Speed of sound in disordered Bose-Einstein condensates

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Disorder modifies the sound-wave excitation spectrum of Bose-Einstein condensates. We consider the classical hydrodynamic limit, where the disorder correlation length is much longer than the condensate healing length. By perturbation theory, we compute the phonon lifetime and the correction to the speed of sound. This correction is found to be negative in all dimensions, with universal asymptotics for smooth correlations. Considering in detail optical speckle potentials, we find a quite rich intermediate structure. This has consequences for the average density of states, particularly in one dimension, where we find a "boson dip" next to a sharp "boson peak" as function of frequency. In one dimension, our prediction is verified in detail by a numerical integration of the Gross-Pitaevskii equation.

I. INTRODUCTION

Disorder is a key feature for the understanding of the properties of matter. In a disordered environment, waves can become coherently localized, leading to suppression of transport [1,2]. Also strong interaction can produce an insulator via the Mott-Hubbard transition [3]. The combined effects of interaction and disorder, despite being studied for decades, still hold surprises. Here we are interested in the influence of a spatially correlated disorder potential on the low-energy excitations of an interacting Bose-Einstein condensate (BEC). As Goldstone excitations, these low-energy Bogoliubov excitations feature a linear, phononlike dispersion relation $\omega_k = ck$ with sound velocity c. The sound velocity is of particular interest because it determines the range of superfluidity, according to the Landau criterion, and it determines the density of states, which enters virtually all physically relevant quantities. Moreover, the speed of sound is directly measurable in cold-atom BECs [4], where well-controlled optical speckle potentials with interesting spatial correlations can be studied [2,5].

Calculating the *effective speed of sound* $\bar{c}$ in disordered systems is far from trivial, with different approaches leading to different predictions. Perturbation theory, on the one hand, predicts an *increased* speed of sound due to three-dimensional (3D) uncorrelated disorder [6,7]. On the other hand, within a self-consistent nonperturbative approach, Yukalov and Graham [8] reported numerically a slight *decrease*. For disordered hard-core bosons on a lattice, Zhang [9] found a decrease of c to fourth order in disorder strength without information on the second-order effect.

To clarify the situation with suitable parameters for present-day BEC experiments, we study in this paper phonon excitations of a BEC in the strongly interacting case where the chemical potential μ is much larger than the disorder strength V and where the condensate healing length $\xi = \hbar / \sqrt{2m\mu}$ is much smaller than the disorder correlation length σ (see Fig. 1). Without disorder, the BEC is contained in a very shallow trap and has a constant density $n_0 = \mu / g$ in the region of interest. In the presence of smooth disorder with correlation length $\sigma \gg \xi$, the BEC ground-state density follows the external potential with the Thomas-Fermi profile $n_0(\boldsymbol{r}) = n_0[1 - V(\boldsymbol{r}) / \mu]$. A long-wavelength density deviation

$\delta n(\boldsymbol{r}, t)$ from this ground state obeys the wave equation

$$[\boldsymbol{\nabla} \cdot c^2(\boldsymbol{r}) \, \boldsymbol{\nabla} - \partial_t^2] \delta n = 0, \qquad (1)$$

where $c(\boldsymbol{r}) = c[1 - V(\boldsymbol{r}) / \mu]^{1/2}$ is the local speed of sound deviating from the clean value $c = \sqrt{\mu/m} = \sqrt{gn_0/m}$. This is a prototypical wave equation in a medium with random elasticity but constant mass density [10,11]. Quite often, the opposite case is studied, with random masses and constant elasticity or, equivalently, a fluctuating index of refraction [12–14]. The disorder potential may always be taken at zero average $\overline{V(\boldsymbol{r})} = 0$. Its strength is characterized by the variance $\overline{V(\boldsymbol{r})^2} = V^2$ and we suppose weak disorder with $V \ll \mu$.

Consider now a sound wave with wave vector k evolving on the disordered potential background with correlation length σ (see Fig. 1). If the wavelength is much longer than
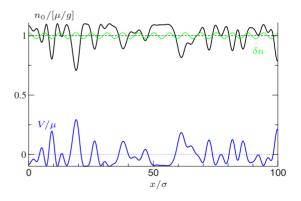

FIG. 1. (Color online) Schematic 1D representation of the system under study: an interacting Bose-Einstein condensate with original homogeneous density $n_0 = \mu / g$ (dashed black line) is exposed to a weak, spatially correlated random potential $V(\boldsymbol{r})$ (solid blue), here a blue-detuned speckle potential with amplitude $V = 0.1\mu$, centered on the mean $\overline{V} = 0$ (dotted blue). We consider the Thomas-Fermi regime where the healing length ξ is much shorter than the disorder correlation length σ. The resulting ground-state density (solid black) [Eq. (9)] mirrors the disorder while leaving the total average density and number of particles constant. On top of this disorder-modified ground state, an elementary plane-wave excitation (green, plotted around 1) propagates with wave vector k, here with $k\sigma = 1$. We calculate its effective speed of sound and the corresponding average density of states.

