitude function that is normalized,

$$1=\int_{\text{open}}d^3w_1\sum_{\zeta_1}\int_{\text{open}}d^3w_2\sum_{\zeta_2}|g_{12}(\mathbf{w}_1,\zeta_1,\mathbf{w}_2,\zeta_2)|^2. \quad (16)$$

A $\zeta=F$ state will be considered to represent a particle input from the left ($\xi_1^3=z_{1a}$ or $\xi_2^3=z_{2a}$) and a $\zeta=B$ state a particle input from the right ($\xi_1^3=z_{1b}$ or $\xi_2^3=z_{2b}$). The reverse ($F\leftrightarrow B$) associations hold for output wave functions.

Before considering two-particle interactions, we need to upgrade Eq. (15) into a wave function in which the one-particle interactions are accounted for. This can be accomplished by means of the Møller wave operators as in Eq. (A25a),

$$\Psi_{12}^{(1\pm)} = \Omega_1^{(1\pm)}\Omega_2^{(1\pm)}\Psi_{12}^{(0)}, \tag{17}$$

where the upper (lower) signs all go together. Let us now presume the existence of a transition operator T_{12} that converts $\Psi_{12}^{(1+)}$ into $\Psi_{12}^{(2+)}$ as follows:

$$\Psi_{12}^{(2+)}(\xi_1,\xi_2) = \Psi_{12}^{(1+)}(\xi_1,\xi_2) + \Delta\Psi_{12}^{(2+)}(\xi_1,\xi_2), \tag{18}$$

where

$$\Delta\Psi_{12}^{(2+)}(\xi_1,\xi_2) = i\hbar\int d^4\xi_1'\int d^4\xi_2'\int d^4\xi_1''\int d^4\xi_2'' \times G_1^{(1+)}(\xi_1;\xi_1')G_2^{(1+)}(\xi_2;\xi_2')T_{12}(\xi_1',\xi_2';\xi_1'',\xi_2'') \times\Psi_{12}^{(1+)}(\xi_1'',\xi_2''). \tag{19}$$

In view of Eqs. (17) and (A18) and the ‡-Hermitian conjugate of Eq. (A29a), we have

$$\Delta\Psi_{12}^{(2+)}(\xi_1,\xi_2) = i\hbar\int d^4\xi_1'\int d^4\xi_2'\int d^4\xi_1''\int d^4\xi_2'' \times G_1^{(0+)}(\xi_1;\xi_1')G_2^{(0+)}(\xi_2;\xi_2')\mathcal{T}_{12}(\xi_1',\xi_2';\xi_1'',\xi_2'') \times\Psi_{12}^{(0)}(\xi_1'',\xi_2''), \tag{20}$$

with the operator definition

$$\mathcal{T}_{12} = \Omega_1^{(1-)\ddagger}\Omega_2^{(1-)\ddagger}T_{12}\Omega_1^{(1+)}\Omega_2^{(1+)}. \tag{21}$$

Let us now infer from probability conservation in the domain (14) the analogs of the optical theorem (A27a) for the operators T_{12} and $\mathcal{T}_{12}$. The components of the operator for probability current are formed as the tensor product of operators of types (A5) and (A6),

$$j_{12}^{\mu_1\mu_2}(\Xi_1,\Xi_2) = j_1^{\mu_1}(\Xi_1)\otimes j_2^{\mu_2}(\Xi_2), \tag{22a}$$

$$J_{12}^{33}(z_1,z_2) = J_1^3(z_1)\otimes J_2^3(z_2). \tag{22b}$$

For any input state to Eq. (18), we want

$$0 = \int_{z_{1a}\le\Xi_1^3\le z_{1b}} d^4\Xi_1\int_{z_{2a}\le\Xi_2^3\le z_{2b}} d^4\Xi_2 \times\frac{\partial^2}{\partial\Xi_1^{\mu_1}\,\partial\Xi_2^{\mu_2}}\langle\Psi_{12}|j_{12}^{\mu_1\mu_2}(\Xi_1,\Xi_2)|\Psi_{12}\rangle = \langle\Psi_{12}|J_{12}^{33}(z_1,z_2)|\Psi_{12}\rangle\Big|_{z_1=z_{1a}}^{z_1=z_{1b}}\Big|_{z_2=z_{2a}}^{z_2=z_{2b}}. \tag{23}$$

With the wave function of Eq. (18) we have

$$0 = [\langle\Psi_{12}^{(1+)}|J_{12}^{33}(z_1,z_2)|\Psi_{12}^{(1+)}\rangle + \langle\Psi_{12}^{(1+)}|J_{12}^{33}(z_1,z_2)|\Delta\Psi_{12}^{(2+)}\rangle + \langle\Delta\Psi_{12}^{(2+)}|J_{12}^{33}(z_1,z_2)|\Psi_{12}^{(1+)}\rangle + \langle\Delta\Psi_{12}^{(2+)}|J_{12}^{33}(z_1,z_2) \times|\Delta\Psi_{12}^{(2+)}\rangle]\Big|_{z_1=z_{1a}}^{z_1=z_{1b}}\Big|_{z_2=z_{2a}}^{z_2=z_{2b}}. \tag{24}$$

It is not difficult to prove that the first summand on the rhs of Eq. (24) yields zero after end-point evaluations, so that terms not involving $\Delta\Psi_{12}^{(2+)}$ in square brackets on the rhs of Eq. (24) can be omitted. It is convenient to define the following associations to specify the input labels P_1,P_2 and the output labels Q_1,Q_2 as functions of the end points, and the inverse functions:

$$P_l(z_l) = \begin{cases} F, & \text{if } z_l = z_{la} \\ B, & \text{if } z_l = z_{lb} \end{cases} \quad l = 1,2, \tag{25a}$$

$$Q_l(z_l) = \begin{cases} B, & \text{if } z_l = z_{la} \\ F, & \text{if } z_l = z_{lb} \end{cases} \quad l = 1,2, \tag{25b}$$

$$z_l(P_l) = \begin{cases} z_{la}, & \text{if } P_l = F \\ z_{lb}, & \text{if } P_l = B \end{cases} \quad l = 1,2, \tag{25c}$$

$$\tilde{z}_l(Q_l) = \begin{cases} z_{la}, & \text{if } Q_l = B \\ z_{lb}, & \text{if } Q_l = F \end{cases} \quad l = 1,2. \tag{25d}$$

To evaluate the remaining three terms in Eq. (24), we observe that generic expressions for $\Psi_{12}^{(1+)}(\mathbf{s}_1,z_1,\mathbf{s}_2,z_2),\Delta\Psi_{12}^{(2+)}(\mathbf{s}_1,z_1,\mathbf{s}_2,z_2)$ in the exterior regions can be inferred from Eqs. (A34), (A35), (20), and (A17) as follows:

$$\Psi_{12}^{(1+)}(\mathbf{s}_1,z_1,\mathbf{s}_2,z_2) = \int_{\text{open}} d^3w_1\sum_{\zeta_1}\int_{\text{open}} d^3w_2\sum_{\zeta_2}\int_{\text{open}} d^3w_1'\sum_{\zeta_1'}\int_{\text{open}} d^3w_2'\sum_{\zeta_2'}\chi_1(\mathbf{s}_1\,.\,z_1;\mathbf{w}_1,\zeta_1)\chi_2(\mathbf{s}_2\,.\,z_2;\mathbf{w}_2,\zeta_2) \times[\delta_{\zeta_1P_1(z_1)}\delta^3(\mathbf{w}_1-\mathbf{w}_1')\delta_{\zeta_1\zeta_1'} + \delta_{\zeta_1Q_1(z_1)}S_1^{(1,z)}(\mathbf{w}_1,\zeta_1;\mathbf{w}_1',\zeta_1')][\delta_{\zeta_2P_2(z_2)}\delta^3(\mathbf{w}_2-\mathbf{w}_2')\delta_{\zeta_2\zeta_2'} + \delta_{\zeta_2Q_2(z_2)}S_2^{(1,z)}(\mathbf{w}_2,\zeta_2;\mathbf{w}_2',\zeta_2')]g_{12}(\mathbf{w}_1'\zeta_1',\mathbf{w}_2',\zeta_2'), \tag{26}$$

$$\Delta\Psi_{12}^{(2+)}(\mathbf{s}_1,z_1,\mathbf{s}_2,z_2) = (i\hbar)^{-1}\int_{\text{open}} d^3w_1\sum_{\zeta_1}\int_{\text{open}} d^3w_2\sum_{\zeta_2}\int_{\text{open}} d^3w_1'\sum_{\zeta_1'}\int_{\text{open}} d^3w_2'\sum_{\zeta_2'}\chi_1(\mathbf{s}_1,z_1;\mathbf{w}_1,\zeta_1)\chi_2(\mathbf{s}_2,z_2;\mathbf{w}_2,\zeta_2)\delta_{\zeta_1Q_1(z_1)} \times\delta_{\zeta_2Q_2(z_2)}\langle\mathbf{w}_1,\zeta_1,\mathbf{w}_2,\zeta_2|\mathcal{T}_{12}|\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2'\rangle g_{12}(\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2'). \tag{27}$$

We can now use the inner product formulas (A13b), including algebraic signs, to evaluate the remaining terms in Eq. (24): dropping an overall factor of $(i\hbar)^{-1}$, and using Eqs. (A13b) and (A21), Eq. (24) reduces to

$$0=\int_{\text{open}} d^3w_1\sum_{\zeta_1}\int_{\text{open}} d^3w_2\sum_{\zeta_2}\int_{\text{open}} d^3w_1'\sum_{\zeta_1'}\int_{\text{open}} d^3w_2'\sum_{\zeta_2'} g_{12}(\mathbf{w}_1,\zeta_1,\mathbf{w}_2,\zeta_2)^* \times\Bigg\{\int_{\text{open}} d^3w_1''\sum_{\zeta_1''}\int_{\text{open}} d^3w_2''\sum_{\zeta_2''}[S_1^{(1,z)\dagger}(\mathbf{w}_1,\zeta_1;\mathbf{w}_1'',\zeta_1'')S_2^{(1,z)\dagger}(\mathbf{w}_2,\zeta_2;\mathbf{w}_2'',\zeta_2'')\langle\mathbf{w}_1'',\zeta_1'',\mathbf{w}_2'',\zeta_2''|\mathcal{T}_{12}|\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2'\rangle - \langle\mathbf{w}_1,\zeta_1,\mathbf{w}_2,\zeta_2|\mathcal{T}_{12}^{\ddagger}|\mathbf{w}_1'',\zeta_1'';\mathbf{w}_2'',\zeta_2''\rangle S_1^{(1,z)}(\mathbf{w}_1'',\zeta_1'';\mathbf{w}_1',\zeta_1')S_2^{(1,z)}(\mathbf{w}_2'',\zeta_2'';\mathbf{w}_2',\zeta_2')] - (i\hbar)^{-1}\langle\mathbf{w}_1,\zeta_1,\mathbf{w}_2,\zeta_2|\mathcal{T}_{12}^{\ddagger}U_1^{(0)}U_2^{(0)}\mathcal{T}_{12}|\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2'\rangle\Bigg\}g_{12}(\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2'). \tag{28}$$

We can now apply Eq. (A40) and its ‡-Hermitian conjugate, in connection with Eq. (21), to absorb the S matrices and convert the partially-on-shell matrix elements of $\Omega_1^{(1-)}$ and $\Omega_2^{(1-)}$ to partially-on-shell matrix elements of $\Omega_1^{(1+)}$ and $\Omega_2^{(1+)}$, respectively, and similarly for the conversion of the ‡-Hermitian conjugates. Therefore, it suffices for the validity of Eq. (24) that the operator equation

$$0=\Omega_1^{(1+)\ddagger}\Omega_2^{(1+)\ddagger}[T_{12}-T_{12}^{\ddagger}-(i\hbar)^{-1}T_{12}^{\ddagger}\Omega_1^{(1-)}\Omega_2^{(1-)}U_1^{(0)}U_2^{(0)} \times\Omega_1^{(1-)\ddagger}\Omega_2^{(1-)\ddagger}T_{12}]\Omega_1^{(1+)}\Omega_2^{(2+)} \tag{29}$$

be valid. If in turn we apply Eq. (A38) and drop the operators exterior to the square brackets on the rhs of Eq. (29), we find a sufficient condition on the operator T_{12} for probability to be conserved in the two-particle scattering problem,

$$0=T_{12}-T_{12}^{\ddagger}-(i\hbar)^{-1}T_{12}^{\ddagger}U_1^{(1)}U_2^{(1)}T_{12}. \tag{30}$$

This is the desired statement of the optical theorem in the two-particle case.

