Evolution of squeezed states under the Fock-Darwin Hamiltonian

Jaime E. Santos*
Centro de Física and Departamento de Física, Universidade do Minho, P-4710-057 Braga, Portugal
CFP and Departamento de Física, Faculdade de Ciências, Universidade do Porto, 4169-007 Porto, Portugal

Nuno M. R. Peres
Centro de Física and Departamento de Física, Universidade do Minho, P-4710-057 Braga, Portugal

João M. B. Lopes dos Santos
CFP and Departamento de Física, Faculdade de Ciências, Universidade do Porto, 4169-007 Porto, Portugal
(Received 16 June 2009; published 2 November 2009)

We develop a complete analytical description of the time evolution of squeezed states of a charged particle under the Fock-Darwin (FD) Hamiltonian and a time-dependent electric field. This result generalizes a relation obtained by Infeld and Plebański for states of the one-dimensional harmonic oscillator. We relate the evolution of a state-vector subjected to squeezing to that of state which is not subjected to squeezing and for which the time evolution under the simple harmonic oscillator dynamics is known (e.g., an eigenstate of the Hamiltonian). A corresponding relation is also established for the Wigner functions of the states, in view of their utility in the analysis of cold-ion experiments. In Appendix A, we compute the response functions of the FD Hamiltonian to an external electric field, using the same techniques as in the main text.

DOI: 10.1103/PhysRevA.80.053401 PACS number(s): 37.10.Vz, 42.50.Dv, 68.65.Hb

I. INTRODUCTION

The study of the time evolution of a nonrelativistic charged particle in homogeneous magnetic and electric fields has a long history in physics. In a quantum context, the treatment of the problem goes back to Darwin [1], who considered the evolution of a Gaussian wave packet in a magnetic field, and Fock [2] who obtained the eigenenergies and eigenstates of a charged particle in an isotropic harmonic potential, subjected to a magnetic field normal to the plane of motion.

If one takes a particle of charge $-e$ and mass m, moving in the xy plane in an harmonic potential of frequency ω_0 and subjected to a magnetic field $\boldsymbol{B}=B\boldsymbol{e}_z$, the Hamiltonian describing the system is given, in the symmetric gauge where $\boldsymbol{A}=\frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}$ by

$$\hat{H}_0=\frac{1}{2m}\left[\left(\hat{p}_x-\frac{eB}{2}\hat{y}\right)^2+\left(\hat{p}_y+\frac{eB}{2}\hat{x}\right)^2\right]+\frac{1}{2}m\omega_0^2(\hat{x}^2+\hat{y}^2), \tag{1}$$

where the operators $\hat{x},\hat{p}_x,\hat{y},\hat{p}_y$ obey the canonical commutation relations.

This simple problem has applications in the context of the quantum Hall effect [3], where disorder and the Coulomb interaction also play a crucial role. Another field for which the study of this Hamiltonian has proved fruitful is that of quantum dots, where the simple Hamiltonian given by Eq. (1) seems to give a good account of the I-V_g curves obtained when a gate voltage is applied to the quantum dot [4] with corrections due to the asymmetry of the confining potential and to the Coulomb interactions also playing a role. For some types of quantum dots, such as InAs/GaAs quantum dots, the agreement between the theoretical and experimental results seems to hold for magnetic fields up to 15 T [5]. Furthermore, the simplicity of Hamiltonian (1) allows for the exact treatment of the problem of the orbital magnetism of noninteracting fermions in a 2d harmonic potential [6,7].

The study of the evolution of Gaussian wave packets also goes back to the first days of quantum mechanics. This study was first undertaken by Schrödinger [8], Kennard [9], and also by Darwin [1] in the context of the harmonic oscillator of a free particle and of a particle in constant electric and magnetic fields. This problem continues to attract attention to the present day in many contexts, see the review by Dodonov [10].

Schrödinger considered the time evolution of a minimal uncertainty state, i.e., a coherent state of the harmonic oscillator in the terminology of Glauber [11]. These states have a wide range of applications in quantum optics (see, e.g., [12]), where they act as the quasiclassical states of the electromagnetic field and in quantum field theory, where they are the basis of the phase-space path integral [13]. Such states and their derivatives have become important in quantum information processing in recent years in the context of the manipulation of cold atoms in traps. It has become possible to reconstruct the Wigner function of a coherent state of the center of mass of an harmonically bound ion [14].

Kennard has considered the evolution of a more general wave packet of the harmonic oscillator, what is now known as a squeezed state. Important early contributions are those of Husimi [15] and Infeld and Plebański [16,17]. Infeld and Plebański introduced the so-called squeezing operator and established a relation between the evolution of initial states for which the time evolution is known ("unsqueezed" states) and states which are derived from such initial states by the application of the squeezing operator (squeezed states). The generalization of this relation to the Fock-Darwin (FD)

*jaime.santos@fisica.uminho.pt

Hamiltonian in an homogeneous time-dependent electric field is the main result of the present paper. Stoler [18,19] proved that squeezed-coherent states are unitarily equivalent to coherent states, thus being minimal uncertainty states, but that their minimal character is not preserved by time evolution although the uncertainty periodically assumes the minimum value compatible with Heisenberg's uncertainty relation. He also showed that the squeezing operator as currently written in terms of quadrature operators is exactly of the form given by Infeld and Plebański. Squeezed states play a proeminent role in quantum optics, see again [12]. Recently, they have also become important in quantum information processing both in a quantum optics context and also through the manipulation of cold atoms [20–29]. Reference [30] gives an up-to-date report of the state of quantum information processing using cold atoms and photons.

Besides the paper of Darwin already referred, the time evolution of states of a nonrelativistic particle in homogeneous electric and magnetic fields was also considered by Malkin, Man'ko, Trifonov, and Dodonov [31–35], who considered the dynamics of a particle in an homogeneous electromagnetic field in terms of coherent states, obtaining an explicit representation of the Green's function, and studied the invariants of the system; by Lewis and Riesenfeld [36], who also considered the invariants of such a system; and by Kim and Weiner [37], who considered the evolution of Gaussian wave packets in a magnetic field, subjected to an isotropic harmonic potential [i.e., the Fock-Darwin Hamiltonian] but also to saddle-point potentials, which are relevant for tunneling problems.

The structure of this paper is as follows: in the next section, we review the notion of squeezing operator and generalize the relation of Infeld and Plebański to states evolving under the Fock-Darwin Hamiltonian. In Sec. III, with a view to applications in the manipulation of cold atoms, we use the relation obtained to establish a relation between the Wigner function of different states and apply it to the special case of squeezed-coherent states. In Sec. IV, we present our conclusions. In Appendix A, we present a derivation of the finite frequency permittivity and conductivity of the FD Hamiltonian that uses the same operator methods that are used in the main text but which lies somewhat outside of the scope of the main text. Finally, in Appendix B, we derive the original Infeld-Plebański relation through elementary means.