the correlation length, $k\sigma \ll 1$, the excitation averages over the potential fluctuations and, to a first approximation, it seems reasonable to replace Eq. (1) by its ensemble average [12]. But then we have no net effect on the speed of sound since $\overline{c^2(\mathbf{r})} = c^2$ exactly. If, on the other hand, the wavelength is much shorter than the correlation length, $k\sigma \gg 1$, the excitation evolves in a locally constant potential, which should result in an average speed of sound $\overline{c} = c\overline{[1 - V(\mathbf{r})/\mu]}^{1/2} \approx c[1 - \frac{1}{8}\overline{V^2}/\mu^2]$. It turns out, however, that both these naïve reasonings fall short.

In order to give the correct answer right away, our main results are briefly summarized in Sec. II. Section III then presents the general hydrodynamical perturbation theory, from which detailed results on the speed of sound are derived in Sec. IV. In Sec. V, we analyze the implications of these results for the disorder-averaged density of states. A short conclusion together with a brief comparison to related works is contained in Sec. VI.

II. MAIN RESULTS

The effective speed of sound in a disordered interacting Bose gas, properly defined as $\omega_k/k = \overline{c}$ from the single-excitation dispersion relation, is affected by scattering processes via virtual intermediate states such that a purely local description fails. We find that the correction $\Delta c = \overline{c} - c$ to the speed of sound has in d dimensions the limiting behavior

$$\frac{\Delta c}{c} = -\frac{\overline{V^2}}{\mu^2}\frac{1}{2d}, \quad k\sigma \ll 1 \qquad (2)$$

$$\frac{\Delta c}{c} = -\frac{\overline{V^2}}{\mu^2}\frac{2+d}{8}, \quad k\sigma \gg 1 \qquad (3)$$

These limits imply that the curves for different dimensions have to intersect around $k\sigma \approx 1$ (see also Fig. 2 below). The precise shape of $\Delta c/c$ at intermediate $k\sigma$ depends on the details of the disorder pair-correlation function. But clearly, there is a negative correction, of order $\overline{V^2}/\mu^2$, in all dimensions and for any disorder with finite correlation length $\sigma \gg \xi$.

A reduced speed of sound implies that the free density of states (DOS) of single excitations,

$$\rho_0(\omega) = \int \frac{d^d k}{(2\pi)^d} \delta(\omega - ck) = \frac{S_d}{(2\pi c)^d}\omega^{d-1}, \qquad (4)$$

is replaced by an enhanced average density of states (AV-DOS) $\overline{\rho}(\omega)$. Our results for this disorder-induced correction can be cast into the form of a function

$$g_d(\omega\sigma/c) = [\overline{\rho}(\omega) - \rho_0(\omega)]/\rho_0(\omega) \qquad (5)$$

that depends only on the reduced momentum $\kappa = \omega\sigma/c$,

$$g_d(\kappa) = -\left[d + \kappa\frac{\partial}{\partial\kappa}\right]\frac{\Delta c}{c} = \frac{\overline{V^2}}{2\mu^2}\begin{cases} 1, & \kappa \ll 1 \\ \frac{d}{4}(2+d), & \kappa \gg 1. \end{cases} \qquad (6)$$

Gurarie and Altland [12] suggested that one should be able to deduce from the asymptotic values and the curvatures of
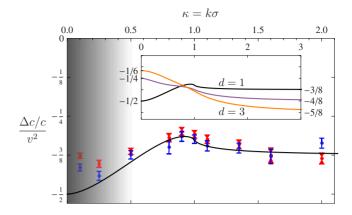

FIG. 2. (Color online) Relative correction to the speed of sound as function of reduced momentum. Black line: analytical prediction (31) for $d=1$ in the limit $\xi/\sigma=0$. Blue and red symbols: data from a numerical integration of the Gross-Pitaevskii equation (with small but finite healing length, such that $k\xi=0.05$) at $v=V/\mu=\pm 0.03$, respectively, averaged over 50 realizations of disorder. In the shaded area, the data stray far from the analytical prediction because its condition of applicability $\sigma \gg \xi$ becomes invalid. (Inset) Theory (28) for $d=1$ (black), $d=2$ (violet), and $d=3$ (orange) together with the limiting values (2) and (3).

such a scaling function whether the AVDOS exhibits a "boson peak" at intermediate frequency $\omega \approx c/\sigma$. The asymptotics of the scaling function in our case allow for a smooth, monotonic transition between the limiting values in any dimension d. Thus, one has no reason to expect any extrema in between, which is indeed found to be the case in two and three dimensions. In $d=1$, however, we find, by analytical calculation for the experimentally relevant case of an optical speckle potential, a quite nonmonotonic AVDOS with an intermediate dip followed by a sharp peak at $\omega\sigma/c=1$.