We investigate now how the transition operator T_{12} relates to ΔH_{12} of Eq. (5). A straightforward generalization of Eqs. (A25b) and (A26a) does not work for the following reason. In fact, let us manipulate Eq. (5) to obtain

$$T_{12}\approx\Delta H_{12}[\mathcal{I}_1\mathcal{I}_2-i\hbar G_1^{(1+)}G_2^{(1+)}\Delta H_{12}]^{-1}. \tag{31}$$

Operator algebra along the lines of Eq. (A27a) now shows that this particular T_{12} only satisfies the optical theorem (30) to first order in the ‡-Hermitian operator ΔH_{12}, since

$$U_1^{(1)}U_2^{(1)}=(i\hbar)^2[G_1^{(1+)}-G_1^{(1-)}][G_2^{(1+)}-G_2^{(1-)}] \tag{32a}$$

$$\neq(i\hbar)^2[G_1^{(1+)}G_2^{(1+)}+G_1^{(1-)}G_2^{(1-)}]. \tag{32b}$$

Evidently what is needed in T_{12} is second- and higher-order terms in ΔH_{12} that involve the intermediate "crossed" Green's function $G_{12}^{(1\Updownarrow)}$, defined as follows:

$$G_{12}^{(1\Updownarrow)}=-i\hbar[G_1^{(1+)}G_2^{(1-)}+G_1^{(1-)}G_2^{(1+)}] \tag{33a}$$

$$=-G_{12}^{(1\Updownarrow)\ddagger}. \tag{33b}$$

This Green's function is zero if we assume equal times as in Eq. (6), corresponding to instantaneous interparticle interactions. Noninstantaneous interactions between the particles seem to entail unavoidably the use of some anticausal Green's functions in order to secure global probability conservation.

I have discovered a compact expression for a correction operator to ΔH_{12} in Eq. (5), such that the resulting Bethe-Salpeter equation conserves probability flow overall. I do not know of a concise derivation of the result from simple beginnings, however, so I shall just state the result and verify the optical theorem. Instead of Eq. (5), we shall propose the following, given in operator form, and preceded by three nested definitions:

$$M_{12}=\Delta H_{12}G_{12}^{(1\Updownarrow)}, \tag{34a}$$

$$K_{12}=\mathcal{I}_1\mathcal{I}_2+[\mathcal{I}_1\mathcal{I}_2+M_{12}M_{12}]^{1/2}, \tag{34b}$$

$$R_{12}=M_{12}(K_{12})^{-1}\Delta H_{12}=-R_{12}^{\ddagger}, \tag{34c}$$

$$\Psi_{12}^{(2+)}=\Psi_{12}^{(1+)}+i\hbar G_1^{(1+)}G_2^{(1+)}[\Delta H_{12}\mathcal{I}_1\mathcal{I}_2+R_{12}]\Psi_{12}^{(2+)}. \tag{34d}$$

The second part of Eq. (34c) requires some computation. Note that in a space-time position coordinate representation of the rhs's of Eqs. (34a)–(34d), the operators depend on 16 variables, as in $R_{12}(\xi_1',\xi_2';\xi_1'',\xi_2'')$, so $\Delta H_{12}\mathcal{I}_1\mathcal{I}_2$ stands for

$$\Delta H_{12}(\xi_1',\xi_2')\,\delta^4(\xi_1'-\xi_1'')\,\delta^4(\xi_2'-\xi_2''). \tag{35}$$

We can now infer a transition operator,

$$T_{12}=(\Delta H_{12}\mathcal{I}_1\mathcal{I}_2+R_{12})[\mathcal{I}_1\mathcal{I}_2-i\hbar G_1^{(1+)}G_2^{(1+)}(\Delta H_{12}\mathcal{I}_1\mathcal{I}_2+R_{12})]^{-1}, \tag{36}$$

and undertake to verify Eq. (30) using Eq. (36) for T_{12}. Manipulating as in Eq. (A27a), and using Eqs. (34c), (32a), and (33a), we work the rhs of Eq. (30) into

$$
\begin{aligned}
&2R_{12} - (\Delta H_{12}\mathcal{I}_1\mathcal{I}_2 - R_{12})G_{12}^{(1\Updownarrow)}(\Delta H_{12}\mathcal{I}_1\mathcal{I}_2 + R_{12}) \\
&\quad = [2M_{12}K_{12}^{-1} - (\mathcal{I}_1\mathcal{I}_2 - M_{12}K_{12}^{-1}) \\
&\quad \times M_{12}(\mathcal{I}_1\mathcal{I}_2 + M_{12}K_{12}^{-1})]\Delta H_{12} \\
&\quad = [2K_{12} - (K_{12} - M_{12})(K_{12} + M_{12})]M_{12}K_{12}^{-2}\Delta H_{12} = 0,
\end{aligned} \tag{37}
$$

where we used the fact that M_{12} and K_{12} commute, and the last step follows from straightforward algebra.

We can combine Eqs. (26) and (27) to define an S matrix for two particles,

$$
\begin{aligned}
&S_{12}^{(2,z)}(\mathbf{w}_1,\zeta_1,\mathbf{w}_2,\zeta_2;\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2') \\
&\quad = S_1^{(1,z)}(\mathbf{w}_1\zeta_1;\mathbf{w}_1',\zeta_1')S_2^{(1,z)}(\mathbf{w}_2\zeta_2;\mathbf{w}_2',\zeta_2') \\
&\qquad + (i\hbar)^{-1}\langle \mathbf{w}_1,\zeta_1,\mathbf{w}_2,\zeta_2|\mathcal{T}_{12}|\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2'\rangle.
\end{aligned} \tag{38}
$$

That $S_{12}^{(2,z)}$ is unitary is implied by the unitarity of S_1 and S_2, and by Eq. (28). We can now state the total wave function in the exterior regions in a simplified form as follows:

$$
\begin{aligned}
\Psi_{12}^{(2+)}(\mathbf{s}_1,z_1,\mathbf{s}_2,z_2) &= \int_{\text{open}} d^3w_1 \sum_{\zeta_1} \int_{\text{open}} d^3w_2 \sum_{\zeta_2} \int_{\text{open}} d^3w_1' \sum_{\zeta_1'} \int_{\text{open}} d^3w_2' \sum_{\zeta_2'} \chi_1(\mathbf{s}_1,z_1;\mathbf{w}_1,\zeta_1)\chi_2(\mathbf{s}_2,z_2;\mathbf{w}_2,\zeta_2) \\
&\times[\delta_{\zeta_1 P_1(z_1)}\delta_{\zeta_2 P_2(z_2)}\delta^3(\mathbf{w}_1-\mathbf{w}_1')\delta_{\zeta_1\zeta_1'}\delta^3(\mathbf{w}_2-\mathbf{w}_2')\delta_{\zeta_2\zeta_2'} + \delta_{\zeta_1 Q_1(z_1)}\delta_{\zeta_2 P_2(z_2)}S_1^{(1,z)}(\mathbf{w}_1,\zeta_1;\mathbf{w}_1',\zeta_1')\delta^3(\mathbf{w}_2-\mathbf{w}_2')\delta_{\zeta_2\zeta_2'} \\
&+ \delta_{\zeta_1 P_1(z_1)}\delta_{\zeta_2 Q_2(z_2)}\delta^3(\mathbf{w}_1-\mathbf{w}_1')\delta_{\zeta_1\zeta_1'}S_2^{(1,z)}(\mathbf{w}_2,\zeta_2;\mathbf{w}_2',\zeta_2') + \delta_{\zeta_1 Q_1(z_1)}\delta_{\zeta_2 Q_2(z_2)} \\
&\times S_{12}^{(2,z)}(\mathbf{w}_1,\zeta_1,\mathbf{w}_2,\zeta_2;\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2')]g_{12}(\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2').
\end{aligned} \tag{39}
$$

IV. TWO-PARTICLE DWELL-CORRELATION TIMES

We will now derive a formula for the dwell-correlation time for two particles scattering from one another and from background vector potentials. According to the principle established in [6], the 16-component current of product time is just $j^{\mu_1\mu_2}(\xi_1,\xi_2)\xi_1^0\xi_2^0$, and the total amount of product time created in the domain (14) is the integral over the domain of the double four-divergence of this current. Therefore, given the input function g_{12} of Eq. (15), the total product time created in the domain is, following an application of the divergence theorem [compare Eqs. (23) and (24)],

$$
\langle t_1 t_2\rangle_{g_{12}} = \langle \Psi_{12}^{(2+)}|J_{12}^{33}(z_1,z_2)t_1 t_2|\Psi_{12}^{(2+)}\rangle\Big|_{z_1=z_{1a}}^{z_{1b}}\Big|_{z_2=z_{2a}}^{z_{2b}}. \tag{40}
$$

Next, let us define differential operators that come up repeatedly in the following:

$$
\Upsilon_l(\mathbf{w}_l,\zeta_l,z_l) = \frac{1}{i}\frac{\partial}{\partial w_l^0} + \sigma(\zeta_l)\frac{m_l z_l}{\hbar K_l^3(\mathbf{w}_l)}, \quad \text{for } l=1,2. \tag{41}
$$

The operators t_l (with $l=1,2$) will be represented by Υ_l, as can be inferred from Eq. (C15).

The open-channel ingredients of $\Psi_{12}^{(2+)}$ in the region exterior to the scattering domain are given in Eq. (39). In evaluating Eq. (40), we shall omit contributions arising from interference between ingoing and outgoing waves. (he latter contributions were accounted for in the one-particle study of [6] [Eqs. (63b) and (91)], as were the contributions from closed-channel output in [6] [Eqs. (64b) and (93)]). This means that there will be no overlap between the four groups of terms on the rhs of Eq. (39), which involve in turn the unit, the $S_1^{(1,z)}$, the $S_2^{(1,z)}$, and the $S_{12}^{(2,z)}$ matrices. Accordingly, Eq. (40) reduces to

$$
\begin{aligned}
\langle t_1 t_2\rangle_{g_{12}} &= \int d^3w_1 \sum_{P_1=B}^{F} \int d^3w_2 \sum_{P_2=B}^{F} [g_{12}(\mathbf{w}_1,P_1,\mathbf{w}_2,P_2)^* \Upsilon_1(\mathbf{w}_1,P_1,z_1(P_1))\Upsilon_2(\mathbf{w}_2,P_2,z_2(P_2))g_{12}(\mathbf{w}_1,P_1,\mathbf{w}_2,P_2)] \\
&- \int d^3w_1' \sum_{\zeta_1'} \int d^3w_1 \sum_{Q_1=B}^{F} \int d^3w_2 \sum_{P_2=B}^{F} \int d^3w_1'' \sum_{\zeta_1''} [g_{12}(\mathbf{w}_1',\zeta_1',\mathbf{w}_2,P_2)^* S_1^{(1,z)\dagger}(\mathbf{w}_1',\zeta_1';\mathbf{w}_1,Q_1)\Upsilon_1(\mathbf{w}_1,Q_1,\tilde{z}_1(Q_1)) \\
&\times \Upsilon_2(\mathbf{w}_2,P_2,z_2(P_2))S_1^{(1,z)}(\mathbf{w}_1,Q_1;\mathbf{w}_1'',\zeta_1'')g_{12}(\mathbf{w}_1'',\zeta_1'',\mathbf{w}_2,P_2)] \\
&- \int d^3w_2' \sum_{\zeta_2'} \int d^3w_1 \sum_{P_1=B}^{F} \int d^3w_2 \sum_{Q_2=B}^{F} \int d^3w_2'' \sum_{\zeta_2''} [g_{12}(\mathbf{w}_1,P_1,\mathbf{w}_2',\zeta_2')^* S_2^{(1,z)\dagger}(\mathbf{w}_2',\zeta_2';\mathbf{w}_2,Q_2)\Upsilon_1(\mathbf{w}_1,P_1,z_1(P_1)) \\
&\times \Upsilon_2(\mathbf{w}_2,Q_2,\tilde{z}_2(Q_2))S_2^{(1,z)}(\mathbf{w}_2,Q_2;\mathbf{w}_2'',\zeta_2'')g_{12}(\mathbf{w}_1,P_1,\mathbf{w}_2'',\zeta_2'')]
\end{aligned}
$$