II. EVOLUTION OF SQUEEZED STATES UNDER THE FD HAMILTONIAN

We consider the time evolution of the state $|\bar{\psi}_t\rangle$ that obeys the time-dependent Schrödinger equation $i\hbar\partial_t|\bar{\psi}_t\rangle=\hat{H}(t)|\bar{\psi}_t\rangle$, where $\hat{H}(t)=\hat{H}_0+\hat{H}_1(t)$, with $\hat{H}_0$ being given by Eq. (1) and where the interaction Hamiltonian $\hat{H}_1(t)$ of the charge with the external electric field is given, in the Schrödinger picture, by $\hat{H}_1(t)=e[E_x(t)\hat{x}+E_y(t)\hat{y}]$.

If one now expands the squares and groups the different terms of Eq. (1), one obtains

$$\hat{H}_0=\frac{1}{2m}(\hat{p}_x^2+\hat{p}_y^2)+\frac{1}{2}m\omega_R^2(\hat{x}^2+\hat{y}^2)+\frac{\omega_L}{2}\hat{L}_z, \tag{2}$$

where $\omega_L=\frac{eB}{m}$ is the gyration frequency and $\omega_R^2=\omega_0^2+\omega_L^2/4$, with $\hat{L}_z=\hat{x}\hat{p}_y-\hat{y}\hat{p}_x$ being the angular momentum component along the z axis. One should note that one can write $\hat{H}_0=\hat{h}_0+\frac{\omega_L}{2}\hat{L}_z$, where $\hat{h}_0$ is the Hamiltonian of the isotropic harmonic oscillator with frequency ω_R and also that $[\hat{h}_0,\hat{L}_z]=0$.

Given the state vector $|\bar{\psi}_t\rangle$, one can define the corresponding state vector $|\phi_t\rangle=e^{i\hat{H}_0t/\hbar}|\bar{\psi}_t\rangle$, in the interaction representation, such that the two vectors coincide at $t=0$. This state vector evolves according to the Hamiltonian $\hat{H}_1^{int}(t)=e^{i\hat{H}_0t/\hbar}\hat{H}_1(t)e^{-i\hat{H}_0t/\hbar}$, which given that $\hat{h}_0$ and $\hat{L}_z$ in $\hat{H}_0$ commute, one can also write as

$$\hat{H}_1^{int}(t)=e^{i\hat{h}_0t/\hbar}e^{i\omega_Lt\hat{L}_z/2\hbar}\hat{H}_1(t)e^{-i\omega_Lt\hat{L}_z/2\hbar}e^{-i\hat{h}_0t/\hbar}. \tag{3}$$

If one now applies the time-dependent rotation, encoded by $\hat{L}_z$ to $\hat{H}_1(t)$, followed by the dynamics of the isotropic harmonic oscillator, encoded in $\hat{h}_0$, one obtains for $\hat{H}_1^{int}(t)$:

$$\hat{H}_1^{int}(t)=e\{E_x'(t)[\hat{x}\cos(\omega_Rt)+\hat{p}_x\sin(\omega_Rt)]+E_y'(t)[\hat{y}\cos(\omega_Rt)+\hat{p}_y\sin(\omega_Rt)]\}, \tag{4}$$

where

$$E_x'(t)=E_x(t)\cos\left(\frac{\omega_L}{2}t\right)+E_y(t)\sin\left(\frac{\omega_L}{2}t\right), \tag{5}$$

$$E_y'(t)=-E_x(t)\sin\left(\frac{\omega_L}{2}t\right)+E_y(t)\cos\left(\frac{\omega_L}{2}t\right), \tag{6}$$

are the components of the electric field in the rotated frame.

The wave equation for $|\phi_t\rangle$ can be formally integrated in terms of time-ordered products of the integral of $\hat{H}_1^{int}(t)$, i.e., $|\phi_t\rangle=T\exp[-\frac{i}{\hbar}\int_0^t du\hat{H}_1^{int}(u)]|\bar{\psi}_0\rangle$. Since the commutator of the operators $\hat{H}_1^{int}(u)$ at different times is a c number, one can write the time-ordered operator above as

$$T\exp\left[-\frac{i}{\hbar}\int_0^t du\hat{H}_1^{int}(u)\right] =\exp\left[-\frac{i}{\hbar}\int_0^t du\hat{H}_1^{int}(u)\right]\times\exp\left\{-\frac{1}{2\hbar^2}\int_0^t du\int_0^u dv[\hat{H}_1^{int}(u),\hat{H}_1^{int}(v)]\right\}, \tag{7}$$

where the second term on the right-hand side is a phase factor.

Collecting the several terms, one obtains for the evolution of $|\bar{\psi}_t\rangle$

$$|\bar{\psi}_t\rangle = \exp\left\{ \frac{ie^2}{2\hbar m\omega_R}\int_0^t du \int_0^u dv\, \sin[\omega_R(u-v)] \times [E'_x(u)E'_x(v) + E'_y(u)E'_y(v)] \right\} \times \exp\left(-\frac{i\omega_L t}{2\hbar}\hat{L}_z\right)\exp\left(-\frac{i}{\hbar}\hat{h}_0 t\right) \times \exp\left(-\frac{i}{\hbar}\int_0^t du \hat{H}_1^{int}(u)\right)|\bar{\psi}_0\rangle. \tag{8}$$

One now assumes, following Infeld and Plebański [17], that the initial state $|\bar{\psi}_0\rangle$ is related to a certain initial state $|\psi_0\rangle$, for which the time evolution under the isotropic harmonic oscillator Hamiltonian $\hat{h}_0$ is known, by

$$|\bar{\psi}_0\rangle = \exp\left[\frac{i}{\hbar}(P_x^0\hat{x} + P_y^0\hat{y} - X_0\hat{p}_x - Y_0\hat{p}_y)\right] \times \exp\left[\frac{i}{2\hbar}r(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}} + \hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})\right]|\psi_0\rangle, \tag{9}$$

where the first operator is a translation operator in phase space, with X_0, Y_0, P_x^0, P_y^0 being arbitrary real constants, and the second operator is the squeezing operator, with r being a real constant that indicates the degree of squeezing.

Substituting Eq. (9) into Eq. (8) one can combine the operators $e^{-(i/\hbar)\int_0^t du \hat{H}_1^{int}(u)}$ and $e^{(i/\hbar)(P_x^0\hat{x}+P_y^0\hat{y}-X_0\hat{p}_x-Y_0\hat{p}_y)}$, since the commutator of their exponents is a c number. This operation merely generates a phase factor, coming from the commutator. One can then commute the resulting operator to the left-hand side, through $e^{-i\omega_L t\hat{L}_z/2\hbar}e^{i\hat{h}_0 t/\hbar}$, using the time evolution of $\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y$ under $\hat{h}_0$ and under the time-dependent rotation around the z axis. Combining the resulting phase factors and operators, we obtain

$$|\bar{\psi}_t\rangle = \exp\left\{-\frac{ie}{2\hbar}\int_0^t du[E_x(u)x_c(u) + E_y(u)y_c(u)]\right\} \times \exp\left\{\frac{i}{\hbar}[p_x^c(t)\hat{x} + p_y^c(t)\hat{y} - x_c(t)\hat{p}_x - y_c(t)\hat{p}_y]\right\} \times \exp\left(-\frac{i\omega_L t}{2\hbar}\hat{L}_z\right)\exp\left(-\frac{i}{\hbar}\hat{h}_0 t\right) \times \exp\left[\frac{i}{2\hbar}r(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}} + \hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})\right]|\psi_0\rangle. \tag{10}$$