III. CLASSICAL HYDRODYNAMIC THEORY

We start our detailed analysis of the mean-field BEC order parameter $\Psi = \sqrt{n}e^{i\phi}$ in terms of the hydrodynamic variables condensate density $n = |\Psi|^2$ and phase ϕ, which determines the superfluid velocity $\mathbf{v} = \frac{\hbar}{m}\boldsymbol{\nabla}\phi$ [15,16]. The grand-canonical Gross-Pitaevskii energy functional for the BEC in presence of an external potential $V(\mathbf{r})$ is

$$E[n,\phi] = \int d^d r \left\{ \frac{\hbar^2}{2m}[(\boldsymbol{\nabla}\sqrt{n})^2 + n(\boldsymbol{\nabla}\phi)^2] + (V(\mathbf{r}) - \mu)n + \frac{g}{2}n^2 \right\}. \qquad (7)$$

The saddle-point equations $\delta E/\delta n|_0 = 0$ and $\delta E/\delta\phi|_0 = 0$ imply that the ground state has constant phase ϕ_0 or zero superfluid velocity $\mathbf{v}_0 = 0$ and a density $n_0(\mathbf{r})$ that obeys the stationary Gross-Pitaevskii equation

$$-\frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\sqrt{n_0(\mathbf{r})}}{\sqrt{n_0(\mathbf{r})}} + gn_0(\mathbf{r}) = \mu - V(\mathbf{r}). \qquad (8)$$

We now restrict our analysis to the case where the healing length ξ is much shorter than the disorder correlation length

σ. In this regime, the quantum pressure in Eq. (8) is negligible and the external disorder potential leaves the Thomas-Fermi imprint

$$n_0(\mathbf{r}) = [\mu - V(\mathbf{r})]/g. \tag{9}$$

This solution is also directly obtained if one drops the density-gradient contribution in Eq. (7) to use

$$E'[n, \phi] = \int d^d r \left\{ \frac{\hbar^2}{2m} n(\nabla \phi)^2 + [V(\mathbf{r}) - \mu]n + \frac{g}{2}n^2 \right\}. \tag{10}$$

Formally, this formulation corresponds to the limit $\xi \to 0$ and all further results can only depend on the reduced momentum $k\sigma$ [17].

The speed of sound characterizes the dynamics of small deviations $\delta n(\mathbf{r}, t) = n(\mathbf{r}, t) - n_0(\mathbf{r})$ and $\delta \phi(\mathbf{r}, t) = \phi(\mathbf{r}, t) - \phi_0$ from the ground state in the long-wavelength regime $k\xi \ll 1$. We can therefore develop the energy functional (10) to second order around the ground-state solution, $E' = E'_0 + F'[\delta n, \delta \phi]$, to obtain the relevant quadratic energy functional

$$F'[\delta n, \delta \phi] = \frac{1}{2} \int d^d r \left\{ \frac{\hbar^2}{m} n_0(\mathbf{r})(\nabla \delta \phi)^2 + g \delta n^2 \right\}. \tag{11}$$

Importantly, the external disorder potential has shifted the ground-state solution according to Eq. (9) around which we now consider the dynamics of fluctuations. Density and phase are conjugate variables with the equations of motion

$$\hbar \partial_t \delta n = \frac{\delta F'}{\delta(\delta \phi)}, \qquad -\hbar \partial_t \delta \phi = \frac{\delta F'}{\delta(\delta n)}. \tag{12}$$

In terms of density and superfluid velocity, they read

$$\partial_t \delta n + \nabla \cdot [n_0(\mathbf{r})\mathbf{v}] = 0, \tag{13}$$

$$\partial_t \mathbf{v} = -\frac{g}{m} \nabla \delta n, \tag{14}$$

and are recognized as the linearized versions of continuity equation and Euler's equation for an ideal compressible fluid, respectively. These can be combined to a single classical wave equation

$$[c^2 \nabla^2 - \partial_t^2]\delta n = \frac{1}{m} \nabla \cdot [V(\mathbf{r}) \nabla \delta n]. \tag{15}$$

This equation is equivalent to Eq. (1), but now written in a form amenable to systematic perturbation theory for a weak external disorder potential $V(\mathbf{r})$.

A. Perturbation theory

Translation invariance of the free equation suggests using a Fourier representation in space and time,

$$[\omega^2 - c^2 k^2]\delta n_k = \int \frac{d^d k'}{(2\pi)^d} \mathcal{V}_{kk'} \delta n_{k'}. \tag{16}$$

The disorder potential causes scattering $\mathbf{k} \to \mathbf{k}'$ of plane waves with an amplitude

$$\mathcal{V}_{kk'} = -\frac{1}{m}(\mathbf{k} \cdot \mathbf{k}')V_{k-k'}. \tag{17}$$

The factor $\mathbf{k} \cdot \mathbf{k}'$ originates from the mixed gradient in Eq. (15) and implies pure p-wave scattering of sound waves [18] in contrast to s-wave scattering of independent particles [19].