$$+\int d^3w_1'\sum_{\zeta_1'}\int d^3w_2'\sum_{\zeta_2'}\int d^3w_1\sum_{Q_1=B}^{F}\int d^3w_2\sum_{Q_2=B}^{F}\int d^3w_1''\sum_{\zeta_1''}\int d^3dw_2''\sum_{\zeta_2''}[g_{12}(\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2')^*$$
$$\times S_{12}^{(2,z)\dagger}(\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2';\mathbf{w}_1,Q_1,\mathbf{w}_2,Q_2)\Upsilon_1(\mathbf{w}_1,Q_1,\tilde{z}_1(Q_1))\Upsilon_2(\mathbf{w}_2,Q_2,\tilde{z}_2(Q_2))$$
$$\times S_{12}^{(2,z)}(\mathbf{w}_1,Q_1,\mathbf{w}_2,Q_2;\mathbf{w}_1'',\zeta_1'',\mathbf{w}_2'',\zeta_2'')g_{12}(\mathbf{w}_1'',\zeta_1'',\mathbf{w}_2'',\zeta_2'')]. \tag{42}$$

In order to compute the expected product dwell time with the centered times, that is,

$$\langle(t_1-\langle t_1\rangle_{g_{12}})(t_2-\langle t_2\rangle_{g_{12}})\rangle_{g_{12}}=\langle t_1t_2\rangle_{g_{12}}-2\langle t_1\rangle_{g_{12}}\langle t_2\rangle_{g_{12}} \tag{43}$$

(the factor of 2 arises because the expectation value of a constant is zero here), we need the expectation values of the operators $t_1\mathcal{I}_2$ and $\mathcal{I}_1t_2$. These can be obtained by replacing the operators Υ_2 and Υ_1, respectively, in all the integrands in Eq. (42) with the unit operator. In particular, in calculating $\langle t_1\rangle_{g_{12}}$, the summand involving no S matrices cancels with the summand involving the $S_2^{(1,z)}$ matrix due to the unitarity of the latter. The result is

$$\langle t_1\rangle_{g_{12}}=-\int d^3w_1'\sum_{\zeta_1'}\int d^3w_1\sum_{Q_1=B}^{F}\int d^3w_2\sum_{P_2=B}^{F}\int d^3w_1''\sum_{\zeta_1''}[g_{12}(\mathbf{w}_1',\zeta_1',\mathbf{w}_2,P_2)^*S_1^{(1,z)\dagger}(\mathbf{w}_1',\zeta_1';\mathbf{w}_1,Q_1)\Upsilon_1(\mathbf{w}_1,Q_1,\tilde{z}_1(Q_1))$$
$$\times S_1^{(1,z)}(\mathbf{w}_1,Q_1;\mathbf{w}_1'',\zeta_1'')g_{12}(\mathbf{w}_1'',\zeta_1'',\mathbf{w}_2,P_2)]$$
$$+\int d^3w_1'\sum_{\zeta_1'}\int d^3w_2'\sum_{\zeta_2'}\int d^3w_1\sum_{Q_1=B}^{F}\int d^3w_2\sum_{Q_2=B}^{F}\int d^3w_1''\sum_{\zeta_1''}\int d^3dw_2''\sum_{\zeta_2''}[g_{12}(\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2')^*$$
$$\times S_{12}^{(2,z)\dagger}(\mathbf{w}_1',\zeta_1',\mathbf{w}_2',\zeta_2';\mathbf{w}_1,Q_1,\mathbf{w}_2,Q_2)\Upsilon_1(\mathbf{w}_1,Q_1,\tilde{z}_1(Q_1))S_{12}^{(2,z)}(\mathbf{w}_1,Q_1,\mathbf{w}_2,Q_2;\mathbf{w}_1'',\zeta_1'',\mathbf{w}_2'',\zeta_2'')g_{12}(\mathbf{w}_1'',\zeta_1'',\mathbf{w}_2'',\zeta_2'')]. \tag{44}$$

We note that, in Eq. (44), in the term involving $S_1^{(1,z)}$, the integral and sum of g_{12}'s over $\mathbf{w}_2,\zeta_2$ can be replaced with a Hermitian density matrix,

$$\rho_1(\mathbf{w}_1'',\zeta_1'';\mathbf{w}',\zeta_1')=\int_{\text{open}}d^3w_2\sum_{\zeta_2}g_{12}(\mathbf{w}_1'',\zeta_1'',\mathbf{w}_2,\zeta_2)\times g_{12}(\mathbf{w}_1',\zeta_1',\mathbf{w}_2,\zeta_2)^*. \tag{45}$$

A similar combination of formulas holds for $\langle t_2\rangle_{g_{12}}$. We shall attempt to correlate formulas (42) and (44) with prospective physical measurements in the discussion of the next section.

V. DISCUSSION

We shall first discuss the physical and operational meanings of the results derived in Secs. III and IV. Let us consider the probability conservation of Eq. (23). In physical terms, we have the following scenario. A large number of trials, each comprising the launch of a pair of particles into the interaction zone, are affected with the common normalized correlated input amplitude $g_{12}(\mathbf{w}_1,\zeta_1,\mathbf{w}_2,\zeta_2)$, and with the input-to-geometry correlations of Eq. (25). We have four quantum states corresponding to the four summands within the square brackets on the rhs of Eq. (39): (i) the input state, (ii) the state in which particle 1 evolves and egresses but particle 2 remains in its input state, (iii) the state in which particle 1 remains in its input state while particle 2 evolves and egresses, and (iv) the state in which both particles are permitted to evolve such that both have egressed from the zone of interaction. Each of these four states in a sense has unit probability; but, in the computation of the overall result, (i) and (iv) contribute +1, while (ii) and (iii) contribute −1. A convenient rule can be inferred (we leave a detailed physical argument to the reader): when particle 1 is entering the zone, apply a minus sign; if exiting, a plus sign; multiply by the corresponding sign for particle 2; the product is the algebraic sign for all contributions from the associated substate. To show this mathematically, we substitute Eq. (39) into Eq. (23). The first through fourth summands in square brackets in Eq. (39) lead to mutually orthogonal states anywhere completely outside the zone of interaction since the P's and Q's are complementary. Each of these four parts of the wave function therefore contributes separately. The minus sign for the second and third parts arises from the products of the ± signs arising from the χ-state norms, and the ± signs arising from the end and evaluation points. The overall sign at each evaluation point is $\sigma(P_1(z_1))\sigma(P_2(z_2))\sigma(Q_1(z_1))\sigma(Q_2(z_2))$ for the contributions to (i), $\sigma(Q_1(z_1))\sigma(P_2(z_2))\sigma(Q_1(z_1))\sigma(Q_2(z_2))$ for the contributions to (ii), $\sigma(P_1(z_1))\sigma(Q_2(z_2))\sigma(Q_1(z_1))\sigma(Q_2(z_2))$ for the contributions to (iii), and $\sigma(Q_1(z_1))\sigma(Q_2(z_2))\sigma(Q_1(z_1))\sigma(Q_2(z_2))$ for the contributions to (iv); this algebraic sign is the same for each contribution to (i), (ii), (iii), and (iv), and is +1, −1, −1, +1, respectively. Hence, probability is conserved.

We now consider some aspects of the computation or measurement of the expectation value of a nontrivial operator, in particular t_1t_2. The theoretical prediction of this result is Eq. (42). The contributions arising from interference between ingoing and outgoing waves were assumed to be of negligible importance; correspondingly, a measurement apparatus for testing the theory is presumed to be able to respect this separation of effects. The operators Υ_l (with $l=1,2$) represent crossing times. To measure $\langle t_1 t_2\rangle_{g_{12}}$, we measure four times for each trial in an ensemble: $(t_1)_{\text{in}}$ and $(t_2)_{\text{in}}$, which are the respective ingress times of particles 1 and 2, and $(t_1)_{\text{ex}}$ and $(t_2)_{\text{ex}}$, which are the egress times. If N is the number of trials, we expect that

$$\langle t_1 t_2\rangle_{g_{12}} = \lim_{N\to\infty} N^{-1}\sum_{\text{trials}} [(t_1)_{\text{ex}} - (t_1)_{\text{in}}][(t_2)_{\text{ex}} - (t_2)_{\text{in}}] \tag{46}$$

$$= \lim_{N\to\infty} N^{-1}\sum_{\text{events}} [(t_1)_{\text{in}}(t_2)_{\text{in}} - (t_1)_{\text{ex}}(t_2)_{\text{in}} - (t_1)_{\text{in}}(t_2)_{\text{ex}} + (t_1)_{\text{ex}}(t_2)_{\text{ex}}]. \tag{47}$$

The four summands within square brackets on the rhs of Eq. (47) correspond one by one to the four summands on the rhs of Eq. (42).

Let us also consider the expectation values of the operators $t_1\mathcal{I}_2$ and $\mathcal{I}_1 t_2$. The first of these is given in Eq. (44). The physical meaning, and expected experimental correlate, of this prediction seems to be the following: given the same input amplitude g_{12}, do N trials with both particles launched into the interaction zone and determine the average value $(t_1)_{\text{ex}}$. Do another N trials with the second particle held in abeyance, where the input of particle 1 is specified by the density matrix of Eq. (45); again determine an average $(t_1)_{\text{ex}}$. Subtract the two averages. This is the predicted value for Eq. (44). A formula for large N is

$$\langle t_1\rangle_{g_{12}} = N^{-1}\sum_{\text{trials}} \{(t_1)_{\text{ex}}|\text{both particles launched}\} - N^{-1}\sum_{\text{trials}} \{(t_1)_{\text{ex}}|\text{particle 2 not launched; density matrix for input of particle 1}\}. \tag{48}$$

The BSE dates to the early 1950s [9,10]; a brief history of its origins was given recently by Salpeter [11]. The equation was directed primarily to the relativistic description of bound states in quantum mechanics and was based on Feynman's theory of positrons [12]. A subsequent publication [13] derived the BSE from quantum field theory. In the intervening years much work with the BSE has been done on relativistic bound states, but we shall not consider that work here.

Starting with the paper [14] in 1980, the BSE has been applied to nonrelativistic condensed-matter physics, in particular to the dynamics of excitons, or bound electron-hole pairs, and thereby to the optical properties of solids. There is now a website [15] dedicated to discussion of applications of the Bethe-Salpeter equation to the latter subjects. The work in hand in its present development has little overlap either with the theory of bound states or with the applications of the BSE to condensed matter.

Finally, although this possibility has not been investigated, it may be that the integral (34d) can treat systems in which one particle enters the interaction zone across a $t=$constant surface, while the other enters across a cylindrical surface in spacetime that contains the zone of interaction, with a radius as the evolution parameter. This construction could model joint problems involving bound states of the former and scattering or dwell times of the latter, problems that are conventionally described in the space-energy domain.