where $x_c(t)$, $y_c(t)$, $p_x^c(t)$, and $p_y^c(t)$ are the classical solutions of the equations of motion for the Fock-Darwin problem with initial positions X_0 and Y_0 and initial momenta P_x^0 and P_y^0. These solutions are given by

$$x_c(t) = \left[X_0\cos\left(\frac{\omega_L}{2}t\right) - Y_0\sin\left(\frac{\omega_L}{2}t\right)\right]\cos(\omega_R t) + \frac{1}{m\omega_R}\left[P_x^0\cos\left(\frac{\omega_L}{2}t\right) - P_y^0\sin\left(\frac{\omega_L}{2}t\right)\right]\sin(\omega_R t) - \frac{e}{2m\omega_R}\int_0^t du\{E_x(u)\{\sin[\omega_+(t-u)] + \sin[\omega_-(t-u)]\} + E_y(u)\{\cos[\omega_+(t-u)] - \cos[\omega_-(t-u)]\}\}, \tag{11}$$

$$y_c(t) = \left[X_0\sin\left(\frac{\omega_L}{2}t\right) + Y_0\cos\left(\frac{\omega_L}{2}t\right)\right]\cos(\omega_R t) + \frac{1}{m\omega_R}\left[P_x^0\sin\left(\frac{\omega_L}{2}t\right) + P_y^0\cos\left(\frac{\omega_L}{2}t\right)\right]\sin(\omega_R t) - \frac{e}{2m\omega_R}\int_0^t du\{-E_x(u)\{\cos[\omega_+(t-u)] - \cos[\omega_-(t-u)]\} + E_y(u)\{\sin[\omega_+(t-u)] + \sin[\omega_-(t-u)]\}\}, \tag{12}$$

and

$$p_x^c(t) = \left[P_x^0\cos\left(\frac{\omega_L}{2}t\right) - P_y^0\sin\left(\frac{\omega_L}{2}t\right)\right]\cos(\omega_R t) - m\omega_R\left[X_0\cos\left(\frac{\omega_L}{2}t\right) - Y_0\sin\left(\frac{\omega_L}{2}t\right)\right]\sin(\omega_R t) - \frac{e}{2}\int_0^t du\{E_x(u)\{\cos[\omega_+(t-u)] + \cos[\omega_-(t-u)]\} - E_y(u)\{\sin[\omega_+(t-u)] - \sin[\omega_-(t-u)]\}\}, \tag{13}$$

$$p_y^c(t) = \left[P_x^0\sin\left(\frac{\omega_L}{2}t\right) + P_y^0\cos\left(\frac{\omega_L}{2}t\right)\right]\cos(\omega_R t) - m\omega_R\left[X_0\sin\left(\frac{\omega_L}{2}t\right) + Y_0\cos\left(\frac{\omega_L}{2}t\right)\right]\sin(\omega_R t) - \frac{e}{2}\int_0^t du\{E_x(u)\{\sin[\omega_+(t-u)] - \sin[\omega_-(t-u)]\} + E_y(u)\{\cos[\omega_+(t-u)] + \cos[\omega_-(t-u)]\}\}. \tag{14}$$

One can read the classical dielectric permittivity of the system from Eqs. (11) and (12), and one obtains the same results as in Appendix A, which shows that the quantum and classical results are identical, as one would expect for a linear system. The classical velocities $v_x^c(t), v_y^c(t)$ can be easily computed by derivation of Eqs. (11) and (12) with respect to time or, using the relations $v_x^c(t) = \frac{1}{m}[p_x^c(t) - \frac{eB}{2}y_c(t)]$ and $v_y^c(t) = \frac{1}{m}[p_y^c(t) + \frac{eB}{2}x_c(t)]$, from Eqs. (11)–(14). One can then read the classical conductivity of the system from the resulting expression, and this result again coincides with that of Appendix A, i.e., the classical and quantum results are identical. Finally, one can also show, after derivation of the velocity expressions with respect to time, that the solution given by Eqs. (11) and (12) obeys the classical equations of motion in two dimensions $\ddot{\boldsymbol{r}}=-\frac{e}{m}[\boldsymbol{E}(t)+\dot{\boldsymbol{r}}\times\boldsymbol{B}]-\omega_0^2\boldsymbol{r}$, with $\boldsymbol{B}=B\boldsymbol{e}_z$.

The left most operator in Eq. (10) is again a translation operator in phase space. If one considers the wave function in the coordinate representation, $\bar{\psi}(x,y,t)=\langle x,y|\bar{\psi}_t\rangle$, and one applies this translation operator to $\langle x,y|$, followed by the rotation operator $e^{-i\omega_L t\hat{L}_z/2\hbar}$, one obtains the following result:

$$\bar{\psi}(x,y,t)=\exp\left(-\frac{i}{2\hbar}\left\{p_x^c(t)x_c(t)+p_y^c y_c(t)+e\int_0^t du[E_x(u)x_c(u)+E_y(u)y_c(u)]\right\}\right)\times\exp\left\{\frac{i}{\hbar}[p_x^c(t)x+p_y^c(t)y]\right\}\times\langle\mathcal{R}_t^{-1}\cdot[\boldsymbol{r}-\boldsymbol{r}_c(t)]|e^{-i\hat{h}_0 t/\hbar}e^{ir(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})/2\hbar}|\psi_0\rangle, \tag{15}$$

with $\boldsymbol{r}_c(t)$ being the classical solutions of the equations of motion (11) and (12) and $\mathcal{R}_t$ being the rotation matrix in two dimensions by an angle of $\omega_L t/2$.

For the wave function in the momentum representation $\bar{\psi}(p_x,p_y,t)=\langle p_x,p_y|\bar{\psi}_t\rangle$, one obtains, applying the translation operator to $\langle p_x,p_y|$, followed again by the rotation operator,

$$\bar{\psi}(p_x,p_y,t)=\exp\left(\frac{i}{2\hbar}\left\{p_x^c(t)x_c(t)+p_y^c y_c(t)-e\int_0^t du[E_x(u)x_c(u)+E_y(u)y_c(u)]\right\}\right)\times\exp\left\{-\frac{i}{\hbar}[p_x x_c(t)+p_y y_c(t)]\right\}\times\langle\mathcal{R}_t^{-1}\cdot[\boldsymbol{p}-\boldsymbol{p}^c(t)]|e^{-i\hat{h}_0 t/\hbar}e^{ir(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})/2\hbar}|\psi_0\rangle, \tag{16}$$

with $\boldsymbol{p}^c(t)$ being the classical solutions of the equations of motion (13) and (14).

