Equivalence of continuous-variable stabilizer states under local Clifford operations

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In this paper, continuous variable stabilizer state is described by its generator matrix. Then the fact that any stabilizer state could be reduced to the corresponding weighed graph state under local unitary operation in the Clifford group is proved. A matrix equation, which is used to determine whether two stabilizer states are equivalent or not under local unitary operation in the Clifford group, is proposed. Then we apply the method to the five-mode unweighed graph states and demonstrate that our theory could be able to correctly and quickly find the equivalent classes of five-mode unweighed graph states.

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I. INTRODUCTION

Stabilizer states [1] and (local) unitary operations in the Clifford group play an important role in numerous applications in quantum information and quantum computation [2], such as quantum error correction codes and fault-tolerant quantum computation. A stabilizer state is a kind of multiqubit pure state, which is a simultaneous eigenstate with eigenvalue 1 of a complete set of commutable Pauli operators. The Clifford group is a set of operators that map a Pauli group to a Pauli group. In the theory of quantum coding, the stabilizer formalism is used to construct stabilizer code to protect the quantum system from decoherence effects [3]. Graph state, which is a special case of stabilizer state, is used to implement universal quantum computation based on quantum measurement, namely, the so-called one-way quantum computation [4,5]. Van den Nest *et al.* [6] and Hein *et al.* [7] investigated the local Clifford (LC) equivalence of stabilizer states for discrete variable (DV) and drew conclusions that the stabilizer state is equivalent to graph state under LC operation, and that the local complement operation for graph, which correspond to LC operations for the corresponding graph state, could be used to conveniently judge whether two-graph states are local unitary equivalent or not.

DV graph state is generalized to continuous variable (CV) graph state by Zhang *et al.* [8]. CV graph state can be stabilized by a certain set of CV Pauli operators determined by the corresponding graph. CV graph state can be built by squeezed state and linear optics [9]. CV graph state, together with Clifford group and photon counting detection, can be used to implement universal CV quantum computation [10]. Universal quantum teleportation network is implemented based on a CV graph state by Ren *et al.* [11]. It is important to investigate the properties of CV graph state, especially the equivalence of graph states under LC operations. Recently, Zhang investigated the local complementation rule for CV unweighed graph states with up to four-mode [12], and presented graphical description of LC operations for CV weighted graph states [13]. However, an universal method to determine whether two CV stabilizer states with finite modes are equivalent or not under LC operation is an open problem. The standard algorithm to find the corresponding LC operations which transform a graph state to another equivalent graph state is also unknown. In this paper, we first present a matrix representation of CV stabilizer state, similar to that of DV stabilizer state. Then a matrix equation which is able to determine whether two stabilizer states are equivalent or not under LC operations is proposed. By solving the matrix equation, we could find the corresponding LC operations that transform a graph state to another equivalent graph state.

This paper is organized as follows. In Sec. II, we first review the basic concepts of Weyl-Heisenberg group and local Clifford group, then give the vector description of CV stabilizer state generator and the matrix representation of local Clifford operator. In Sec. III, we propose the matrix representation of CV stabilizer state. In Sec. IV, we show that CV stabilizer state is LC equivalent to CV weighted graph state. We present the method to determine whether two stabilizer states are equivalent under LC operations in Sec. V. In Sec. VI, we apply this rule to five-mode unweighted graph states.

II. REPRESENTATIONS OF CV PAULI OPERATORS AND LC OPERATORS

The CV operations are first reviewed. Here, our use of CVs follows the standard prescription given in [14]. For CV systems, Weyl-Heisenberg group of phase-space displacement is a Lie group with generators $\hat{x}=\frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^\dagger)$ (quadrature amplitude) and $\hat{p}=\frac{1}{\sqrt{2}i}(\hat{a}-\hat{a}^\dagger)$ (quadrature phase) of the electromagnetic field. The commutation relation reads as $[\hat{x},\hat{p}]=i$, where $\hbar=1$. Similar to the qubit Pauli operators, the single mode Pauli operators are defined as $X(s)=\exp[-is\hat{p}]$ and $Z(t)=\exp[it\hat{x}]$, where $s,t\in\mathbb{R}$. The Pauli operator $X(s)$ is a position-translation operator, which acts on the computational basis of position eigenstates $\{|q\rangle;q\in\mathbb{R}\}$ as $X(s)|q\rangle=|q+s\rangle$. Similarly Z is a momentum-translation operator acting on the computational basis of momentum eigenstates