APPENDIX A: SINGLE-PARTICLE SCATTERING

In this appendix we shall describe that single-particle scattering theory needed for the two-particle theory advanced in the main text. We do this on the grounds that we want to show (1) that formal scattering theory extends straightforwardly to the case that the zone of interaction is compact in spacetime, so that energy is not conserved, and (2) that important dynamical ingredients—Green's functions and transition operators—are the same for the evolution of the wave function along families of $z=$constant planes as along $t=$constant planes. Unlike the formalism in [6], we shall not carry along the z derivative of the wave function as a separate entity in a two-component wave function: the wave function will be a single-component complex-scalar-valued function of the particle's four space-time coordinates, such that z derivatives are to be computed first and specialization to a particular z value is effected subsequently.

Let the particle's mass be m and its electrical charge be e, and let there be a given background four-vector electromagnetic potential field $(A^0(\xi),\mathbf{A}(\xi))$, where $\mathbf{A}=(A^1,A^2,A^3)$, and $(\xi)=(t,\mathbf{r})=(\mathbf{s},z)=(t,x,y,z)$. The four A fields are supposed to have compact support contained between the planes $t=t_a$ and $t=t_b$, and between the planes $z=z_a$ and $z=z_b$, with $t_a<t_b$ and $z_a<z_b$. The Schrödinger dynamics is governed by a Hamiltonian $H^{(1)}$ that is the sum of a free-particle Hamiltonian $H^{(0)}$ and an interaction Hamiltonian ΔH,

$$H^{(1)}(A^0,\mathbf{A}) = H^{(0)} + \Delta H(A^0,\mathbf{A}), \tag{A1a}$$

$$H^{(0)} = -\frac{\hbar^2}{2m}\nabla\cdot\nabla, \tag{A1b}$$

$$\Delta H(A^0,\mathbf{A}) = eA^0 + \frac{ie\hbar}{2mc}(\mathbf{A}\cdot\nabla + \nabla\cdot\mathbf{A}) + \frac{e^2}{2mc^2}\mathbf{A}\cdot\mathbf{A}. \tag{A1c}$$

(The parentheses around a superscript on H are to distinguish this signifier from a vector index.) After some kinematical preliminaries, we shall treat the dynamics of a particle scattering from this background field.

Let $\phi(\xi)$ and $\psi(\xi)$ be two kinematical wave functions, that is, having compact support or having rapidly decreasing magnitude in all directions in spacetime. Then in contrast to the usual inner product we define $\langle\phi|\psi\rangle$ in terms of a four-dimensional integral,

$$\langle \phi | \psi \rangle = \int d^4\xi \phi(\xi)^* \psi(\xi), \tag{A2}$$

where the integral is over all spacetime. We can define matrix elements of operators accordingly, including operators involving time derivatives. The ‡-Hermitian conjugate $\Xi^\ddagger$ of an operator Ξ will be defined as follows: for all acceptable kinematical wave functions ϕ and ψ, we must have

$$\langle \phi | \Xi^\ddagger | \psi \rangle = \langle \psi | \Xi | \phi \rangle^*. \tag{A3}$$

We call an operator ‡ Hermitian if it equals its ‡-Hermitian conjugate; the operators $i\hbar(\partial/\partial t)$, $H^{(0)}$, and ΔH are all ‡ Hermitian. Note that the ‡ operation here is not the same as Lippmann's ([16], Eq. (4)) similarly named conjugation [see also [17], Eq. (1.2.18b)]. The operators and matrix elements for the components of four-momentum p^μ are

$$p^\mu = \frac{\hbar}{i}\frac{\partial}{\partial \xi^\mu}, \quad \mu = 0,1,2,3, \tag{A4a}$$

$$\langle \phi | p^\mu | \psi \rangle = \int d^4\xi \psi(\xi)^* \frac{\hbar}{i}\frac{\partial \phi}{\partial \xi^\mu}(\xi). \tag{A4b}$$

(There is no 4×4 metric tensor in this nonrelativistic spacetime—all vector and tensor indices will be superscripts; $-p^0$ will be the energy, $+p^1$ will be the x component of linear momentum, etc.) The operators associated with the probability current density at the space-time point Ξ are $j^\mu(\Xi)$, which have the following space-time matrices:

$$\langle \xi | j^0(\Xi) | \xi' \rangle = \delta^4(\xi - \Xi)\delta^4(\Xi - \xi'), \tag{A5a}$$

$$\langle \xi | j^k(\Xi) | \xi' \rangle = \frac{\hbar}{2im}\delta^4(\xi - \Xi)\left(\underset{\longrightarrow}{\frac{\partial}{\partial \Xi^k}} - \underset{\longleftarrow}{\frac{\partial}{\partial \Xi^k}}\right)\delta^4(\Xi - \xi'),$$

$$k = 1,2,3. \tag{A5b}$$

The underarrows indicate the direction the differentiation acts when the operator is sandwiched into a matrix element with functions of ξ's. We shall have need of integrals of the normal component of the current over t=constant slices and over z=constant slices of spacetime, so we define the operators $J^0(t)$ and $J^3(z)$ as follows:

$$\langle \xi | J^0(t) | \xi' \rangle = \delta^1(\xi^0 - t)\delta^1(t - \xi^{0'})\delta^3(\mathbf{r} - \mathbf{r}'), \tag{A6a}$$

$$\langle \xi | J^3(z) | \xi' \rangle = \frac{\hbar}{2im}\delta^1(\xi^3 - z)\left(\underset{\longrightarrow}{\frac{\partial}{\partial z}} - \underset{\longleftarrow}{\frac{\partial}{\partial z}}\right) \times \delta^1(z - \xi^{3'})\delta^3(\mathbf{s} - \mathbf{s}'). \tag{A6b}$$

Matrix elements of $J^0(t)$ entail four-dimensional integrals that reduce to three-dimensional integrals over the t slice of spacetime, and hence correspond to a conventional inner product introduced into nonrelativistic quantum mechanics; if another operator Γ that commutes with $J^0(t)$ is involved, the matrix element of $J^0(t)\Gamma$ reduces to the conventional matrix element of Γ on a slice t in spacetime. Matrix elements $\langle \phi | J^3(z) | \psi \rangle$ yield the total z current across the z slice of spacetime that is associated with the transition $\psi \to \phi$. If the operator Γ commutes with $J^3(z)$, the matrix element $\langle \psi | J^3(z)\Gamma | \psi \rangle$ will be construed to be the "expectation value" in the state ψ of the current of the entity represented by Γ across the z plane; in this paper, we shall sometimes take $\Gamma = t$.

Let us define complete sets of on-shell plane-wave states for t propagation and for z propagation of solutions of the Schrödinger equation. (Note that completeness here is constrained to mean completeness within the subset of on-shell states, not to completeness for general complex-valued functions on spacetime.) This equation is

$$i\hbar\frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\psi = 0. \tag{A7}$$

In both cases we have a wave four-vector (k^μ), but the specialization to the on-shell condition is different. For t evolution we take

$$(k^\mu) = (K^0(\mathbf{k}), \mathbf{k}), \tag{A8a}$$

$$K^0(\mathbf{k}) = [\hbar/(2m)]\mathbf{k}\cdot\mathbf{k} \geq 0. \tag{A8b}$$

[Note that $\hbar k^0 = \hbar K^0(\mathbf{k})$ is the non-negative kinetic energy, and so is the negative of the eigenvalue of momentum p^0.] For z evolution we have four subcases, corresponding to forward and backward propagations and to open and closed channels. For open channels we define

$$(k^\mu) = (\mathbf{w}, \pm K^3(\mathbf{w})), \tag{A9a}$$

$$K^3(\mathbf{w}) = [(2m/\hbar)w^0 - (w^1)^2 - (w^2)^2]^{1/2} > 0, \tag{A9b}$$

and for closed channels we have

$$(k^\mu) = (\mathbf{w}, \pm i\kappa^3(\mathbf{w})), \tag{A10a}$$

$$\kappa^3(\mathbf{w}) = [(w^1)^2 + (w^2)^2 - (2m/\hbar)w^0]^{1/2} > 0. \tag{A10b}$$

The borderline cases that $\kappa^3(\mathbf{w})=0$ are studied in [6] [Eqs. (53) and (54)] but will be ignored here as being of measure zero on the w space. We have the normalized wave functions for t propagation,

$$\psi(t,\mathbf{r};\mathbf{k}) = (2\pi)^{-3/2}\exp[-iK^0(\mathbf{k})t + i\mathbf{k}\cdot\mathbf{r}], \tag{A11a}$$

$$\langle \psi(\mathbf{k}) | J^0(t_a) | \psi(\mathbf{k}') \rangle = \delta^3(\mathbf{k} - \mathbf{k}'), \quad \text{for any } t_a. \tag{A11b}$$

For z propagation we define, as in [6],

$$\zeta = \begin{cases} F, & \text{label for forward propagation along } z \\ B, & \text{label for backward propagation along } z, \end{cases} \tag{A12a}$$

$$\sigma(F) = +1, \tag{A12b}$$

$$\sigma(B) = -1. \tag{A12c}$$

We then have [be reminded that $(\mathbf{s}) = (t,x,y)$, and $(\mathbf{w}) = (w^0, w^1, w^2) = (k^0, k^1, k^2)$]

$$\chi(\mathbf{s},z;\mathbf{w},\zeta) = (2\pi)^{-3/2}\exp[i(-w^0t + w^1x + w^2y)] \times \begin{cases} \{m/[\hbar K^3(\mathbf{w})]\}^{1/2}\exp[i\sigma(\zeta)K^3(\mathbf{w})z], & \text{for } \mathbf{w} \text{ open} \\ \{m/[\hbar\kappa^3(\mathbf{w})]\}^{1/2}\exp\{-\sigma(\zeta)[i\pi/4 + \kappa^3(\mathbf{w})z]\}, & \text{for } \mathbf{w} \text{ closed,} \end{cases} \tag{A13a}$$

$$\langle\chi(\mathbf{w},\zeta)|J^3(z_a)|\chi(\mathbf{w}',\zeta')\rangle = \begin{cases} \sigma(\zeta)\delta_{\zeta\zeta'}\delta^3(\mathbf{w}-\mathbf{w}'), & \text{for } \mathbf{w} \text{ open} \\ (\delta_{\zeta F}\delta_{\zeta' B} + \delta_{\zeta B}\delta_{\zeta' F})\delta^3(\mathbf{w}-\mathbf{w}'), & \text{for } \mathbf{w} \text{ closed.} \end{cases} \tag{A13b}$$

The normalization and product integrals are independent of z_a.