One can now apply the relation of Infeld and Plebański [17], relating the evolution of $\langle x,y|e^{-i\hat{h}_0 t/\hbar}e^{ir(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})/2\hbar}|\psi_0\rangle$ to that of $\langle x,y|e^{-i\hat{h}_0 t/\hbar}|\psi_0\rangle$ and the evolution of $\langle p_x,p_y|e^{-i\hat{h}_0 t/\hbar}e^{ir(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})/2\hbar}|\psi_0\rangle$ to that of $\langle p_x,p_y|e^{-i\hat{h}_0 t/\hbar}|\psi_0\rangle$, to Eqs. (15) and (16), respectively. This relation is derived by elementary means in Appendix B.

One obtains for Eq. (15) the result

$$\bar{\psi}(x,y,t)=\mu_r^{-1}(t)\exp\left(-\frac{i}{2\hbar}\left\{p_x^c(t)x_c(t)+p_y^c y_c(t)+e\int_0^t du[E_x(u)x_c(u)+E_y(u)y_c(u)]\right\}\right)\times\exp\left\{\frac{i}{\hbar}[p_x^c(t)x+p_y^c(t)y]\right\}\exp\left(\frac{im\omega_R\sinh(2r)\sin(2\omega_R t)}{2\hbar\mu_r^2(t)}\{[x-x_c(t)]^2+[y-y_c(t)]^2\}\right)\psi(\mathcal{R}_t^{-1}\cdot[\boldsymbol{r}-\boldsymbol{r}_c(t)]/\mu_r(t),\tau_r), \tag{17}$$

where $\mu_r(t)=\sqrt{e^{-2r}\cos^2(\omega_R t)+e^{2r}\sin^2(\omega_R t)}$ and $\tau_r=\frac{1}{\omega_R}\arctan[e^{2r}\tan(\omega_R t)]$.

For Eq. (16), one obtains

$$\bar{\psi}(p_x,p_y,t)=\mu_{-r}^{-1}(t)\exp\left(\frac{i}{2\hbar}\left\{p_x^c(t)x_c(t)+p_y^c y_c(t)-e\int_0^t du[E_x(u)x_c(u)+E_y(u)y_c(u)]\right\}\right)\exp\left\{-\frac{i}{\hbar}[p_x x_c(t)+p_y y_c(t)]\right\}\times\exp\left(-\frac{i\sinh(2r)\sin(2\omega_R t)}{2\hbar m\omega_R\mu_{-r}^2(t)}\{[p_x-p_x^c(t)]^2+[p_y-p_y^c(t)]^2\}\right)\psi(\mathcal{R}_t^{-1}\cdot[\boldsymbol{p}-\boldsymbol{p}^c(t)]/\mu_{-r}(t),\tau_{-r}), \tag{18}$$

where $\mu_{-r}(t)=\sqrt{e^{2r}\cos^2(\omega_R t)+e^{-2r}\sin^2(\omega_R t)}$ and $\tau_{-r}=\frac{1}{\omega_R}\arctan[e^{-2r}\tan(\omega_R t)]$. Expressions (17) and (18) constitute the main result of this paper and generalize those of Infeld and Plebański as presented in [17] to states evolving under the Fock-Darwin Hamiltonian subjected to a time-dependent electric field in the plane of the system.

III. WIGNER FUNCTION FOR A SQUEEZED STATE

One can also establish a relation between the Wigner function of a state subjected to squeezing in the presence of electric and magnetic fields and the Wigner function of an "unsqueezed" state evolving under the isotropic harmonic oscillator dynamics.

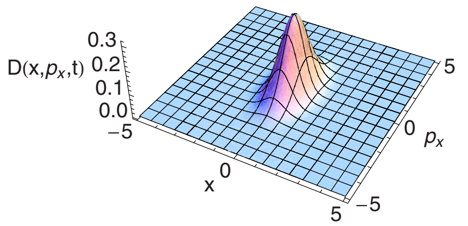

FIG. 1. (Color online) x and p_x section of Wigner function of squeezed-coherent state as given by Eq. (21) at $t=0$, evolving under the Fock-Darwin Hamiltonian. The length of the x, p_x axis is, respectively, 10^{-8} m and 10^{-26} kg ms^{-1} and the Wigner function was multiplied by $\hbar^2$.

The Wigner function is defined as [38]

$$D(\boldsymbol{r},\boldsymbol{p},t) = \frac{1}{(2\pi\hbar)^2}\int d^2v e^{-i\mathbf{p}\cdot\mathbf{v}/\hbar}\bar{\psi}^*(\boldsymbol{r}-\boldsymbol{v}/2,t)\bar{\psi}(\boldsymbol{r}+\boldsymbol{v}/2,t). \quad (19)$$

The Wigner function gives, after integration with respect to the momentum $\boldsymbol{p}$ or to the coordinate $\boldsymbol{r}$, the coordinate or momentum distribution function of the system and can be considered a quantum generalization of the Boltzmann distribution function. However, the Wigner function is not a *bonafide* distribution since it can assume negative values.

Substituting the result of Eq. (17) into Eq. (19), one obtains, after a substitution of variable on $\boldsymbol{v}$

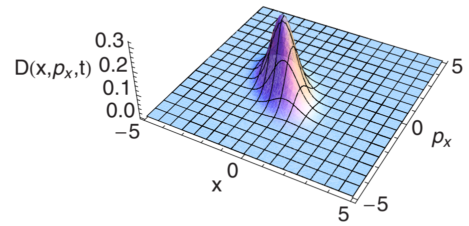

FIG. 3. (Color online) Same as 1 at time $t=3\pi/(4\omega_R)$.

$$D(\boldsymbol{r},\boldsymbol{p},t) = D_0(\mathcal{R}_t^{-1}\cdot[\boldsymbol{r}-\boldsymbol{r}_c(t)]/\mu_r(t),\mathcal{R}_t^{-1}\cdot\boldsymbol{u}(\boldsymbol{r},\boldsymbol{p},t),\tau_r), \quad (20)$$

where $\boldsymbol{u}(\boldsymbol{r},\boldsymbol{p},t)=\mu_r(t)[\boldsymbol{p}-\boldsymbol{p}^c(t)]-m\omega_R\sinh(2r)\sin(2\omega_R t)[\boldsymbol{r}-\boldsymbol{r}_c(t)]/\mu_r(t)$ and where $D_0(\boldsymbol{r},\boldsymbol{p},t)$ is the Wigner function of the unsqueezed initial state $\psi_0(x,y)$ evolving under the isotropic harmonic oscillator dynamics in the absence of magnetic or electric fields.

If the unsqueezed initial state $|\psi_0\rangle$ in Eq. (9) is the vacuum of the isotropic harmonic oscillator, then the state $|\bar{\psi}_0\rangle$ is a squeezed-coherent state, evolving under the Fock-Darwin Hamiltonian. In the context of cold ion experiments, one can produce squeezed states of the ion center of mass, either by quenching of the trap frequency or parametric amplification [25], as well as multichromatic excitation of the ion [24,26,29].