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$\{|p\rangle; p \in \mathbb{R}\}$ as $Z(t)|p\rangle = |p+t\rangle$. These operators are noncommutative and obey the identity

$$X(s)Z(t) = e^{-ist}Z(t)X(s). \tag{1}$$

Next, we introduce some basic concepts of CV Weyl-Heisenberg group and local Clifford group. The Pauli operators for one mode can construct a set of Pauli operators $\{X_i(s_i), Z_i(t_i); i=1,2,\ldots,n\}$ for n-mode system, which generate Weyl-Heisenberg group $\mathcal{C}_1$. An element G of $\mathcal{C}_1$ can be generally written as the following form:

$$G = e^{i\theta}\prod_k X_k(s_k)Z_k(t_k), \quad G \in \mathcal{C}_1, \ \theta \in [0,2\pi), \tag{2}$$

where θ stands for a phase factor. The Clifford group $\mathcal{C}_2$ is the normalizer of $\mathcal{C}_1$, i.e., it is the group of unitary operators U satisfying $UGU^\dagger \in \mathcal{C}_1$, $\forall G \in \mathcal{C}_1$. The local Clifford operators on one mode consists of the following operators and the products among them [14]: the position displacement operator $X(s)$, the Fourier transform

$$F = \exp[i(\pi/4)(\hat{x}^2 + \hat{p}^2)],$$

and the phase operator

$$P(\eta) = \exp[i(\eta/2)\hat{x}^2].$$

We could easily verify that $X(s)GX^\dagger(s) \in \mathcal{C}_1$, $FGF^\dagger \in \mathcal{C}_1$, and $P(\eta)GP^\dagger(\eta) \in \mathcal{C}_1$, $\forall G \in \mathcal{C}_1$.

Similar to the DV condition [6], we will first introduce the vector description of the CV Pauli operators, then give the matrix description of CV LC operators. The mapping σ between the n-mode Pauli group and the set of $2n$-dimension real column vectors is defined as

$$\sigma(G) = \sigma\left[e^{i\theta}\prod_k X_k(s_k)Z_k(t_k)\right] \triangleq (t_1, \ldots, t_n, s_1, \ldots, s_n)^T. \tag{3}$$

The multiply operation of Pauli operators is correspondingly mapped to the addition operation of vectors, i.e,

$$\sigma(G_1 \times G_2) = \sigma(G_1) + \sigma(G_2), \quad G_1, G_2 \in \mathcal{C}_1. \tag{4}$$

Thus a $2n$-dimension real column vectors can represent an n-mode Pauli operator up to a phase factor.

The sufficient and necessary condition under which two Pauli operators G_1 and G_2 are commutable is described as follows. Two Pauli operators G_1 and G_2 are commutable if and only if the following condition is satisfied,

$$\sigma_1^T P \sigma_2 = 2m\pi, \tag{5}$$

where

$$\sigma_i = \sigma(G_i),$$

$$P = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

I is an identity matrix, m is an integer.

Proof. Suppose $G_1 = e^{i\theta}\prod X_k(u_k)Z_k(v_k)$ and $G_2 = e^{i\varphi}\prod X_k(\alpha_k)Z_k(\beta_k)$. Since $G_1G_2 = e^{i(\theta+\varphi)}e^{i(\sum_k v_k\alpha_k)}\prod X_k(u_k+\alpha_k)Z_k(v_k+\beta_k)$ and $G_2G_1 = e^{i(\theta+\varphi)}e^{i(\sum_k u_k\beta_k)}\prod X_k(u_k+\alpha_k)Z_k(v_k+\beta_k)$. If G_1 and G_2 are commutable, then $G_1G_2 = G_2G_1$. It can be concluded that $\sum_k v_k\alpha_k - \sum_k u_k\beta_k = 2m\pi$, where m is an integer. Obviously, the above identity can be rewritten as $\sigma_1^T P\sigma_2 = 2m\pi$, where $2n\times 2n$ matrix P is defined as $P = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$, I is a $n\times n$ identity matrix,

$$\sigma_1 = \sigma(G_1) = (v_1, \ldots, v_n, u_1, \ldots, u_n)^T,$$

$$\sigma_2 = \sigma(G_2) = (\beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_n)^T.$$