Let us now define Green's functions for the free particle. We want

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\right)G^{(0\pm)}(t,\mathbf{r};t',\mathbf{r}') = \delta^1(t-t')\delta^3(\mathbf{r}-\mathbf{r}'), \tag{A14}$$

with $G^{(0+)}$ and $G^{(0-)}$ being the causal and anticausal Green's functions, respectively. We have

$$G^{(0\pm)}(t,\mathbf{r};t',\mathbf{r}') = \int d^4k\frac{\exp[-ik^0(t-t') + i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]}{\hbar(k^0 \pm i\epsilon) - [\hbar^2/(2m)]\mathbf{k}\cdot\mathbf{k}}. \tag{A15}$$

(ϵ is the usual small positive quantity that specifies the integration contour around the singularity.) If in Eq. (A15) we do the integral over k^0, we find that

$$G^{(0\pm)}(t,\mathbf{r};t',\mathbf{r}') = (\pm i\hbar)^{-1}\theta(\pm(t-t')) \times\int d^3k\psi(t,\mathbf{r};\mathbf{k})\psi(t',\mathbf{r}';\mathbf{k})^*, \tag{A16}$$

where θ is the unit step function. If in Eq. (A15) we do the integral over k^3, we find that

$$\begin{aligned}G^{(0\pm)}(\mathbf{s},z;\mathbf{s}',z') = (\pm i\hbar)^{-1}\int_{\text{open}} d^3w[\theta(\pm(z-z'))\\ \times\chi(\mathbf{s},z;\mathbf{w},F)\chi(\mathbf{s}',z';\mathbf{w},F)^*\\ + \theta(\pm(z'-z))\chi(\mathbf{s},z;\mathbf{w},B)\chi(\mathbf{s}',z';\mathbf{w},B)^*]\\ + (i\hbar)^{-1}\int_{\text{closed}} d^3w[\theta(z-z')\\ \times\chi(\mathbf{s},z;\mathbf{w},F)\chi(\mathbf{s}',z';\mathbf{w},B)^* - \theta(z'-z)\\ \times\chi(\mathbf{s},z;\mathbf{w},B)\chi(\mathbf{s}',z';\mathbf{w},F)^*].\end{aligned} \tag{A17}$$

Note that in either environment

$$G^{(0-)}(\xi;\xi') = G^{(0+)}(\xi';\xi)^* = G^{(0+)\ddagger}(\xi;\xi'). \tag{A18}$$

We shall have need of the $U^{(0)}$ function, which takes the place of the operators $\exp[-iH^{(0)}(t-t')/\hbar]$ or $2\pi\hbar\delta^1(E-H^{(0)})$ that occur in formal scattering theory when energy is conserved [see, e.g., [17], Eq. (2.5.29)],

$$U^{(0)} = (i\hbar)[G^{(0+)} - G^{(0-)}] = U^{(0)\ddagger}. \tag{A19}$$

In coordinates, we have for the t-evolution environment

$$U^{(0)}(t,\mathbf{r};t',\mathbf{r}') = \int d^3k\psi(t,\mathbf{r};\mathbf{k})\psi(t',\mathbf{r}';\mathbf{k})^*, \tag{A20}$$

and for z evolution

$$U^{(0)}(\mathbf{s},z;\mathbf{s}',z') = \int_{\text{open}} d^3w\sum_{\zeta=B}^{F}\chi(\mathbf{s},z;\mathbf{w},\zeta)\chi(\mathbf{s}',z';\mathbf{w},\zeta)^*. \tag{A21}$$

There is no contribution to $U^{(0)}$ from closed-channel states.

For the dynamics of Eq. (A1), we can infer integral equations for the forward and backward evolving wave functions,

$$\Psi^{(1\pm)}(\xi) = \Psi^{(0)}(\xi) + \int d^4\xi' G^{(0\pm)}(\xi;\xi') \times\Delta H(A^0(\xi'),\mathbf{A}(\xi'))\Psi^{(1\pm)}(\xi'). \tag{A22}$$

In the above, the superscript "(1)" means that at most one-body interactions are involved. The input state $\Psi^{(0)}(\xi)$ is defined in the two cases as follows:

$$\Psi^{(0)}(t,\mathbf{r}) = \int d^3k\psi(t,\mathbf{r};\mathbf{k})f(\mathbf{k}), \quad \text{for } t \text{ evolution,} \tag{A23a}$$

$$\Psi^{(0)}(\mathbf{s},z) = \int_{\text{open}} d^3w\sum_{\zeta=B}^{F}\chi(\mathbf{s},z;\mathbf{w},\zeta)g(\mathbf{w},\zeta), \quad \text{for } z \text{ evolution,} \tag{A23b}$$

where in Eq. (A23b) the $\zeta=F$ contribution is input from the left ($z=z_a$) of the slab, and the $\zeta=B$ part is input from the right ($z=z_b$). We are assuming that there is no closed-channel input; this simplification avoids the case that the problem is divided such that there are two adjacent or closely parallel slabs with faces $z_a<z_m<z_b$, for then at the intermediate face $z=z_m$, closed-channel states can propagate from left to right or vice versa as output from one slab and input to the other. The weight functions f and g are both normalized to 1 for unit input current,

$$\int d^3k|f(\mathbf{k})|^2 = 1, \tag{A24a}$$

$$\int_{\text{open}} d^3w \sum_{\zeta=B}^{F} |g(\mathbf{w},\zeta)|^2 = 1. \quad \text{(A24b)}$$

I have not succeeded in establishing a continuously evolving density-matrix formulation for the z evolution case; however, once the input-to-output mapping is obtained for the needed wave functions, partially incoherent mixtures of input, and accordingly of output, states can be constructed as dictated by the physics of the system. Moreover, I have not tackled the problem that a measurement is performed at an intermediate $z=z_m$ plane, which would lead to a partial collapse of the wave function (or density matrix) for the F or the B components, or both, at $z=z_m$. The resulting feedback loops, involving one or more reflections, would lead to a change in the input and the output states at $z=z_m$; something like an iterated self-consistent analysis might be needed to converge to the result.

We can solve Eq. (A22) for $\Psi^{(1\pm)}$ in terms of a formal power series, condensed as follows:

$$\Psi^{(1\pm)} = \Omega^{(1\pm)}\Psi^{(0)}, \quad \text{(A25a)}$$

$$\Omega^{(1\pm)} = [\mathcal{I} - G^{(0\pm)}\Delta H]^{-1}. \quad \text{(A25b)}$$

Here $\Omega^{(1\pm)}$ are the Møller wave operators, $\mathcal{I}$ is represented by the four-dimensional delta function as on the rhs of Eq. (A14), and each operator product entails a four-dimensional integral over spacetime. If we define the transition operator $T^{(1)}$ and manipulate its ‡-Hermitian conjugate, we have

$$T^{(1)} = \Delta H \Omega^{(1+)} = \Omega^{(1-)\ddagger}\Delta H, \quad \text{(A26a)}$$

$$T^{(1)\ddagger} = \Omega^{(1+)\ddagger}\Delta H = \Delta H \Omega^{(1-)}. \quad \text{(A26b)}$$

Considered as a function, $T^{(1)}(\xi;\xi')$ has its support in each (ξ) and (ξ') the same compact domain in spacetime as that of ΔH. We can now prove an optical theorem at the operator level [compare [17], Eq. (2.5.29)],

$$\begin{aligned} T^{(1)} - T^{(1)\ddagger} &= \Delta H[\mathcal{I} - G^{(0+)}\Delta H]^{-1} - [\mathcal{I} - \Delta H G^{(0-)}]^{-1}\Delta H \\ &= [\mathcal{I} - \Delta H G^{(0-)}]^{-1}\Delta H[G^{(0+)} - G^{(0-)}]\Delta H[\mathcal{I} \\ &\quad - G^{(0+)}\Delta H]^{-1} = (i\hbar)^{-1}T^{(1)\ddagger}U^{(0)}T^{(1)} \quad \text{(A27a)} \\ &= [\mathcal{I} - \Delta H G^{(0+)}]^{-1}\Delta H - \Delta H[\mathcal{I} - G^{(0-)}\Delta H]^{-1} \\ &= (i\hbar)^{-1}T^{(1)}U^{(0)}T^{(1)\ddagger}. \quad \text{(A27b)} \end{aligned}$$

Further manipulations of Eqs. (A25) and (A26) yield

$$\Omega^{(1+)} = \mathcal{I} + G^{(0+)}T^{(1)}, \quad \text{(A28a)}$$

$$\Omega^{(1-)} = \mathcal{I} + G^{(0-)}T^{(1)\ddagger}, \quad \text{(A28b)}$$

$$\Omega^{(1-)\ddagger} = \mathcal{I} + T^{(1)}G^{(0+)}. \quad \text{(A28c)}$$

We shall need the complete Green's functions and the U operator for the interacting case,

$$G^{(1\pm)} = \Omega^{(1\pm)}G^{(0\pm)} = G^{(1\mp)\ddagger}, \quad \text{(A29a)}$$

$$U^{(1)} = (i\hbar)[G^{(1+)} - G^{(1-)}] = U^{(1)\ddagger}. \quad \text{(A29b)}$$

The operator $U^{(1)}$ corresponds to $\exp[-iH^{(1)}(t-t')/\hbar]$ or $2\pi\hbar\,\delta^1(E-H^{(1)})$ of the energy-conserving case. The S matrix is defined only for on-shell open-channel states as follows:

$$S^{(1,t)}(\mathbf{k};\mathbf{k}') = \delta^3(\mathbf{k}-\mathbf{k}') + (i\hbar)^{-1}\langle\psi(\mathbf{k})|T^{(1)}|\psi(\mathbf{k}')\rangle, \quad \text{(A30a)}$$

for t evolution, with any $\mathbf{k},\mathbf{k}'$, and

$$S^{(1,z)}(\mathbf{w},\zeta;\mathbf{w}',\zeta') = \delta_{\zeta\zeta'}\delta^3(\mathbf{w}-\mathbf{w}') + (i\hbar)^{-1}\langle\chi(\mathbf{w},\zeta)|T^{(1)} \times|\chi(\mathbf{w}',\zeta')\rangle \quad \text{(A30b)}$$

for z evolution, $\mathbf{w},\mathbf{w}'$ open.

Both of these S matrices prove to be unitary. In fact, we have for z evolution (note that since S's are matrices rather than space-time operators, we use "†" rather than the "‡" to denote Hermitian conjugation),

$$\begin{aligned} &\int_{\text{open}} d^3w'' \sum_{\zeta''} S^{(1,z)\dagger}(\mathbf{w},\zeta;\mathbf{w}'',\zeta'') \\ &\quad \times S^{(1,z)}(\mathbf{w}'',\zeta'';\mathbf{w}',\zeta') - \delta^3(\mathbf{w}-\mathbf{w}')\delta_{\zeta\zeta'} \\ &= (i\hbar)^{-1}\langle\chi(\mathbf{w},\zeta)|[T^{(1)} - T^{(1)\ddagger} - (i\hbar)^{-1}T^{(1)\ddagger} \\ &\quad \times \int_{\text{open}} d^3w'' \sum_{\zeta''} |\chi(\mathbf{w}'',\zeta'')\rangle\langle\chi(\mathbf{w}'',\zeta'')|T^{(1)}]|\chi(\mathbf{w}',\zeta')\rangle = 0, \end{aligned} \quad \text{(A31a)}$$

where the last step follows from Eqs. (A21) and (A27a). It follows from Eqs. (A21) and (A27b) that

$$\int_{\text{open}} d^3w'' \sum_{\zeta''} S^{(1,z)}(\mathbf{w},\zeta;\mathbf{w}'',\zeta'')S^{(1,z)\dagger}(\mathbf{w}'',\zeta'';\mathbf{w}',\zeta') = \delta^3(\mathbf{w}-\mathbf{w}')\delta_{\zeta\zeta'}. \quad \text{(A31b)}$$

One can prove similarly that both

$$\int d^3k'' S^{(1,t)\dagger}(\mathbf{k};\mathbf{k}'')S^{(1,t)}(\mathbf{k}'';\mathbf{k}') = \delta^3(\mathbf{k}-\mathbf{k}'), \quad \text{(A32a)}$$

$$\int d^3k'' S^{(1,t)}(\mathbf{k};\mathbf{k}'')S^{(1,t)\dagger}(\mathbf{k}'';\mathbf{k}') = \delta^3(\mathbf{k}-\mathbf{k}'). \quad \text{(A32b)}$$

We shall now show that unitarity of the S matrices entails probability current conservation. We shall first need expressions for the wave functions in the exterior regions, that is, at $t=t_a,t_b$ in the t-evolution case, and $z=z_a,z_b$ in the z-evolution case. Combining Eqs. (A16), (A17), and (A22), we find that the input at $\Psi^{(1+)}(t_a,\mathbf{r})$ is given by Eq. (A23a), and the output at $t=t_b$ is

$$\Psi^{(1+)}(t_b,\mathbf{r}) = \int d^3k \int d^3k'\, \psi(t_b,\mathbf{r};\mathbf{k}) S^{(1,t)}(\mathbf{k};\mathbf{k}') f(\mathbf{k}'). \tag{A33}$$