In this case, $D_0(\boldsymbol{r},\boldsymbol{p},t)=(1/\pi^2\hbar^2)e^{-m\omega_R\mathbf{r}^2/\hbar-\mathbf{p}^2/\hbar m\omega_R}$, and we obtain for the Wigner function of the squeezed-coherent state the result

$$D(\boldsymbol{r},\boldsymbol{p},t) = \frac{1}{\pi^2\hbar^2}e^{-[m\omega_R\mu_{-r}^2(t)/\hbar][\mathbf{r}-\mathbf{r}_c(t)]^2-[\mu_r^2(t)/\hbar m\omega_R][\mathbf{p}-\mathbf{p}^c(t)]^2+(2/\hbar)\sinh(2r)\sin(2\omega_R t)[\mathbf{p}-\mathbf{p}^c(t)]\cdot[\mathbf{r}-\mathbf{r}_c(t)]}, \quad (21)$$

which represents an asymmetric Gaussian whose shape is preserved by the dynamics of the system, rotating in phase space with its center determined by the solutions of the classical equations of motion [Eqs. (11)–(14)]. Note that for $r=0$, i.e., for a coherent state, the Wigner function decouples into a product of functions that depend only on the coordinates or the momenta. Also, note that result (21) is strictly positive, but this is not the case in general, e.g., if we had taken $|\psi_0\rangle$ to be an excited state of the harmonic oscillator.

A group of still pictures of the x,p_x section of the Wigner function (21) is shown in Figs. 1–4 for a system subjected to a right-handed circularly polarized wave, incident along the zz axis and aligned with the xx axis at $t=0$. The intensity of the field is $E_0=100$ V m^{-1}, with frequency equal to $\Omega=1.42\times10^9$ Hz. The harmonic frequency of the trap is $\omega_0=7.04\times10^7$ Hz and the mass of the particle $m=1.50\times10^{-26}$ kg is that of a $^9Be^+$ ion, with the applied magnetic field being $B=6.60$ T, which gives a gyration frequency $\omega_L=7.06\times10^7$ Hz and $\omega_R=7.88\times10^7$ Hz [39]. Finally, the squeezing parameter $r=0.35$.

The integration of expression (21) with respect to $\boldsymbol{p}$ or $\boldsymbol{r}$ yields the coordinate or momentum distribution. The wave packets in coordinate or position space are centered around the classical solutions of the equa-

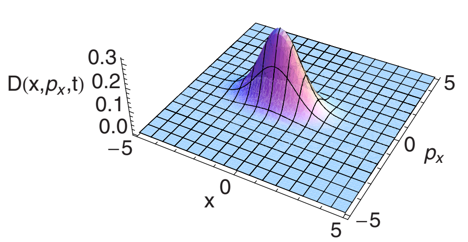

FIG. 2. (Color online) Same as 1 at time $t=\pi/(2\omega_R)$.

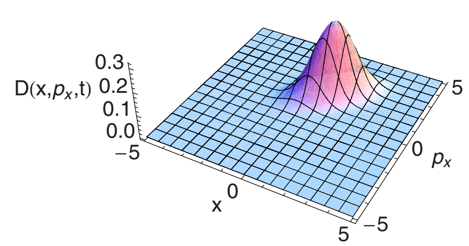

FIG. 4. (Color online) Same as 1 at time $t=5\pi/(4\omega_R)$.

tions of motion, the uncertainty in, e.g., x, p_x being given by $\langle\Delta x^2\rangle_t=(\hbar/2m\omega_R)[\cos^2(\omega_R t)e^{-2r}+\sin^2(\omega_R t)e^{2r}]$, $\langle\Delta p_x^2\rangle_t=(\hbar m\omega_R/2)[\cos^2(\omega_R t)e^{2r}+\sin^2(\omega_R t)e^{-2r}]$, i.e., the uncertainties oscillate with period $2\pi/\omega_R$. Their product is given by

$$\langle\Delta x^2\rangle_t\langle\Delta p_x^2\rangle_t=\frac{\hbar^2}{4}[1+\sinh^2(2r)\sin^2(2\omega_R t)]\geq\frac{\hbar^2}{4}, \quad (22)$$

in agreement with Heisenberg's uncertainty relation. Note that the uncertainty in x and p_x oscillate in opposition, i.e., one increases while the other is decreasing. Also note that, unlike a coherent state, the squeezed-coherent state is not a minimum uncertainty state for these two canonical variables except when $t=n\pi/2\omega_R$ [19].

IV. CONCLUSIONS

In this paper, we have considered the time evolution of general squeezed states evolving under the Fock-Darwin Hamiltonian in an homogeneous time-dependent electric field. We have generalized a relation of Infeld and Plebański between the time evolution of states of the harmonic oscillator subjected to squeezing and states not subjected to squeezing, for which the time evolution is known, to states evolving under the FD Hamiltonian in two dimensions. A corresponding relation was also established for the Wigner functions of the states. Finally, in Appendix A, we computed the response functions of the FD Hamiltonian to an external electric field using the same techniques as in the main text.

ACKNOWLEDGMENTS

We acknowledge helpful discussions with A. Ferreira. The authors were supported by FCT under Grant No. PTDC/FIS/64404/2006.

APPENDIX A: CALCULATION OF THE DIELECTRIC PERMITTIVITY AND OPTICAL CONDUCTIVITY

We will first indicate how to diagonalize Hamiltonian (1) in the operator representation. The diagonalization of this Hamiltonian and the determination of its eigenfunctions was first obtained by Fock [2].

Introducing the annihilation and creation operators $\hat{a}_x,\hat{a}_y,\hat{a}_x^\dagger,\hat{a}_y^\dagger$ through $\hat{x}=(\hbar/2m\omega_R)^{1/2}(\hat{a}_x+\hat{a}_x^\dagger)$, $\hat{p}_x=i(\hbar m\omega_R/2)^{1/2}(\hat{a}_x-\hat{a}_x^\dagger)$, $\hat{y}=(\hbar/2m\omega_R)^{1/2}(\hat{a}_y+\hat{a}_y^\dagger)$, and $\hat{p}_y=i(\hbar m\omega_R/2)^{1/2}(\hat{a}_y-\hat{a}_y^\dagger)$, one can write such Hamiltonian in the form

$$\hat{H}_0=\hbar\omega_R(\hat{a}_x^\dagger\hat{a}_x+\hat{a}_y^\dagger\hat{a}_y+1)+\frac{\omega_L}{2}\hat{L}_z, \quad (A1)$$

with $\hat{L}_z=i\hbar(\hat{a}_y^\dagger\hat{a}_x-\hat{a}_x^\dagger\hat{a}_y)$ being the only operator that is nondiagonal in the annihilation and creation operators in the above expression. Introducing the "circular polarization" operators $\hat{a}_+,\hat{a}_-,\hat{a}_+^\dagger,\hat{a}_-^\dagger$ through $\hat{a}_x=\frac{1}{\sqrt{2}}(\hat{a}_++\hat{a}_-)$, $\hat{a}_y=\frac{i}{\sqrt{2}}(\hat{a}_+-\hat{a}_-)$, $\hat{a}_x^\dagger=\frac{1}{\sqrt{2}}(\hat{a}_+^\dagger+\hat{a}_-^\dagger)$, and $\hat{a}_y^\dagger=-\frac{i}{\sqrt{2}}(\hat{a}_+^\dagger-\hat{a}_-^\dagger)$ [40], one has $\hat{L}_z=\hbar(\hat{a}_+^\dagger\hat{a}_+-\hat{a}_-^\dagger\hat{a}_-)$ and one can write $\hat{H}_0$ as

$$\hat{H}_0=\hbar\omega_+\hat{a}_+^\dagger\hat{a}_++\hbar\omega_-\hat{a}_-^\dagger\hat{a}_-+\hbar\omega_R, \quad (A2)$$

where $\omega_+=\omega_R+\omega_L/2$ and $\omega_-=\omega_R-\omega_L/2$. The energy levels are now given in terms of the occupation numbers of the modes $+,-$ by $E=\hbar\omega_+n_++\hbar\omega_-n_-+\hbar\omega_R$.