Now the matrix description of LC operations is given. The one-mode LC operation is first discussed, then is generalized to n-mode systems. Corresponding to mapping σ, mapping δ maps the n-mode local Clifford $\mathcal{C}_2^n$ group to the set M^n, consisting of some $2n\times 2n$ real matrices. First consider one-mode local Clifford group $\mathcal{C}_2^1$ and the set M^1.

$$\delta(F) \triangleq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \delta(P(\eta)) \triangleq \begin{bmatrix} 1 & \eta \\ 0 & 1 \end{bmatrix}, \tag{6}$$

$$\delta(X(u)) \triangleq I, \quad \delta(Z(v)) \triangleq I. \tag{7}$$

Correspondingly, δ also maps the multiply operation between Clifford operators to the multiply operation between the matrices, which means that $\delta(U_1U_2) = \delta(U_1)\delta(U_2)$, U_1, $U_2 \in \mathcal{C}_2$. Because of the following identities:

$$Fe^{i\theta}X(u)Z(v)F^\dagger = e^{i(\theta - vu)}X(-v)Z(u), \tag{8}$$

$$P(\eta)e^{i\theta}X(u)Z(v)P^\dagger(\eta) = e^{i(\theta + u^2\eta/2)}X(u)Z(v+\eta u), \tag{9}$$

$$X(s)e^{i\theta}X(u)Z(v)X^\dagger(s) = e^{i(\theta - sv)}X(u)Z(v), \tag{10}$$

$$Z(t)e^{i\theta}X(u)Z(v)Z^\dagger(t) = e^{i(\theta + ut)}X(u)Z(v), \tag{11}$$

it can be easily verified that $\sigma(FGF^\dagger) = \delta(F)\sigma(G)$, $\sigma(P(\eta)GP^\dagger(\eta)) = \delta(P(\eta))\sigma(G)$, $\sigma(X(u)GX^\dagger(u)) = \delta(X(u))\sigma(G)$, and $\sigma(Z(v)GZ^\dagger(v)) = \delta(Z(v))\sigma(G)$, where G is a one-mode Pauli operator. Because any one-mode LC operator U can be written as the production of $X(u)$, $P(\eta)$, and F, $\sigma(UGU^\dagger) = \delta(U)\sigma(G)$. For convenience of presentation, another operator $P_x(\eta) = FP(\eta)F^\dagger$ is defined, $P_x(\eta)$ obeys the following identity:

$$\delta(P_x(\eta)) = \begin{bmatrix} 1 & 0 \\ -\eta & 1 \end{bmatrix}. \tag{12}$$

Because $\delta(U_1U_2) = \delta(U_1)\delta(U_2)$, $\delta(I) = \delta(UU^\dagger) = \delta(U)\delta(U^\dagger)$, and $\delta(I) = I$, it can be found that $\delta(U^\dagger) = (\delta(U))^{-1}$. It can be easily proven that any matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $\det(A) = 1$ satisfies that

$$A = \begin{cases} \delta\left[P\left(\frac{b}{d}\right)P_x(-cd)FP_x\left(-\frac{1}{d}\right)P(-d)P_x\left(-\frac{1}{d}\right)\right] & \text{if } d \neq 0 \\ \delta\left[P(-ab)P_x\left(\frac{1}{b}\right)P(b)P_x\left(\frac{1}{b}\right)\right] & \text{if } d = 0. \end{cases} \tag{13}$$

Because $\det[\delta(F)] = \det[\delta(P(\eta))] = \det[\delta(X(u))] = 1$, thus,

$\det[\delta(U)]=1$, which means that the set M^1 consists of 2×2 matrices whose determinates equal one.

Now it can be easily generalized to n-mode LC group $\mathcal{C}_2^n$. δ maps n-mode LC operator U to the matrix $Q=\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, where A, B, C, and D are diagonal matrices, i.e., $A=\mathrm{diag}(a_1,\ldots,a_n)$, $B=\mathrm{diag}(b_1,\ldots,b_n)$, $C=\mathrm{diag}(c_1,\ldots,c_n)$, and $D=\mathrm{diag}(d_1,\ldots,d_n)$, which satisfy that $a_k\times d_k-c_k\times b_k=1$ $(k=1,\ldots,n)$. The fact means that $\left[\begin{smallmatrix} a_k & b_k \\ c_k & d_k \end{smallmatrix}\right]$ $(k=1,\ldots,n)$ is the LC operator on the kth mode.