The single-excitation dispersion relation can be derived from the corresponding Green's function. The free Green's function is diagonal in $\mathbf{k}$,

$$G_0(k, \omega) = [\omega^2 - c^2 k^2 + i0]^{-1}. \tag{18}$$

Taking the disorder average of the full Green's function $G = [G_0^{-1} - \mathcal{V}]^{-1}$ leads in the standard way to [14,19]

$$\bar{G}(k, \omega) = [G_0(k, \omega)^{-1} - \Sigma(k, \omega)]^{-1}. \tag{19}$$

The poles of this average Green's function at $\omega^2 = c^2 k^2 + \Sigma(k, \omega)$ now determine the effective dispersion relation. The so-called self-energy $\Sigma(k, \omega)$ is given to leading order in disorder strength by the Born approximation

$$\Sigma(k, \omega) = \int \frac{d^d k'}{(2\pi)^d} \overline{[\mathcal{V}\mathcal{V}]}_{kk'} G_0(k', \omega). \tag{20}$$

The scattering potential correlator

$$\overline{[\mathcal{V}\mathcal{V}]}_{kk'} = m^{-2}V^2[\mathbf{k} \cdot \mathbf{k}']^2 \sigma^d P_d(|\mathbf{k}' - \mathbf{k}|\sigma) \tag{21}$$

involves the dimensionless k-space correlator of the bare potential

$$P_d(\kappa) = \int d^d \rho e^{-i\kappa \cdot \rho} C_d(\rho). \tag{22}$$

Its real-space correlator $C_d(r/\sigma) = \overline{V(\mathbf{r})V(0)}/V^2$ is assumed to be isotropic. We will consider correlated potentials for which $C(r/\sigma)$ decays from $C_d(0) = 1$ to 0 on the length scale σ. The smoothness of $V(\mathbf{r})$ implies that the power spectrum $P_d(\kappa)$ decreases rapidly to 0 as function of $\kappa = k\sigma$.

Applying Sokhotsky's formula $(x + i0)^{-1} = P\frac{1}{x} - i\pi\delta(x)$ to the free Green's function in Eq. (20), we can evaluate the real part and the imaginary part of the self-energy separately. The imaginary part determines the lifetime γ^{-1} of the excitations, whereas its real part shifts the speed of sound by Δc,

$$\frac{\Sigma(k, ck)}{2c^2 k^2} = \frac{\Delta c}{c} - i\frac{\gamma}{2ck}. \tag{23}$$

To leading order in V, the on-shell dispersion $\omega = ck$ is used for evaluating the self-energy.

B. Scattering rate and 1D localization length

Calculating the imaginary part in Eq. (20), the scattering rate at frequency $\omega = ck$ can be expressed as

$$\gamma(\omega) = \frac{\pi V^2}{2\mu^2} \omega^2 \rho_0(\omega) \sigma^d f_d(\omega\sigma/c). \tag{24}$$

The last factor is the angular average of the correlation function on the energy shell

$$f_d(\kappa) = S_d^{-1} \int d\Omega_d (\cos\theta)^2 P_d\left(2\kappa\sin\frac{\theta}{2}\right). \quad (25)$$

The squared cosine under the integral goes back to the $[\boldsymbol{k}\cdot\boldsymbol{k'}]^2$ in the potential correlator (21), being again characteristic for p-wave scattering of sound waves.

In one dimension, there are only the two contributions $\theta = 0, \pi$ of forward- and backscattering, respectively, such that

$$\gamma(\omega) = \frac{V^2\sigma}{4\mu^2 c}\omega^2[P_1(0) + P_1(2\omega\sigma/c)]. \quad (26)$$

We note in passing that the one-dimensional (1D) backscattering process $k \mapsto -k$ described by the second contribution $P_1(2\kappa)$ is known to induce strong Anderson localization of the excitation in the disordered potential [20]. The backscattering rate is directly proportional to the inverse localization length $\Gamma_{\mathrm{loc}} = \gamma_{\mathrm{bs}}/2c_0$ describing exponential localization [21]. Taking the backscattering contribution of Eq. (26), we find

$$\Gamma_{\mathrm{loc}} = \sigma\frac{V^2}{8\mu^2}\frac{\omega^2}{c^2}P_1(2k\sigma), \quad (27)$$

which agrees with the findings of a hydrodynamic theory similar to ours [22] and also with the sound-wave limit of Bogoliubov excitations considered in [23]. It should be noted that these latter approaches employ the phase formalism that is particularly suited for 1D systems, whereas our Green's-function theory permits to go to higher dimensions without conceptual difficulties.