At $z=z_a$ we have, dropping any closed-channel output,

$$\Psi^{(1+)}(\mathbf{s},z_a) = \int_{\text{open}} d^3w \bigg[\chi(\mathbf{s},z_a;\mathbf{w},F) g(\mathbf{w},F) + \int_{\text{open}} d^3w' \sum_{\zeta'} \chi(\mathbf{s},z_a;\mathbf{w},B) \times S^{(1,z)}(\mathbf{w},B;\mathbf{w}',\zeta') g(\mathbf{w}',\zeta') \bigg], \tag{A34}$$

and at $z=z_b$, again dropping closed-channel output,

$$\Psi^{(1+)}(\mathbf{s},z_b) = \int_{\text{open}} d^3w \bigg[\chi(\mathbf{s},z_b;\mathbf{w},B) g(\mathbf{w},B) + \int_{\text{open}} d^3w' \sum_{\zeta'} \chi(\mathbf{s},z_b;\mathbf{w},F) \times S^{(1,z)}(\mathbf{w},F;\mathbf{w}',\zeta') g(\mathbf{w}',\zeta') \bigg]. \tag{A35}$$

It is well known that the wave function for the dynamical system represented by Eq. (A1) admits a locally conserved current [see [18], Eq. (57.2)]. The current derived from the same wave function and the operator (A5) is globally conserved, however, in that if we integrate the four-divergence of that current over a space-time domain that covers the region of significant interaction, we get zero as a result of the ‡-Hermitian property of ΔH. Therefore, with an application of the divergence theorem, we have for t evolution (μ is to be summed from 0 to 3)

$$0 = \int_{t_a\le \Xi^0 \le t_b} d^4\Xi \frac{\partial}{\partial \Xi^\mu} \langle \Psi^{(1+)} | j^\mu(\Xi) | \Psi^{(1+)} \rangle = \langle \Psi^{(1+)} | J^0(\Xi^0) | \Psi^{(1+)} \rangle \Big|_{\Xi^0=t_a}^{\Xi^0=t_b} \tag{A36a}$$

$$= \int d^3k' \int d^3k \int d^3k'' f(\mathbf{k}')^* S^{(1,t)\dagger}(\mathbf{k}';\mathbf{k}) \times S^{(1,t)}(\mathbf{k};\mathbf{k}'') f(\mathbf{k}'') - \int d^3k f(\mathbf{k})^* f(\mathbf{k}), \tag{A36b}$$

which is consistent with the unitarity of $S^{(1,t)}$. For z propagation we have, accounting for the algebraic signs in Eq. (A13b),

$$0 = \int_{z_a\le \Xi^3 \le z_b} d^4\Xi \frac{\partial}{\partial \Xi^\mu} \langle \Psi^{(1+)} | j^\mu(\Xi) | \Psi^{(1+)} \rangle = \langle \Psi^{(1+)} | J^3(\Xi^3) | \Psi^{(1+)} \rangle \Big|_{\Xi^3=z_a}^{\Xi^3=z_b} \tag{A37a}$$

$$= \int_{\text{open}} d^3w' \sum_{\zeta'} \int_{\text{open}} d^3w \sum_{\zeta} \int_{\text{open}} d^3w'' \times \sum_{\zeta''} g(\mathbf{w}',\zeta')^* S^{(1,z)\dagger}(\mathbf{w}',\zeta';\mathbf{w},\zeta) \times S^{(1,z)}(\mathbf{w},\zeta;\mathbf{w}'',\zeta'') g(\mathbf{w}'',\zeta'') - \int_{\text{open}} d^3w \sum_{\zeta} g(\mathbf{w},\zeta)^* g(\mathbf{w},\zeta), \tag{A37b}$$

which is consistent with the unitarity of $S^{(1,z)}$.

We shall need some other results for the main text. First, we show that operatorwise (and, hence, in both environments)

$$\Omega^{(1\pm)} U^{(0)} \Omega^{(1\pm)\ddagger} = U^{(1)}, \tag{A38}$$

where the upper (lower) signs correspond. In fact, we can use Eqs. (A27a) to simplify

$$\begin{aligned}\Omega^{(1-)} U^{(0)} \Omega^{(1-)\ddagger} &= (i\hbar)[\mathcal{I} + G^{(0-)} T^{(1)\ddagger}][G^{(0+)} - G^{(0-)}] \times [\mathcal{I} + T^{(1)} G^{(0+)}] \\ &= (i\hbar)[G^{(0+)} - G^{(0-)} + G^{(0+)} T^{(1)} G^{(0+)} - G^{(0-)} T^{(1)} G^{(0+)} + G^{(0-)} T^{(1)\ddagger} G^{(0+)} - G^{(0-)} T^{(1)\ddagger} G^{(0-)} + (i\hbar)^{-1} G^{(0-)} T^{(1)\ddagger} U^{(0)} T^{(1)} G^{(0+)}] \\ &= (i\hbar)[G^{(1+)} - G^{(1-)}]. \end{aligned} \tag{A39}$$

Similarly, Eq. (A27b) can be used to show the validity of Eq. (A38) with $\Omega^{(1+)}$'s.

Second, we want to show that

$$\int_{\text{open}} d^3w \sum_{\zeta} \Omega^{(1-)} |\chi(\mathbf{w},\zeta)\rangle S^{(1,z)}(\mathbf{w},\zeta;\mathbf{w}',\zeta') = \Omega^{(1+)} |\chi(\mathbf{w}',\zeta')\rangle. \tag{A40}$$

The above is to be true in general, that is, whatever be a compactly supported test function $\langle \phi |$ applied to both sides on the left. In fact, we can use Eqs. (A28b) and (A30b) to write out the lhs of Eq. (A40) and manipulate as follows:

$$\begin{aligned}&\int_{\text{open}} d^3w \sum_{\zeta} [\mathcal{I} + G^{(0-)} T^{(1)\ddagger}] |\chi(\mathbf{w},\zeta)\rangle [\delta_{\zeta\zeta'} \delta^3(\mathbf{w}-\mathbf{w}') + (i\hbar)^{-1} \langle \chi(\mathbf{w},\zeta) | T^{(1)} | \chi(\mathbf{w}',\zeta')\rangle] \\ &= \{\mathcal{I} + G^{(0-)} T^{(1)\ddagger} + (i\hbar)^{-1} [U^{(0)} T^{(1)} + G^{(0-)} T^{(1)\ddagger} U^{(0)} T^{(1)}]\} \times |\chi(\mathbf{w}',\zeta')\rangle = [\mathcal{I} + G^{(0-)} T^{(1)\ddagger} + (G^{(0+)} - G^{(0-)}) T^{(1)} + G^{(0-)} \times (T^{(1)} - T^{(1)\ddagger})] |\chi(\mathbf{w}',\zeta')\rangle \\ &= [\mathcal{I} + G^{(0+)} T^{(1)}] |\chi(\mathbf{w}',\zeta')\rangle \\ &= \Omega^{(1+)} |\chi(\mathbf{w}',\zeta')\rangle. \end{aligned} \tag{A41}$$

Third, we consider what gauge invariance entails in the dynamical system of Eq. (A1), in the integral equation presentation of Eqs. (A22). We define the Schrödinger operator $\mathcal{S}^{(1)}(A^0,\mathbf{A})$ as follows:

$$\mathcal{S}^{(1)}(A^0,\mathbf{A}) = i\hbar\frac{\partial}{\partial t} - H^{(1)}(A^0,\mathbf{A}). \tag{A42}$$

Let $\Lambda(\xi)$ be a smooth real-valued function on spacetime. Then we have the Schrödinger operator with gauge-transformed four-vector potential satisfying

$$\mathcal{S}^{(1)}(A^0 - \partial\Lambda/\partial t,\mathbf{A} + c\,\boldsymbol{\nabla}\Lambda) = \exp[ie\Lambda/\hbar]\mathcal{S}^{(1)}(A^0,\mathbf{A})\exp[-ie\Lambda/\hbar]. \tag{A43}$$

We define the gauge-transformed wave functions, Green's functions, and interaction part of the Hamiltonian with superscript Λ's as follows:

$$\Psi^{(\Lambda,1\pm)}(\xi) = \exp[ie\Lambda(\xi)/\hbar]\Psi^{(1\pm)}(\xi), \tag{A44a}$$

$$\Psi^{(\Lambda,0)}(\xi) = \exp[ie\Lambda(\xi)/\hbar]\Psi^{(0)}(\xi), \tag{A44b}$$

$$G^{(\Lambda,0\pm)}(\xi;\xi') = \exp[ie\Lambda(\xi)/\hbar]G^{(0\pm)}(\xi;\xi')\exp[-ie\Lambda(\xi')/\hbar], \tag{A44c}$$

$$\Delta H^{(\Lambda)}(A^0(\xi),\mathbf{A}(\xi)) = \exp[ie\Lambda(\xi)/\hbar][\mathcal{S}^{(1)}(0,\mathbf{0}) - \mathcal{S}^{(1)}(A^0,\mathbf{A})]\exp[-ie\Lambda(\xi)/\hbar] \tag{A44d}$$

$$= \exp[ie\Lambda(\xi)/\hbar]\Delta H(A^0,\mathbf{A}) \times\exp[-ie\Lambda(\xi)/\hbar] \tag{A44e}$$

$$\neq \Delta H(A^0 + \partial\Lambda/\partial t,\mathbf{A} + c\,\boldsymbol{\nabla}\Lambda). \tag{A44f}$$

The inequality (A44f) says that the gauge transformation of $\Delta H(A^0,\mathbf{A})$ is not achieved by transforming its ingredient four-vector potential components, unlike Eq. (A43). We now ask whether it is true that the integral (A22) is satisfied by the gauge-transformed entities, as in

$$\Psi^{(\Lambda,1\pm)}(\xi) = \Psi^{(\Lambda,0)}(\xi) + \int d^4\xi' G^{(\Lambda,0\pm)}(\xi;\xi') \times\Delta H^{(\Lambda)}(A^0(\xi'),\mathbf{A}(\xi'))\Psi^{(\Lambda,1\pm)}(\xi'). \tag{A45}$$

In fact Eq. (A45), after the substitutions of Eqs. (A44), yields Eq. (A22). We now recognize the following operator gauge transformation results:

$$\Omega^{(\Lambda,1\pm)}(\xi;\xi') = \exp[ie\Lambda(\xi)/\hbar]\Omega^{(1\pm)}(\xi;\xi')\exp[-ie\Lambda(\xi')/\hbar], \tag{A46a}$$

$$T^{(\Lambda,1)}(\xi;\xi') = \exp[ie\Lambda(\xi)/\hbar]T^{(1)}(\xi;\xi')\exp[-ie\Lambda(\xi')/\hbar], \tag{A46b}$$

$$T^{(\Lambda,1)\ddagger}(\xi;\xi') = \exp[ie\Lambda(\xi)/\hbar]T^{(1)\ddagger}(\xi;\xi')\exp[-ie\Lambda(\xi')/\hbar], \tag{A46c}$$

$$G^{(\Lambda,1\pm)}(\xi;\xi') = \exp[ie\Lambda(\xi)/\hbar]G^{(1\pm)}(\xi;\xi')\exp[-ie\Lambda(\xi')/\hbar], \tag{A46d}$$

$$j^{(\Lambda)\mu}(\Xi) = \exp[ie\Lambda(\xi)/\hbar]j^{\mu}(\Xi)\exp[-ie\Lambda(\xi')/\hbar] \tag{A46e}$$

$$= \begin{cases} j^0(\Xi), & \text{if } \mu = 0 \\ j^k(\Xi) - \delta^4(\xi - \Xi)\dfrac{e}{m}\dfrac{\partial\Lambda(\Xi)}{\partial\Xi^k}\delta^4(\Xi - \xi'), & \text{if } \mu = k = 1,2,3. \end{cases} \tag{A46f}$$

Finally, we shall partially restate a result from [6] [Eq. (65)] on the matrix elements of the time current for a wave function of type (A23b). We have

$$\langle\Psi^{(0)}|J^3(z)t|\Psi^{(0)}\rangle = \int_{\text{open}} d^3w\sum_{\zeta} g(\mathbf{w},\zeta)^*\sigma(\zeta)\left[\frac{1}{i}\frac{\partial g}{\partial w^0}(\mathbf{w},\zeta) + \sigma(\zeta)\frac{mz}{\hbar K^3(\mathbf{w})}g(\mathbf{w},\zeta)\right], \tag{A47}$$

wherein terms arising from interference between F and B states were omitted.