In this section, we give the vector representation of Pauli group and the matrix representation of LC group up to a certain phase factor. We will prove later the fact that the phase factor can be revised to any value by Pauli operators, which enables us not to take the phase factor into account.

III. MATRIX REPRESENTATION OF CV STABILIZER STATE

An n-mode stabilizer state $|\Psi\rangle$ is defined as the simultaneous eigenstate with eigenvalue 1 of n commutable and independent Pauli group elements [3,15]. The set $S=\{G\in\mathcal{C}_1, G|\Psi\rangle=|\Psi\rangle\}$ is called the stabilizer of the state $|\Psi\rangle$. Though the number of generators of S is infinite, they in fact belong to n kinds of different sets, each of which can be written as $\{G_k=e^{i\theta}\Pi_k X_k(ru_k)Z_k(rv_k), r\in\mathbb{R}\}$, and u_k and v_k are certain numbers that are related to the state $|\Psi\rangle$. The reason is that if $e^{i\theta}\Pi X_k(u_k)Z_k(v_k)|\Psi\rangle=|\Psi\rangle$, then $e^{ir\theta}\exp[i\frac{r(r-1)}{2}\Sigma u_k v_k]\Pi X_k(ru_k)Z_k(rv_k)|\Psi\rangle=|\Psi\rangle$, $\forall r\in\mathbb{R}$, which implies that not the value of u_k and v_k, but the proportion of them determines the stabilizer state. This leads to the idea that one can choose one of the operators, which have the same proportion between their u_k and v_k, to be the representative of these operators without loss of generality. Thus the stabilizer of a stabilizer state can be represented by n n-mode independent and commutable Pauli operators with different proportion between their u_k and v_k.

According to the vector representation of Pauli operators, a stabilizer which consists of n commutable and independent generators can be represented by a $2n\times n$ matrix Θ, whose columns are the vectors mapped from the Pauli generator operator [16]. Once the matrix Θ is given, the generators S can be determined up to some phase factors, then the stabilize state can be determined. Since the column vectors $C_1=(v_1,\ldots,v_n,u_1,\ldots,u_n)^T$ and $C_2(rv_1,\ldots,rv_n,ru_1,\ldots,ru_n)^T$, where $r\in\mathbb{R}$, represent the same Pauli operator, and substituting $G_1(G_1\in S)$ by $G_1G_2(G_2\in S)$ retains the stabilizer same, any matrix Θ' that is obtained by elementary column transformation on Θ represents the same stabilizer as Θ. Here, the elementary column transformation stands for the following operations: multiplication of a column vector by a constant, addition of two column vectors to set the values of one column and interchanging the position of two column vectors. For $n\times n$ matrix, performing an elementary column transformation on it equals to right multiplying an $n\times n$ full-ranked matrix on it.

The property of generation matrix Θ. If there exists a simultaneous eigenstate $|\Psi\rangle$ with eigenvalue 1 of G_1 and G_2, $\sigma_1=\sigma(G_1)$ and $\sigma_2=\sigma(G_2)$ should satisfy the following condition that $\sigma_1^T P\sigma_2=0$, and the generation matrix Θ should satisfy the following relation:

$$\Theta^T P\Theta = 0. \tag{14}$$

Proof. Because u_k and v_k can be scaled by the same factor r while remaining the stabilizer operator of the eigenstate. For $G_1^r=e^{i\theta}\Pi X_k(ru_k)Z(rv_k)$, $r\in\mathbb{R}$ satisfies that $G_1^r|\Psi\rangle=|\Psi\rangle$, if it is commutable with $G_2=e^{i\varphi}\Pi X_k(\alpha_k)Z(\beta_k)$, then $\Sigma rv_k\alpha_k-\Sigma ru_k\beta_k=2m\pi$. The equation is correct if and only if $m=0$ for any value of r. This means $\sigma_1^T P\sigma_2=0$, with σ_1 and σ_2 being the column of Θ. It can be verified easily that $\Theta^T P\Theta=0$.