In any dimension, the phase function $f_d(\kappa)$ in Eq. (25) tends to a constant for small κ and the scattering rate of low-energy excitations rate tends to zero as $\gamma/\omega \propto \omega^d$. Also the localization length in 1D diverges as $\Gamma_{\mathrm{loc}}^{-1} \propto \omega^{-2}$ at low frequency. In higher dimensions, it is known to be even larger, if not infinite [13]. This assures that low-energy excitations are long lived and extended. It is thus meaningful to discuss their effective sound velocity.

IV. EFFECTIVE SPEED OF SOUND

From Eqs. (20) and (23), the speed-of-sound shift Δc in any dimension d is obtained as a Cauchy principal-value integral over the potential correlation

$$\frac{\Delta c}{c} = -\frac{1}{2}\frac{V^2}{\mu^2}\mathrm{P}\int\frac{d^d(k'\sigma)}{(2\pi)^d}\frac{[\boldsymbol{k}\cdot\boldsymbol{k'}]^2 P_d(|\boldsymbol{k'}-\boldsymbol{k}|\sigma)}{k^2(k'^2-k^2)}. \quad (28)$$

In the limits $k\sigma \gg 1$ and $k\sigma \ll 1$ where the potential appears very smooth or δ correlated, respectively, over a wavelength of the propagating excitation, this correction is independent of the precise form of the bare potential correlator $P_d(\kappa)$ [see Eqs. (2) and (3) above and the detailed derivation in Sec. IV C below]. Let us then discuss the interesting, detailed form of this correction as function of $\kappa = k\sigma$ in $d=1$. For concreteness, we study the case of an optical speckle potential, which has recently been successfully used in experiments on Anderson localization of matter waves [2].

A. Speckle potential

By focusing a laser beam through a diffusor, the condensate is subject to a random lightshift potential proportional to the intensity of the laser field [24]. The one-point potential value $V(\boldsymbol{r})$ of a speckle pattern [25] has the skewed probability distribution

$$P(w)dw = \Theta(1+w)\exp[-(1+w)]dw \quad (29)$$

for $w = V(\boldsymbol{r})/V$. For this one-sided exponential, odd moments, such as $\overline{V(\boldsymbol{r})^3}$, are different from zero. A blue-detuned lightshift potential with $V > 0$ features repulsive peaks (this case is depicted in Fig. 1), whereas a red-detuned one with $V < 0$ consists of attractive wells. As far as spatial correlations are concerned, the laws of optics forbid variations on a length scale shorter than the correlation length σ, which depends on the laser wavelength and the geometry of the imaging system, but typically ranges around 1 μm. In one dimension, the correlation function is

$$P_1(\kappa) = \frac{\pi}{2}(2-|\kappa|)\Theta(2-|\kappa|). \quad (30)$$

Its bounded support in k space implies that within the Born approximation, backscattering and inverse localization length vanish for $k\sigma > 1$; however, exponential localization still prevails due to higher orders in perturbation theory [5].

B. In dimension $d=1$

The principal-value integral (28) over the piecewise linear function (30) is elementary and we find a speed-of-sound correction at $\kappa = k\sigma$ of

$$\frac{\Delta c}{c} = -\frac{V^2}{2\mu^2}\left(1 + \frac{\kappa}{4}\ln\left|\frac{1-\kappa}{1+\kappa}\right| - \frac{\kappa^2}{4}\ln\left|\frac{1-\kappa^2}{\kappa^2}\right|\right). \quad (31)$$

Its limiting values are $\Delta c/c = -\frac{1}{2}V^2/\mu^2$ for small κ and $\Delta c/c = -\frac{3}{8}V^2/\mu^2$ for large κ, as stated in Eqs. (2) and (3). This correction to the speed of sound, plotted in Fig. 2 as function of $\kappa = k\sigma$, shows a rather intricate, nonmonotonic behavior. Notably, there is a logarithmic nonanalyticity at $\kappa = 1$, the value beyond which backscattering is suppressed. The speed-of-sound correction is clearly *negative* for all κ, which may come as a surprise in view of [6,7].

In order to check this prediction in detail, we have numerically integrated the full Gross-Pitaevskii equation describing an elementary excitation with fixed k on top of the numerically determined ground state in a speckle potential with $V = \pm 0.03\mu$ and variable σ. This simulation operates at a small but finite value of $k\xi = 0.05$ and includes the full quantum pressure. Moreover, it does not rely on a linearization for small excitations nor perturbation theory in V. We extract the effective dispersion ω_k by monitoring the phase of $\Psi_k(t)$ and then find $\bar{c} = \omega_k/k$. As shown in Fig. 2, the data agree beautifully with Eq. (31) in its realm of validity, $\sigma \gg \xi$. When the correlation length decreases toward the healing length (shaded area in Fig. 2), the correction vanishes at fixed disorder strength because the condensate density is smoothed with respect to the Thomas-Fermi profile [26]. But in any case, only negative corrections are found in $d=1$.