We will now compute the dielectric permittivity and optical conductivity of the system in the quantum regime by considering the Hamiltonian $\hat{H}(t)$ of the system interacting with an homogeneous time-dependent electric field, as given in Sec. II. Expressing the operators $\hat{x}$ and $\hat{y}$ in terms of $\hat{a}_+,\hat{a}_-,\hat{a}_+^\dagger,\hat{a}_-^\dagger$, we have the following expression for the interaction Hamiltonian $\hat{H}_1(t)$:

$$\hat{H}_1(t)=e(\hbar/2m\omega_R)^{1/2}[\mathcal{E}_+(t)\hat{a}_++\mathcal{E}_-(t)\hat{a}_+^\dagger+\mathcal{E}_-(t)\hat{a}_-+\mathcal{E}_+(t)\hat{a}_-^\dagger], \quad (A3)$$

where $\mathcal{E}_+(t)=\frac{1}{\sqrt{2}}[E_x(t)+iE_y(t)]$ and $\mathcal{E}_-(t)=\frac{1}{\sqrt{2}}[E_x(t)-iE_y(t)]$.

If $|\psi_t\rangle$ is a solution of the time-dependent Schrödinger equation, one defines, as above, the state vector $|\phi_t\rangle=e^{i\hat{H}_0t/\hbar}|\psi_t\rangle$, in the interaction representation, such that the two vectors coincide at $t=0$, when the field is turned on. Given that in the interaction representation the annihilation and creation operators contained in $\hat{H}_1(t)$ evolve in time through multiplication by a phase factor $e^{\pm i\omega_\pm t}$, $\hat{H}_1^{int}(t)$ is given by

$$\hat{H}_1^{int}(t)=e(\hbar/2m\omega_R)^{1/2}[\mathcal{E}_+(t)e^{-i\omega_+t}\hat{a}_++\mathcal{E}_-(t)e^{i\omega_+t}\hat{a}_+^\dagger+\mathcal{E}_-(t)e^{-i\omega_-t}\hat{a}_-+\mathcal{E}_+(t)e^{i\omega_-t}\hat{a}_-^\dagger]. \quad (A4)$$

One can now write, as in Sec. II, the formal solution $|\phi_t\rangle=T\exp[-\frac{i}{\hbar}\int_0^t du\hat{H}_1^{int}(u)]|\psi_0\rangle$. The time-ordered operator above is then written using identity (7), the second term in this product being a phase factor that can be discarded when computing expectation values.

We can write for $|\psi_t\rangle$ the result

$$|\psi_t\rangle=e^{-i\hat{H}_0t/\hbar}e^{[z_+(t)\hat{a}_+^\dagger-z_+^*(t)\hat{a}_+]+[z_-(t)\hat{a}_-^\dagger-z_-^*(t)\hat{a}_-]}|\psi_0\rangle, \quad (A5)$$

with $z_\pm(t)$ being given by

$$z_+(t)=-\frac{ie}{\sqrt{2m\hbar\omega_R}}\int_0^t du\mathcal{E}_-(u)e^{i\omega_+u}, \quad (A6)$$

$$z_-(t)=-\frac{ie}{\sqrt{2m\hbar\omega_R}}\int_0^t du\mathcal{E}_+(u)e^{i\omega_-u}, \quad (A7)$$

and where we have discarded the phase factor referred above. The operator $\hat{D}(z_+,z_-)=e^{[z_+(t)\hat{a}_+^\dagger-z_+^*(t)\hat{a}_+]+[z_-(t)\hat{a}_-^\dagger-z_-^*(t)\hat{a}_-]}$ is the displacement operator for the annihilation and creation operators, i.e., $\hat{D}^\dagger\hat{a}_\pm\hat{D}=\hat{a}_\pm+z_\pm$ and $\hat{D}^\dagger\hat{a}_\pm^\dagger\hat{D}=\hat{a}_\pm^\dagger+z_\pm^*$. Therefore, using the above representation of $|\psi_t\rangle$, one can show that the average value of any operator $\hat{A}(\hat{a}_+,\hat{a}_+^\dagger,\hat{a}_-,\hat{a}_-^\dagger)$, in the Schrödinger representation, is given by

$$\langle\hat{A}\rangle_t = \langle\psi_t|\hat{A}(\hat{a}_+,\hat{a}_+^\dagger,\hat{a}_-,\hat{a}_-^\dagger)|\psi_t\rangle$$
$$= \langle\psi_0|\hat{D}^\dagger\hat{A}(\hat{a}_+e^{-i\omega_+t},\hat{a}_+^\dagger e^{i\omega_+t},\hat{a}_-e^{-i\omega_-t},\hat{a}_-^\dagger e^{i\omega_-t})\hat{D}|\psi_0\rangle$$
$$= \langle\psi_0|\hat{A}\{[\hat{a}_+ + z_+(t)]e^{-i\omega_+t},[\hat{a}_+^\dagger + z_+^*(t)]e^{i\omega_+t},[\hat{a}_- + z_-(t)]e^{-i\omega_-t},[\hat{a}_-^\dagger + z_-^*(t)]e^{i\omega_-t}\}|\psi_0\rangle. \quad (A8)$$

In particular, if $\hat{A}=\beta_+^*\hat{a}_+ + \beta_+\hat{a}_+^\dagger + \beta_-^*\hat{a}_- + \beta_-\hat{a}_-^\dagger$, i.e., for a linear function of the annihilation or creation operators, such as $\hat{x},\hat{y},\hat{p}_x,\hat{p}_y$, which is the case that will concerns us below, one has that

$$\delta\langle\hat{A}\rangle_t = \beta_+^* z_+(t)e^{-i\omega_+t} + \beta_+ z_+^*(t)e^{i\omega_+t} + \beta_-^* z_-(t)e^{-i\omega_-t} + \beta_- z_-^*(t)e^{i\omega_-t}, \quad (A9)$$

where $\delta\langle\hat{A}\rangle_t=\langle\hat{A}\rangle_t-\langle\hat{A}\rangle_t^0$ is the difference between the average value in presence and absence of the applied electric field.