APPENDIX B: CLASSICAL VARIATIONAL PRINCIPLES FOR ONE PARTICLE

The Hamiltonian formalism in this appendix goes back at least to the work of Lanczos ([19], Chap. VI.10). We shall introduce an action functional for the one-particle problem that involves constraints and associated Lagrange multipliers, such that it reduces to either the Lagrangian form or the Hamiltonian form of the variational principle if one or another set of variational equations is factored back into the action functional. We begin with the action functional $\mathcal{A}$ (summation convention on j,k from 1 to 3, and τ is a parameter),

$$\mathcal{A} = \int_{\tau_a}^{\tau_b} d\tau \left[p^0 \left(\frac{d\xi^0}{d\tau} - u^0 \right) + p^j \left(\frac{d\xi^j}{d\tau} - u^j \right) + \frac{m}{2} \frac{u^j u^j}{u^0} - eA^0 u^0 + \frac{e}{c} A^j u^j \right]. \tag{B1}$$

The desired equations are obtained by the calculus of variations: $\mathcal{A}$ is supposed to be stationary under independent variations of the momenta p^0, p^j to recover the constraints, of u^0, u^j to obtain the momenta as functions of u's, and of the ξ^0, ξ^j to obtain differential equations of motion. In these continuously differentiable variations, the p's and u's can be varied without end-point limitations, but variations in ξ's must go to zero at the end points. We find that

$$\frac{d\xi^0}{d\tau} - u^0 = 0, \tag{B2a}$$

$$\frac{d\xi^j}{d\tau} - u^j = 0, \quad j = 1,2,3, \tag{B2b}$$

$$-p^0 - \frac{m}{2} \frac{u^j u^j}{(u^0)^2} - eA^0 = 0, \tag{B3a}$$

$$-p^j + m \frac{u^j}{u^0} + \frac{e}{c} A^j = 0, \quad j = 1,2,3, \tag{B3b}$$

$$-\frac{dp^0}{d\tau} - e \frac{\partial A^0}{\partial \xi^0} u^0 + \frac{e}{c} \frac{\partial A^j}{\partial \xi^0} u^j = 0, \tag{B4a}$$

$$-\frac{dp^j}{d\tau} - e \frac{\partial A^0}{\partial \xi^j} u^0 + \frac{e}{c} \frac{\partial A^k}{\partial \xi^j} u^k = 0, \quad j = 1,2,3. \tag{B4b}$$

The Lagrangian form of the variational principle is recovered if we use Eqs. (B2a) and (B2b) to eliminate all u's in Eq. (B1) and use the fact that, in the result, the parameter τ can be eliminated in favor of $t = \xi^0$. The Hamiltonian form of the variational principle is recovered if we use Eq. (B3b) to eliminate u^j/u^0 (with $j=1,2,3$). Let us carry out the latter procedure: we find that

$$\frac{u^j}{u^0} = \frac{1}{m} \left(p^j - \frac{e}{c} A^j \right). \tag{B5}$$

With the replacement (B5), the action functional becomes $\mathcal{A}'$, where

$$\mathcal{A}' = \int_{\tau_a}^{\tau_b} d\tau \left\{ p^0 \frac{d\xi^0}{d\tau} + p^j \frac{d\xi^j}{d\tau} - u^0 \left[p^0 + \frac{1}{2m} \left(p^j - \frac{e}{c} A^j \right) \times \left(p^j - \frac{e}{c} A^j \right) + eA^0 \right] \right\}. \tag{B6}$$

The quantity in square brackets on the rhs of Eq. (B6) is p^0 plus the Hamiltonian; this quantity serves as a classical model for the Schrödinger operator that annihilates dynamical wave functions [see [6], Eqs. (5)–(9)].

The equations of motion are effectively gauge invariant: if the gauge function is $\Lambda(\xi)$ and we substitute into Eqs. (B3) and (B4),

$$p^0 \to p^0 + e\, \partial \Lambda / \partial \xi^0, \tag{B7a}$$

$$p^j \to p^j + e\, \partial \Lambda / \partial \xi^j, \tag{B7b}$$

$$A^0 \to A^0 - \partial \Lambda / \partial \xi^0, \tag{B7c}$$

$$A^j \to A^j + c\, \partial \Lambda / \partial \xi^j, \tag{B7d}$$

then Eqs. (B3) are gauge invariant and Eqs.(B4) are gauge invariant modulo the constraints (B2).

APPENDIX C: PHYSICS OF ONE-PARTICLE CURRENTS

We shall evaluate and interpret currents of particle presence and particle time for a generic one-free-particle state (A23b). We find, according to Eq. (A13a), that

$$\langle \Psi^{(0)} | J^3(z) | \Psi^{(0)} \rangle = \int_{\text{open}} d^3 w [|g(\mathbf{w}, F)|^2 - |g(\mathbf{w}, B)|^2], \tag{C1}$$

which is just the net forward flow of particle presence minus the net backward flow of particle presence. For the time operator from Eq. (A47), having neglected interference terms between forward and backward flows, we find that

$$\langle \Psi^{(0)} | J^3(z) t | \Psi^{(0)} \rangle \approx \int_{\text{open}} d^3 w \left\{ \left[g(\mathbf{w}, F)^* \left(\frac{1}{i} \frac{\partial}{\partial w^0} + \frac{mz}{\hbar K^3(\mathbf{w})} \right) g(\mathbf{w}, F) \right] - \left[g(\mathbf{w}, B)^* \left(\frac{1}{i} \frac{\partial}{\partial w^0} - \frac{mz}{\hbar K^3(\mathbf{w})} \right) g(\mathbf{w}, B) \right] \right\}. \tag{C2}$$

This is the net current of time across the given z=constant surface in the positive z direction. We shall argue in the next paragraph that this current represents a mean time at which a forward-moving particle crosses the given z plane minus a mean time at which a backward-moving particle crosses the same z plane. If there are N total trials, counting both F- and B-type launches, the average current of time at z is to be taken by division by N, as follows:

$$\langle \Psi^{(0)} | J^3(z) t | \Psi^{(0)} \rangle \leftrightarrow (N)^{-1} \Big[\sum (\text{time of each forward crossing at } z) - \sum (\text{time of each backward crossing at } z) \Big]. \tag{C3}$$

In order to verify Eq. (C3), it is convenient to simplify to two space-time dimensions with coordinates (t,z); then the wave vector is (we take open channels only)

$$(w^0, \pm K^3(w^0)), \quad \text{with} \quad w^0 \geq 0,$$

$$K^3(w^0) = (2mw^0/\hbar)^{1/2} \geq 0. \tag{C4}$$

Let the free-particle wave function be

$$\Psi^{(0)}(t,z) = \int_0^\infty dw^0 \left[\frac{m}{2\pi\hbar K^3(w^0)}\right]^{1/2} \exp(-iw^0 t) \times\{\exp[iK^3(w^0)z]g(w^0,F) + \exp[-iK^3(w^0)z]g(w^0,B)\}, \tag{C5}$$

with the normalization condition [compare Eq. (A24b)]

$$1 = \int_0^\infty dw^0[|g(w^0,F)|^2 + |g(w^0,B)|^2]. \tag{C6}$$

Let us make changes of variables in the rhs's of Eqs. (C5) and (C6), both from (w^0,ζ) to k_z and from $g(w^0,\zeta)$ to $r(k_z)$,

$$k_z = \sigma(\zeta)K^3(w^0), \tag{C7a}$$

$$r(k_z) = \begin{cases} (\hbar k_z/m)^{1/2} g(\hbar k_z^2/2m,F), & \text{for } k_z > 0 \\ (\hbar |k_z|/m)^{1/2} g(\hbar k_z^2/2m,B), & \text{for } k_z < 0. \end{cases} \tag{C7b}$$

After some manipulations, Eqs. (C5) and (C6) become

$$\Psi^{(0)}(t,z) = \int_{-\infty}^{+\infty} dk_z (2\pi)^{-1/2} \exp\left[-i\frac{\hbar k_z^2}{2m}t + ik_z z\right] r(k_z), \tag{C8}$$

$$1 = \int_{-\infty}^{+\infty} dk_z |r(k_z)|^2. \tag{C9}$$

That is, $\Psi^{(0)}$ is a free-particle wave function that is normalized to 1 in the conventional sense on any t=constant slice of spacetime. This suggests that we calculate the average value of z on any t=constant slice of spacetime,

$$z_{\text{av}}(t) = \int_{-\infty}^{+\infty} dz \Psi^{(0)}(t,z)^* z \Psi^{(0)}(t,z). \tag{C10}$$

Substituting Eq. (C8) into Eq. (C10), we find next

$$z_{\text{av}}(t) = \int_{-\infty}^{+\infty} dk_z r(k_z)^* \left[i\frac{dr}{dk_z}(k_z) + \frac{\hbar k_z t}{m} r(k_z)\right]. \tag{C11}$$

Now let us apply the transformations (C7a) and (C7b) in reverse to Eq. (C11). If we define

$$S(w^0) = (2\hbar w^0/m)^{1/2} = \hbar K^3(w^0)/m, \tag{C12}$$

where $S(w^0)$ is the speed of a particle of energy $\hbar w^0$, we find now that

$$z_{\text{av}}(t) = \int_0^\infty dw^0 S(w^0) g(w^0,F)^* \left[i\frac{dg}{dw^0}(w^0,F) + \left(\frac{i}{4w^0} + t\right) g(w^0,F)\right] - \int_0^\infty dw^0 S(w^0) g(w^0,B)^* \times\left[i\frac{dg}{dw^0}(w^0,B) + \left(\frac{i}{4w^0} + t\right) g(w^0,B)\right]. \tag{C13}$$

Suppose that $g(w^0,F)$ and $g(w^0,B)$ are concentrated in magnitude around w_F^0 and w_B^0, respectively, and that the speed is slowly varying at those values of w^0. We also neglect $(i/4w^0)$ in both integrands. Then we have

$$z_{\text{av}}(t) \approx S(w_F^0)\int_0^\infty dw^0 g(w^0,F)^* \left[i\frac{dg}{dw^0}(w^0,F) + tg(w^0,F)\right] - S(w_B^0)\int_0^\infty dw^0 g(w^0,B)^* \left[i\frac{dg}{dw^0}(w^0,B) + tg(w^0,B)\right]. \tag{C14}$$

If, respectively, $g(w^0,B)$ and $g(w^0,F)$ are zero identically, we solve for t in these two cases and obtain

$$t_F \approx \int_0^\infty dw^0 g(w^0,F)^* \left[\frac{1}{i}\frac{dg}{dw^0}(w^0,F) + \frac{z_{\text{av}}(t_F)}{S(w_F^0)} g(w^0,F)\right], \tag{C15a}$$

$$t_B \approx \int_0^\infty dw^0 g(w^0,B)^* \left[\frac{1}{i}\frac{dg}{dw^0}(w^0,B) - \frac{z_{\text{av}}(t_B)}{S(w_B^0)} g(w^0,B)\right]. \tag{C15b}$$

These two expressions for time, respectively, resemble the terms on the rhs of Eq. (C2), except for the overall minus sign in front of the second square bracket in Eq. (C2). If the validity of equating parameter values with averages for both t and z is granted, we can now assert that the conjectured association (C3) has been rendered plausible.