C. In higher dimensions

In higher dimensions, the integral (28) over the correlation functions (for speckle, see [19]) is sufficiently complicated that analytical solutions such as Eq. (31) are not available in general. [As an exception to this rule, we find for the two-dimensional (2D) speckle correlation $\Delta c/c = -\frac{V^2}{8\mu^2}(4 - \kappa^{-2})$ for $\kappa > 1$.] But in all cases, the principle-value integral (28) can be evaluated numerically. The inset of Fig. 2 shows the corresponding curves. Short-range correlated potentials ($k\sigma \ll 1$) affect low dimensions more than high dimensions and vice versa.

The limits (2) and (3) can be calculated analytically as follows. It is useful to rewrite Eq. (28) in terms of $\eta = 1/k\sigma$ as

$$\frac{\Delta c}{c} = -\frac{V^2}{2\mu^2} P \int \frac{d^d q}{(2\pi)^d} \frac{P_d(q)[1 + \eta q \cos \beta]^2}{2 \eta q \cos \beta + \eta^2 q^2}. \quad (32)$$

Denoting the angular part of the integral by $A_d(\eta q)$, one arrives at the radial integral $\int_0^\infty dq q^{d-1} P_d(q) A_d(\eta q)$. In the limit $k\sigma \ll 1$, the parameter ηq tends to infinity nearly everywhere under the integral. Then

$$A_d(\infty) = \int \frac{d\Omega_d}{(2\pi)^d} (\cos \beta)^2 = \frac{S_d}{(2\pi)^d} d^{-1}, \quad (33)$$

and with $\int \frac{d^d q}{(2\pi)^d} P_d(q) = C_d(0) = 1$, we arrive at Eq. (2). In the limit $k\sigma \to \infty$, we proceed similarly with $\eta \to 0$. The angular integrand reduces to $1 + [2\eta \cos \beta + \eta^2]^{-1}$, whose principle-value integral evaluates after some algebra to

$$A_d(0) = \frac{S_d}{(2\pi)^d} \frac{d+2}{4}, \quad (34)$$

which leads to Eq. (3).

D. Numerical investigation beyond Born

The perturbative analysis relies on the Born approximation (20), so that good agreement with the true values is only expected at rather small disorder. Could a larger disorder strength reverse the sign of the correction? We have numerically investigated different values of $v = V/\mu$ at fixed $k\sigma = 1$. In Fig. 3, we show the data divided by v^2 such that the Born approximation shows as a horizontal line. As expected, for small $|v|$, the agreement is very satisfactory. One can distinguish a third-order correction $O(v^3)$ as a linear trend with negative slope; if needed, it could be calculated pushing Eq. (20) beyond the Born approximation [5]. In a Gaussian model with a symmetric probability distribution [27], such a third-order term would be absent.

The error bars in Figs. 2 and 3 indicate the estimated error of the mean after ensemble-averaging over 50 realizations of disorder. Figure 4 displays exemplary histograms of the values obtained for different disorder realizations. Clearly, the probability distributions are single peaked with well-defined averages on the negative side. That the speed of sound has self-averaging character was to be expected since a plane wave samples different spatial regions at once. At strong disorder with $v \gtrsim 0.1$, the speckle disorder with its un-
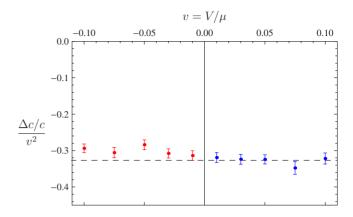

FIG. 3. (Color online) Relative correction to the speed of sound at $k\sigma = 1$, divided by v^2, as function of disorder strength $v = V/\mu$, taken from the numerical integration with $k\xi = 0.05$. (Dashed line) Analytical prediction $\Delta c/(cv^2) = \frac{1}{4}\ln 2 - \frac{1}{2} \approx -0.3267$.

bounded probability distribution is likely to fragment the condensate and the concept of a unique, well-defined speed of sound becomes questionable. As a precursor, we already observed a slight broadening of the probability distribution for $v = +0.1$ (lower right panel). From the data shown, we conclude that the correction to the speed of sound remains *negative* over the entire interval of interest.