Using this result, one can easily show that the induced polarization in the system $P_x(t)=-e\delta\langle\hat{x}\rangle_t$ and $P_y(t)=-e\delta\langle\hat{y}\rangle_t$ is given by $P_i(t)=\int_0^t du\chi_{ij}(t-u)E_j(u)$, where the permittivity of the system is given by

$$\chi_{xx}(t) = \chi_{yy}(t) = \frac{e^2}{2m\omega_R}[\sin(\omega_+t) + \sin(\omega_-t)], \quad (A10)$$

$$\chi_{xy}(t) = -\chi_{yx}(-t) = \frac{e^2}{2m\omega_R}[\cos(\omega_+t) - \cos(\omega_-t)]. \quad (A11)$$

The first equality between the permittivities follows from rotational invariance around the z axis and the second from linear response theory.

The induced current $j_x(t)=-e\delta\langle\hat{v}_x\rangle_t$ and $j_y(t)=-e\delta\langle\hat{v}_y\rangle_t$ is related to $P_x(t)$ and $P_y(t)$ by $j_x(t)=\dot{P}_x(t)$, $j_y(t)=\dot{P}_y(t)$ and given that $\chi_{ij}(0)=0$, one easily obtains for the conductivity, defined by $j_i(t)=\int_0^t du\sigma_{ij}(t-u)E_j(u)$, the result $\sigma_{ij}(t)=\dot{\chi}_{ij}(t)$. Hence,

$$\sigma_{xx}(t) = \sigma_{yy}(t) = \frac{e^2}{2m\omega_R}[\omega_+\cos(\omega_+t) + \omega_-\cos(\omega_-t)], \quad (A12)$$

$$\sigma_{xy}(t) = \sigma_{yx}(-t) = -\frac{e^2}{2m\omega_R}[\omega_+\sin(\omega_+t) - \omega_-\sin(\omega_-t)]. \quad (A13)$$

As above, one can also compute the conductivity from the definition of $\hat{v}_x=\frac{1}{m}(\hat{p}_x-\frac{eB}{2}\hat{y})$ and $\hat{v}_y=\frac{1}{m}(\hat{p}_y+\frac{eB}{2}\hat{x})$ since all the operators involved are linear in the annihilation and creation operators. Note that, since $\omega_++\omega_-=2\omega_R$ and $\sigma_{xx}(0)=\sigma_{yy}(0)=e^2/m$, a result which agrees with the f-sum rule for a single quantum particle. It is interesting to consider this system in two limits, namely, $B\to 0$ (simple harmonic oscillator) and $\omega_0\to 0$ (particle in a magnetic field). In the first case, $\omega_+=\omega_-=\omega_0$ and one obtains $\sigma_{xx}(t)=\sigma_{yy}(t)=\frac{e^2}{m}\cos(\omega_0 t)$, $\sigma_{xy}(t)=\sigma_{yx}(-t)=0$ (this result is obvious, given the lack of transverse response if $B=0$). In the second case, $\omega_+=\omega_L$, $\omega_-=0$, and $\omega_R=\omega_L/2$. One has that $\sigma_{xx}(t)=\sigma_{yy}(t)=\frac{e^2}{m}\cos(\omega_L t)$ and $\sigma_{xy}(t)=\sigma_{yx}(-t)=-\frac{e^2}{m}\sin(\omega_L t)$.

Finally, let us consider the case in which a constant electric field is turned on at $t=0$. In that case, one obtains at large times $t\to\infty$ that the current $j_i=\tilde{\sigma}_{ij}(s\to 0^+)E_j$, where $\tilde{\sigma}_{ij}(s)$ is the Laplace transform of $\sigma_{ij}(t)$. Performing the integrals, one obtains $\tilde{\sigma}_{xx}(0)=\tilde{\sigma}_{yy}(0)=0$. The limit $s\to 0$ requires a bit of care in the transverse conductivity case since $\tilde{\sigma}_{xy}(s)=-\tilde{\sigma}_{yx}(s)=-e^2/2m\omega_R[\omega_+^2/(\omega_+^2+s^2)-\omega_-^2/(\omega_-^2+s^2)]$. We obtain $\tilde{\sigma}_{xy}(0)=-\tilde{\sigma}_{yx}(0)=0$ if $\omega_0\neq 0$. However, we obtain $\tilde{\sigma}_{xy}(0)=-\tilde{\sigma}_{yx}(0)=-e/B$ in the $\omega_0=0$ case ($\omega_-=0$). This result is physically simple to understand if one realizes that, when a harmonic force is present, a constant electric field merely displaces the force center, whether a constant magnetic field is present or not (a shift in the origin of the coordinates merely contributes a constant term to the vector potential that can be simply gauged away). Therefore, one will not observe a response of the velocity to the electric field in that case. However, in the absence of an harmonic force, the electric field "pulls" on the gyration radius center as if it were a free particle and one does observe a transverse response.

One should again note that the quantum and classical results obtained for the susceptibilities computed above and those obtained from the classical equations of motion (11)–(14) are identical and, moreover, that the response to the electric field is purely linear. This result follows from the fact that the classical equations of motion and their quantum counterparts, the Ehrenfest equations, are linear and can therefore be solved with respect to the field and the initial conditions. In this respect, the system behavior is trivial. However, the equality of results between the classical and quantum cases is limited to operators that are linear combinations of the coordinates and momenta. In the case of nonlinear operators, one can still use the methods discussed in this appendix to study their time evolution. Furthermore, if one is interested in the evolution of wave functions, as discussed in the main text, one should keep the phase factors that were discarded in the computation of average values.

APPENDIX B: INFELD-PLEBAŃSKI IDENTITY

We give here an elementary demonstration of the relation of Infeld and Plebański [16]. If $\hat{S}_r=e^{ir(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})/2\hbar}$ is the squeezing operator introduced above, it is easy to show [19] that $\hat{S}_r\hat{\boldsymbol{r}}\hat{S}_r^\dagger=\hat{\boldsymbol{r}}e^r$ and $\hat{S}_r\hat{\boldsymbol{p}}\hat{S}_r^\dagger=\hat{\boldsymbol{p}}e^{-r}$, i.e., $\hat{S}_r$ is a scale transformation operator that preserves the volume of phase space. Using these identities, one can show that

$$\hat{S}_r|\boldsymbol{r}\rangle = e^{-rd/2}|\boldsymbol{r}e^{-r}\rangle, \quad (B1)$$

$$\hat{S}_r|\boldsymbol{p}\rangle = e^{rd/2}|\boldsymbol{p}e^{r}\rangle, \quad (B2)$$

where d is the space dimension (two in this case).