A difficulty remains. Kijowski [20] and Mielnik [21] (the latter based his argument in part on that of the former) pointed out that a state vector made up entirely of a superposition of F-type waves can nevertheless give rise to locally negative currents on a t=constant plane. There appears to be no consistency problem in principle in measuring just the local current density as a function of position, including time, on a z=constant surface; the problem arises in obtaining both the local current density and the overall direction of motion of the particle. It was argued in [6] (Sec. 4) that only a sufficiently encompassing subset of that plane could permit the measurement of the global direction of flow of a particle at a crossing, so that the localized zones of highly oscillating flow would integrate out. We shall now try to make this assertion more quantitative in the present context. Kijowski proposed a free-particle wave function comprised of the superposition of two wave packets with strongly peaked, but distinct, positive z momenta $\hbar k_z$ and $\hbar l_z$ and adjustable relative amplitudes, at t=0,

$$\Psi(0,z) \approx \int_0^\infty dw_z \{[A\phi(w_z - k_z) + B\phi(w_z - l_z)]\exp(iw_z z)\}, \tag{C16}$$

where $\phi(u)$ resembles a Dirac delta function centered at u =0. Then, by a suitable choice of k_z, l_z, A, B, the z component of probability current can come out negative at z=0 due to interference terms between the two packet waves. Let us try to translate Kijowski's model into the formalism proposed herein. We need a wave function and its z derivative as a function of t on the $z=z_b$ line in two-dimensional spacetime. Let A and B be real, and

$$\Psi(t,z_b) = A \exp\left[ik_z z_b - i\frac{\hbar k_z^2}{2m}(t-t_b) - \frac{\alpha^2}{2}(t-t_b)^2\right] + B \exp\left[il_z z_b - i\frac{\hbar l_z^2}{2m}(t-t_b') - \frac{\alpha^2}{2}(t-t_b')^2\right]. \quad \text{(C17)}$$

In order to guarantee purely F-type states, we also need to specify $\partial\Psi/\partial z(t,z_b)$ appropriately. If the transform of Ψ is $\tilde{\Psi}(w^0,z_b)$, that is,

$$\tilde{\Psi}(w^0,z_b) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} dt \exp(iw^0 t)\Psi(t,z_b), \quad \text{(C18)}$$

we take

$$\frac{\partial\tilde{\Psi}}{\partial z}(w^0,z_b) = \begin{cases} i(2mw^0/\hbar)^{1/2}\tilde{\Psi}(w^0,z_b), & \text{if } w^0>0 \\ -(2m|w^0|/\hbar)^{1/2}\tilde{\Psi}(w^0,z_b), & \text{if } w^0<0, \end{cases} \quad \text{(C19)}$$

and apply the inverse of Eq. (C18) to yield $\partial\Psi/\partial z(t,z_b)$. Note that we have assumed that the two packets cross the $z=z_b$ plane at different times t_b and t_b': due to the uncertainty ($\approx\hbar\alpha$) in kinetic energy of each of the two packets, we cannot determine the time of crossing to an accuracy better than $\approx\alpha^{-1}$; therefore, the individual trials in the ensemble each entail a different crossing time for the packets, distributed over a time difference of about

$$|t_b - t_b'| \approx \alpha^{-1}. \quad \text{(C20)}$$

The cross terms in the z current associated with Eqs. (C18) and (C19) will be approximately

$$\left\{\frac{\hbar(k_z+l_z)}{2m}AB \exp\left[i(k_z-l_z)z_b - i\frac{\hbar k_z^2}{2m}(t-t_b) + i\frac{\hbar l_z^2}{2m}(t-t_b') - \frac{\alpha^2}{2}(t-t_b)^2 - \frac{\alpha^2}{2}(t-t_b')^2\right]\right\} + \{\text{complex conjugate}\}. \quad \text{(C21)}$$

As t ranges from t_b to t_b', the phase of the cross term changes by

$$\Delta(\text{phase}) \approx \pm\left[\frac{\hbar^2 k_z^2}{2m} - \frac{\hbar^2 l_z^2}{2m}\right](t_b'-t_b)/\hbar, \quad \text{(C22)}$$

In order for these wave packets to have distinguishable energies, we want the difference in their kinetic energies to be several times greater than their individual energy uncertainties $\hbar\alpha$. Therefore, the phase difference (C22) will be several times greater than 1, so that the interference term (C21) will oscillate considerably in the times $t_b \le t \le t_b'$, say. The upshot seems to be that across the range of uncertainty of time t of departure of the particle (manifested in the form of a quantum-mechanical amplitude and current) from the $z=z_b$ plane, its z current can oscillate a lot, and the oscillatory terms average to zero, leaving a positive residue for this superposition of F-type waves. Similarly, we expect that in observing the time of departure these oscillations will not contribute significantly to a measurement of this quantity within the known range of time uncertainty. The oscillatory terms presumably represent eddies, localized in space and time, of the current of flow of the particle's presence. The detection of these eddies requires a measurement in time and space that is so precise that information on the global direction of motion of the particle is inaccessible or lost.

We emphasize more generally that, with respect to the last paragraph's discussion, the physics of nonprobability measures—as the spatial components of currents across a surface—is inadequately analyzed at present. We reemphasize that the quantities being calculated here are net flows of the currents of particle presence and of particle time across z=constant surfaces; these reduce approximately to conventional averages with probability (i.e., non-negative) measures only in carefully controlled circumstances. Kijowski [20,22], Piron [23], and Mielnik [21] studied the problem of locally negative currents in spatial propagation vs probabilities, but supported quantitatively different explanations from that proposed here. We shall not consider this subject further in this paper.

To conclude, we evaluate the net divergence of the current of time when scattering is present. We return to four-dimensional spacetime, with the exterior wave functions of Eqs. (A34) and (A35). We find, again neglecting $F\times B$ interference, that

$$\begin{aligned}\langle\Psi^{(1+)}|[J^3(z_b)t - J^3(z_a)t]|\Psi^{(1+)}\rangle &= \Bigg\{\int_{\text{open}} d^3w'\sum_{\zeta'}\int_{\text{open}} d^3w''\sum_{\zeta''}\int_{\text{open}} d^3w\, g(\mathbf{w}',\zeta')^* \\ &\times\Bigg[S^{(1,z)\dagger}(\mathbf{w}',\zeta';\mathbf{w},B)\left(\frac{1}{i}\frac{\partial}{\partial w^0} - \frac{mz_a}{\hbar K^3(\mathbf{w})}\right) \\ &\times S^{(1,z)}(\mathbf{w},B;\mathbf{w}'',\zeta'') + S^{(1,z)\dagger}(\mathbf{w}',\zeta';\mathbf{w},F)\left(\frac{1}{i}\frac{\partial}{\partial w^0}\right. \\ &\left.+\frac{mz_b}{\hbar K^3(\mathbf{w})}\right)S^{(1,z)}(\mathbf{w},F;\mathbf{w}'',\zeta'')\Bigg]g(\mathbf{w}'',\zeta'')\Bigg\} \\ &-\Bigg\{\int_{\text{open}} d^3w\Bigg[g(\mathbf{w},B)^*\left(\frac{1}{i}\frac{\partial}{\partial w^0} - \frac{mz_b}{\hbar K^3(\mathbf{w})}\right)g(\mathbf{w},B) \\ &+ g(\mathbf{w},F)^*\left(\frac{1}{i}\frac{\partial}{\partial w^0} + \frac{mz_a}{\hbar K^3(\mathbf{w})}\right)g(\mathbf{w},F)\Bigg]\Bigg\}. \end{aligned} \quad \text{(C23)}$$

We infer from Eqs. (C3) that in observational terms the rhs of Eq. (C23) is the approximate prediction for

$$(N)^{-1}\sum_{\text{trials}}[(\text{time of egress of the particle}) - (\text{time of ingress of the particle})], \quad \text{(C24)}$$

that is, the ensemble average dwell time measured for each of the N trials.

APPENDIX D: FLUX-FLUX CORRELATIONS

Following a suggestion, I have studied the method of FFCs for computing single-particle dwell times. This method was recently investigated by Muñoz *et al.* [24], where the original derivation was given by Pollak and Miller [25].

Taking [24], Eq. (13) as a starting point, I reformulated this equation and used the methods proposed in the present work to rederive the dwell time formulas given in Eq. (C23) and in [6] [Eqs. (89), (90), and (92)]; I did not attempt to rederive the contributions in [6] [Eqs. (91) and (93)] as these involve F-wave against B-wave interference and closed-channel contributions to the dwell time, respectively. Concerning the F-B interference, the terms appear as $\pm\sin kL/kL$ in [24], five lines below Eq. (6); it is clear from [6], Eq. (91) that these contributions are very sensitive to phase and thereby to the position of a detector at either end of the slab within which dwell time is defined and measured. Moreover, a detector would have to be unable to distinguish F-type motion from B-type motion of a particle crossing through the detector at or near a given time, in order for quantum-mechanical interference to occur. I infer that F-B interference in detectors can be ignored for most purposes in computing dwell times, as I have done elsewhere in this paper.

Concerning the rederivation of the dwell time formulas of [6] [Eqs. (89), (90), and (92)], I shall merely give an initial formula and make a few remarks on the computation. We combine [24], Eqs. (13), (14), and (17), which establish an expression for the dwell time T_D for a particle in a slab in two-dimensional space-time with coordinates (t,z), such that the slab is bracketed by $z_a<z<z_b$. Then we have (ψ is the free-particle input state at time zero)

$$T_D = -\frac{1}{2}\int_{-\infty}^{+\infty}|\tau|d\tau\int_{-\infty}^{+\infty}dt\langle\psi|J(t+\tau,z_\alpha)J(t,z_\beta)|\psi\rangle\Big|_{z_\alpha=z_a}^{z_\alpha=z_b}\Big|_{z_\beta=z_a}^{z_\beta=z_b}. \tag{D1}$$

In Eq. (D1), $J(t,z)$ is the $J^3(z)$ operator of Eq. (A6b) in the Heisenberg picture, that is,

$$J(t,z) = \exp[iH^{(1)}t/\hbar]J^3(z)\exp[-iH^{(1)}t/\hbar]. \tag{D2}$$

The factor of 1/2 and $\int_{-\infty}^{+\infty}|\tau|d\tau$ in Eq. (D1) arise from taking the real part of an original expression, following [24] [Eqs. (13) and (14)]. If we re-express Eq. (D1) into the Schrödinger picture for a scattering problem where the Hamiltonian can be time dependent, we find that (all integrals from $-\infty$ to $+\infty$)

$$\begin{aligned}T_D = -\frac{1}{2}\int dt_1\int dz_1\int|\tau|d\tau\int dz_2\int dt\int dz_3\int dt_4\int dz_4\\ \times\Psi^{(1+)}(t_1,z_1)^*\langle t_1,z_1|J^3(z_\alpha)|t+\tau,z_2\rangle U^{(1)}(t+\tau,z_2;t,z_3)\\ \times\langle t,z_3|J^3(z_\beta)|t_4,z_4\rangle\Psi^{(1+)}(t_4,z_4)\Big|_{z_\alpha=z_a}^{z_\alpha=z_b}\Big|_{z_\beta=z_a}^{z_\beta=z_b}.\end{aligned} \tag{D3}$$

The wave function $\Psi^{(1+)}(t,z)$ and the operator $U^{(1)}$ are as in Eqs. (A25a) and (A38), and for z's in the slab's exterior regions they can be expressed in terms of the χ functions of Eq. (A13a) and the S matrices of Eq. (A30b). Note that because of the particular dependence of the integrand on τ and t, and the presence of $|\tau|$, Eq. (D3) is not reducible to simple space-time operator products.

The factor $|\tau|$ in the integrand gives rise to Cauchy principal-value (PV) integrals in the reduction of Eq. (D3) to integrals over the spectrum of open-channel time momenta $w^0>0$. It is worth remarking that in order to get the algebraic signs of these PV integrals to yield the desired results, I had to make the additional restrictions that $z_a<0$ and $z_b>0$; these restrictions are needed neither in the free-particle problem treated in [24], nor in the derivations of [6], Eq. (89), where any $z_a<z_b$ can be used. I have no deeper explanation of this mathematical phenomenon. In view of the length of the computations using the FFC method and of the need of a congeries of approximations to evaluate the PV integrals, I do not recommend the FFC method for calculating dwell times.

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