V. DENSITY OF STATES

Knowing the speed of sound, we can compute the AV-DOS

$$\bar{\rho}(\omega) = \int \frac{d^d k}{(2\pi)^d} \delta(\omega - \omega_k) \quad (35)$$

using the effective dispersion $\omega_k = \bar{c}(k)k$ in the perturbative limit where $\gamma/\omega \ll 1$. Denoting, similarly to [12],

$$\bar{\rho}(\omega) = \rho_0(\omega)[1 + g_d(\omega \sigma/c)], \quad (36)$$
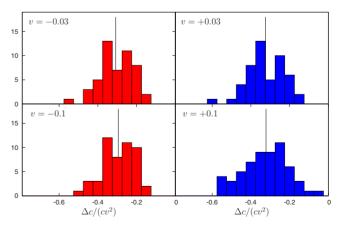
we find for the relative correction

FIG. 4. (Color online) Histograms over 50 realizations of disorder for the relative speed-of-sound correction for different values of $v = V/\mu$ obtained by numerical integration of the Gross-Pitaevskii equation with an excitation at $k\sigma = 1$. Vertical lines indicate the average values plotted in Fig. 3.
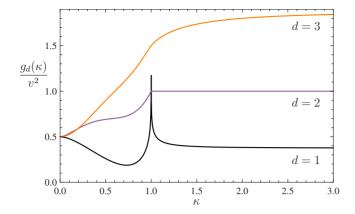
FIG. 5. (Color online) Correction to the density of states $g_d(\kappa)$ $=(\bar{\rho}-\rho_0)/\rho_0$ divided by the squared disorder strength $v=V/\mu$ as function of reduced momentum $\kappa=\omega\sigma/c$ in dimension $d=1,2,3$. At $\kappa=1$, the momentum beyond which elastic backscattering becomes impossible in the Born approximation, there is a logarithmic divergence in $d=1$, a kink in $d=2$, and a curvature discontinuity in $d=3$.

$$g_d(\kappa) = -\left[d + \kappa\frac{\partial}{\partial\kappa}\right]\frac{\Delta c}{c}, \qquad (37)$$

with the limiting values (6) [28]. The scaling functions $g_d(\kappa)$ are plotted in Fig. 5 for dimension $d=1,2,3$. In one dimension,

$$g_1(\kappa) = \frac{v^2}{2}\left(1 + \frac{\kappa}{2}\ln\left|\frac{\kappa-1}{\kappa+1}\right| - \frac{3\kappa^2}{4}\ln\left|\frac{1-\kappa^2}{\kappa^2}\right|\right) \quad (38)$$

shows a pronounced dip around $\kappa\approx0.7$ and a sharp logarithmic divergence at $\kappa=1$. This particular structure is a consequence of the Born approximation, more specifically the nonanalyticity of the speckle pair-correlation function at the boundary of its support. But a local maximum is also found for other correlated potentials with fast-enough decay of $P_d(q)$ such as Gaussian correlation [27]. The existence of this structure could not be inferred from the asymptotics of $g_1(\kappa)$ alone [12]. Indeed, expanding the asymptotic behavior as

$$g_d(\kappa) = v^2 \begin{cases} \beta_d^<(1 + \alpha_d^<\kappa^2 + \cdots), & \kappa \ll 1 \\ \beta_d^>(1 + \alpha_d^>\kappa^{-2} + \cdots), & \kappa \gg 1, \end{cases} \quad (39)$$

we find $\alpha_1^< = -1 - \frac{3}{2}|\ln\kappa| < 0$ and $\alpha_1^> = \frac{1}{18} > 0$ of opposite sign. Together with the fact that $\beta_1^<$ is larger than $\beta_1^>$, these asymptotics would be compatible with a monotonic behavior and thus are not sufficient to infer the existence of intermediate extrema.

In two dimensions, the scaling function is exactly constant for $\kappa > 1$ and thus $\alpha_2^> = 0$ which seems to happen also in other cases [12]. At $\kappa=1$, there is a kink, but overall, $g_2(\kappa)$ shows a monotonic behavior without local extrema. In three dimensions, the logarithmic singularity has moved to the second derivative of $g_3(\kappa)$, which is hardly resolvable in the figure, and $\alpha_3^> < 0$ as expected [12], leaving an all but structureless AVDOS.

VI. CONCLUSIONS

In conclusion, we have perturbatively calculated the influence of a weak spatially correlated disorder potential on the sound-wave spectrum of Bose-Einstein condensates in the hydrodynamic limit $\xi \ll \sigma, 1/k$ and arbitrary dimension. The sound-wave lifetimes are long enough to observe a disorder-induced correction to the speed of sound, which is found to be *reduced*. For the experimentally relevant case of an optical speckle potential, we compute the correction to the speed of sound analytically. A numerical integration of the full mean-field dynamics in $d=1$ confirms our prediction in its range of validity and even allows to access nonperturbative disorder strengths.

The present hydrodynamic theory compares well to results in $d=1$ from the phase-formalism approaches of Bilas and Pavloff [22] and Lugan *et al.* [23]. We find perfect agreement concerning the localization length, which we obtain from the backscattering rate.