We now wish to consider the wave function of a squeezed state evolving under the isotropic harmonic oscillator $\hat{h}_0$, i.e., the matrix element $\langle x,y|e^{-i\hat{h}_0t/\hbar}\hat{S}_r|\psi_0\rangle$ (the discussion in momentum space is completely analogous). Inserting a complete set of position eigenstates, one can write this quantity as

$$\bar{\psi}(x,y,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx_0 dy_0 K_r(x,y,x_0,y_0;t)\psi_0(x_0,y_0), \tag{B3}$$

where $K_r(x,y,x_0,y_0;t)=\langle x,y|e^{-i\hat{h}_0 t/\hbar}\hat{S}_r|x_0,y_0\rangle$. Using identity (B1), one has that

$$K_r(x,y,x_0,y_0;t) = e^{-r}K(x,y,x_0e^{-r},y_0e^{-r};t), \tag{B4}$$

where $K(x,y,x_0,y_0;t)=\langle x,y|e^{-i\hat{h}_0 t/\hbar}|x_0,y_0\rangle$ is the isotropic harmonic oscillator propagator. Note, however, that Eq. (B4) is valid for an arbitrary one-particle Hamiltonian. The harmonic oscillator propagator is given by [41]

$$K(x,y,x_0,y_0;t) = \frac{m\omega_R}{2\pi i\hbar\,\sin(\omega_R t)}\exp\left\{\frac{im\omega_R}{2\hbar}\left[\cot(\omega_R t)(x^2+y^2+x_0^2+y_0^2) - \frac{2(xx_0+yy_0)}{\sin(\omega_R t)}\right]\right\}. \tag{B5}$$

Now, introducing the scaled variables $\tilde{x}=x/\mu_r(t)$ and $\tilde{y}=y/\mu_r(t)$, with $\mu_r(t)=\sqrt{e^{-2r}\cos^2(\omega_R t)+e^{2r}\sin^2(\omega_R t)}$ and $\tau_r=\frac{1}{\omega_R}\arctan[e^{2r}\tan(\omega_R t)]$, one can show, using Eq. (B4), that

$$K_r(x,y,x_0,y_0;t) = \mu_r^{-1}(t)e^{im\omega_R\mu_r^{-2}(t)\sinh(2r)\sin(2\omega_R t)(x^2+y^2)/2\hbar} \times K(x/\mu_r(t),y/\mu_r(t),x_0,y_0;\tau_r). \tag{B6}$$

Substituting Eq. (B6) into Eq. (B3), one obtains

$$\bar{\psi}(x,y,t) = \mu_r^{-1}(t)e^{im\omega_R\mu_r^{-2}(t)\sinh(2r)\sin(2\omega_R t)(x^2+y^2)/2\hbar} \times \psi(x/\mu_r(t),y/\mu_r(t),\tau_r), \tag{B7}$$

which is the Infeld-Plebański relation used in the main text. Since the expression for the propagator in momentum space is entirely analogous to Eq. (B5), the steps are identical to those above, except that r is replaced by $-r$. One obtains

$$\bar{\psi}(p_x,p_y,t) = \mu_{-r}^{-1}(t)e^{-i\mu_{-r}^{-2}(t)\,\sinh(2r)\sin(2\omega_R t)(p_x^2+p_y^2)/2\hbar m\omega_R} \times \psi(p_x/\mu_{-r}(t),p_y/\mu_{-r}(t),\tau_{-r}). \tag{B8}$$

[1] C. G. Darwin, Proc. R. Soc. London, Ser. A **117**, 258 (1927).
[2] V. Fock, Z. Phys. **47**, 446 (1928).
[3] M. Janssen, O. Viehweger, U. Fastenrath, and J. Hajdu, *Introduction to the Theory of the Integer Quantum Hall Effect* (VCH, Weinheim, 1994).
[4] L. P. Kouwenhoven, D. G. Austing, and S. Tarucha, Rep. Prog. Phys. **64**, 701 (2001).
[5] A. Babinski, M. Potemski, S. Raymond, J. Lapointe, and Z. R. Wasilewski, Phys. Status Solidi **3**, 3748 (2006).
[6] Y. Ishikawa and M. Fukuyama, J. Phys. Soc. Jpn. **68**, 2405 (1999).
[7] J. P. Gazeau, P. Y. Hsiao, and A. Jellal, Phys. Rev. B **65**, 094427 (2002).
[8] E. Schrödinger, Naturwiss. **14**, 664 (1926).
[9] E. H. Kennard, Z. Phys. **44**, 326 (1927).
[10] V. V. Dodonov, J. Opt. B **4**, R1 (2002).
[11] R. J. Glauber, Phys. Rev. Lett. **10**, 84 (1963).
[12] U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, England, 1997).
[13] L. Brown, *Quantum Field Theory* (Cambridge University Press, Cambridge, England, 1994).
[14] D. Leibfried, D. M. Meekhof, B. E. King, C. Monroe, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **77**, 4281 (1996).
[15] K. Husimi, Prog. Theor. Phys. **9**, 238 (1953).
[16] L. Infeld and J. Plebański, Acta Phys. Pol. **14**, 41 (1955).
[17] J. Plebański, Phys. Rev. **101**, 1825 (1956).
[18] D. Stoler, Phys. Rev. D **1**, 3217 (1970).
[19] D. Stoler, Phys. Rev. D **4**, 1925 (1971).
[20] G. Adesso and G. Chiribella, Phys. Rev. Lett. **100**, 170503 (2008).
[21] M. Ban, J. Opt. B: Quantum Semiclassical Opt. **1**, L9 (1999).
[22] S. L. Braunstein and H. J. Kimble, Phys. Rev. A **61**, 042302 (2000).
[23] S. L. Braunstein, Phys. Rev. A **71**, 055801 (2005).
[24] J. I. Cirac, A. S. Parkins, R. Blatt, and P. Zoller, Phys. Rev. Lett. **70**, 556 (1993).
[25] D. J. Heinzen and D. J. Wineland, Phys. Rev. A **42**, 2977 (1990).
[26] D. M. Meekhof, C. Monroe, B. E. King, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **76**, 1796 (1996).
[27] E. S. Polzik, Nature (London) **453**, 45 (2008).
[28] H. Vahlbruch, S. Chelkowski, K. Danzmann, and R. Schnabel, New J. Phys. **9**, 371 (2007).
[29] H. Zeng and F. Lin, Phys. Rev. A **52**, 809 (1995).
[30] H. J. Kimble, Nature (London) **453**, 1023 (2008).
[31] V. V. Dodonov, I. A. Malkin, and V. I. Man'ko, Physica **59**, 241 (1972).
[32] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, Phys. Rev. D **2**, 1371 (1970).
[33] I. A. Malkin and V. I. Man'ko, Sov. Phys. JETP **28**, 527 (1969).
[34] I. A. Malkin and V. I. Man'ko, Phys. Lett. **32A**, 243 (1970).
[35] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, Phys. Lett. **30A**, 414 (1969).
[36] H. R. Lewis and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[37] H. Y. Kim and J. H. Weiner, Phys. Rev. B **7**, 1353 (1973).
[38] E. P. Wigner, Phys. Rev. **40**, 749 (1932).
[39] The still pictures were obtained from Eqs. (11)–(14) and (21) by considering that the charge of the particle is given by $+e$ rather than $-e$ as in the main text since the charge of the $^9Be^+$ ion is positive. This amounts to replacing e by $-e$ in these formulas.
[40] J. Schwinger, *Quantum Mechanics-Symbolism of Atomic Measurements* (Springer, New York, 2001).
[41] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).