However, our results are in contrast to the impact of uncorrelated disorder in three dimensions, for which Giorgini *et al.* [6] have predicted a positive correction to the speed of sound. Yet, at present, there appears no contradiction between their results and ours. We have used a simple hydrodynamic description valid for $\xi \ll \sigma, 1/k$ that cannot cover the case of a truly δ-correlated disorder, spatially varying on a scale $\sigma \ll \xi$, considered by Giorgini *et al.* In particular, for such rapidly varying potentials, the Thomas-Fermi approximation Eq. (9) for the ground-state density does not hold anymore and should be replaced by the solution of Eq. (8), which then shows the smoothed imprint of the disorder potential [26].

Furthermore, we have determined the average sound-wave density of states. In low dimensions, its structure is very rich, including a broad dip followed by a sharp peak in $d=1$. As a rule, specific correlation-related features tend to be washed out by integration in higher-dimensional k space. Thus we expect arguments on general grounds [12] to hold more reliably in higher dimensions. Conversely, the low-dimensional behavior may escape a bird's-eye view and require detailed calculations. We have presented such a calculation for spatially correlated speckle disorder, so that our results should be of immediate use for cold-atom experiments.

[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).

[2] J. Billy *et al.*, Nature (London) **453**, 891 (2008).

[3] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Nature (London) **415**, 39 (2002).

[4] M. R. Andrews, D. M. Kurn, H. J. Miesner, D. S. Durfee, C. G. Townsend, S. Inouye, and W. Ketterle, Phys. Rev. Lett. **79**, 553 (1997).

[5] P. Lugan, A. Aspect, L. Sanchez-Palencia, D. Delande, B. Gremaud, C. A. Müller, and C. Miniatura, Phys. Rev. A **80**, 023605 (2009).

[6] S. Giorgini, L. Pitaevskii, and S. Stringari, Phys. Rev. B **49**, 12938 (1994).

[7] G. M. Falco, A. Pelster, and R. Graham, Phys. Rev. A **75**, 063619 (2007).

[8] V. I. Yukalov and R. Graham, Phys. Rev. A **75**, 023619 (2007).

[9] L. Zhang, Phys. Rev. B **47**, 14364 (1993).

[10] V. Gurarie and J. T. Chalker, Phys. Rev. B **68**, 134207 (2003).

[11] C. Gaul and H. Büttner, Phys. Rev. E **76**, 011111 (2007).

[12] V. Gurarie and A. Altland, Phys. Rev. Lett. **94**, 245502 (2005).

[13] S. John and M. J. Stephen, Phys. Rev. B **28**, 6358 (1983).

[14] E. Akkermans and G. Montambaux, *Mesoscopic Physics of Electrons and Photons* (Cambridge University Press, London, 2007).

[15] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).

[16] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, London, 2002).

[17] The full quantum problem with three independent length scales λ, ξ, σ requires a more elaborate formulation that will be published elsewhere.

[18] C. Gaul and C. A. Müller, Europhys. Lett. **83**, 10006 (2008).

[19] R. Kuhn, O. Sigwarth, C. Miniatura, D. Delande, and C. A. Müller, New J. Phys. **9**, 161 (2007).

[20] B. Kramer and A. Mackinnon, Rep. Prog. Phys. **56**, 1469 (1993).

[21] D. J. Thouless, J. Phys. C **6**, L49 (1973).

[22] N. Bilas and N. Pavloff, Eur. Phys. J. D **40**, 387 (2006).

[23] P. Lugan, D. Clément, P. Bouyer, A. Aspect, and L. Sanchez-Palencia, Phys. Rev. Lett. **99**, 180402 (2007).

[24] D. Clément, A. F. Varón, J. A. Retter, L. Sanchez-Palencia, A. Aspect, and P. Bouyer, New J. Phys. **8**, 165 (2006).

[25] J. W. Goodman, in *Laser Speckle and Related Phenomena*, edited by J. C. Dainty (Springer-Verlag, Berlin, 1975).

[26] L. Sanchez-Palencia, Phys. Rev. A **74**, 053625 (2006).

[27] M. Hartung, T. Wellens, C. A. Müller, K. Richter, and P. Schlagheck, Phys. Rev. Lett. **101**, 020603 (2008).

[28] Bilas and Pavloff [22] also calculated the integrated density of states in 1D via the phase formalism within a hydrodynamic theory valid under the same hypotheses as ours. Their Eq. (27) seems to indicate that the density of states is only modified to order V^4. However, this result was obtained for a potential $V(x)=v(x)/\mu$ such that $\eta(x)=v(x)/[1-v(x)]=v(x)+v(x)^2+O(v^3)$ is centered, $\bar{\eta}=0$. This corresponds to a noncentered bare potential $\bar{v}=-V^2/\mu^2$. Reverting to our case of a centered potential $\bar{v}=0$, equivalent to keeping the average density or total number of particles fixed, which seems a reasonable choice, this shift translates to our limiting results $(\bar{\rho}-\rho_0)/\rho_0 = \frac{1}{2}v^2$ and $\Delta c/c = -\frac{1}{2}v^2$ for the effective δ-correlated case $\kappa \ll 1$ as it should.