Applying the matrix representation of LC group, the evolution of a stabilizer under LC operators can be calculated as $\Theta_2=Q\Theta_1$. The stabilizer Θ_1 and Θ_2 are equivalent under LC operation if and only if there exist Q and N satisfying that

$$\Theta_2 = Q\Theta_1 N, \tag{15}$$

where N is a $2n\times 2n$ invertible matrix standing for a elementary column transformation and Q stands for the LC operations.

Phase. In the above discussion, we do not take the phase into account. The matrix Θ represents the stabilizer up to phase factors, which can be changed by any value through Pauli operator. Remembering that $\delta(X(s))=I$ and $\delta(Z(t))=I$, which means that $X(s)$ and $Z(t)$ only change the phase of a Pauli operator, while the displacement in phase space remaining the same. Suppose $G_1=e^{i\theta}\Pi_k X_k(\alpha_k)Z_k(\beta_k)$ is the Pauli operator whose phase factor θ will be adjusted, and $G_2=\Pi_k X_k(u_k)Z_k(v_k)$ applies on G_1, obtaining $G_2G_1G_2^\dagger=e^{i\Sigma_k(\alpha_k v_k-\beta_k u_k)}G_1$. $\Sigma_k(\alpha_k v_k-\beta_k u_k)$ is the amount that phase factor changes. For a stabilizer with the generator $\{G_1,\ldots G_n\}$, where $G_i=e^{i\theta_i}\Pi X_{ik}(\alpha_{ik})Z_{ik}(\beta_{ik})$ $(i=1,\ldots,n)$. We want to adjust the phase $\{\theta_i\}$ of the generator $\{G_i\}$ into $\{\theta_i+\varphi_i\}$ $(i=1,\ldots,n)$ by $G=\Pi_k X_k(u_k)Z_k(v_k)$, one needs to solve the following equations:

$$\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} & -\beta_{11} & \cdots & -\beta_{1n} \\ \alpha_{21} & \cdots & \alpha_{2n} & -\beta_{21} & \cdots & -\beta_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \cdots & \alpha_{nn} & -\beta_{n1} & \cdots & -\beta_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \varphi_n \end{bmatrix}. \tag{16}$$

Because the left matrix of Eq. (16), which is $[Z^T \quad X^T]\left[\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right]=\Theta^T P^T$ is full-ranked with a rank of n, the rank of the coefficient matrix equals the rank of the augmented matrix. Thus the equations above must have a solution, and the dimension of the solution space is n. There must be a Pauli operator $G=\Pi_k X_k(u_k)Z_k(v_k)$ which could be able to change the phase factors $\{\theta_i\}$ of generator $\{G_i\}$ to be any values. So, we could apply local Clifford operations to obtain the desired phases.

IV. LC EQUIVALENCE BETWEEN STABILIZER STATES AND WEIGHTED GRAPH STATES FOR CV

A. CV weighted graph states

A weighted graph state is described by a mathematical graph $G=(V,E)$, i.e., a finite set of n vertices V connected by a set of edges E, where every edge is specified by a factor Ω_{ab} corresponding to the interaction strength between the modes a and b [13]. The method to get n-mode weighted graph states is as following. First to prepare n phase-squeezed states, approximate to the zero-phase eigenstates. Then apply quantum nondemolition (QND) operation $[C_Z(\Omega)]$ operation with different interaction coefficient Ω_{ab} to each pair of modes $\{a,b\}$ connected by a weighted edge in the graph. Because all C_Z operations commute each other, the order in which C_Z are applied does not matter. Now the n-mode state has become n-mode weighted graph state, satisfying that $g_a=(\hat{p}_a-\Sigma\Omega_{ab}\hat{x}_b)\rightarrow 0$, where $a\in V$ is a vertex of the graph and $b\in N_a$ are the neighbor of a. Thus the corresponding stabilizer for the CV weighted graph state is $\{G_a(\xi)=\exp(-i\xi g_a)=X_a(\xi)\Pi Z_b(\Omega_{ab}\xi)\,|\,\xi\in\mathbb{R}\}$. Applying results in Sec. II, the generator matrix of a CV weighted graph state can be expressed as $\Theta=[\begin{smallmatrix}G\\I\end{smallmatrix}]$, with G as a symmetrical matrix standing for the adjacent matrix for the weighted graph and I an n-order identity matrix.

B. Equivalence between stabilizer state and weighted graph state for CV

We have shown that CV weighted graph state is a kind of stabilizer state. In this section, we will illustrate that any stabilizer state can be transformed to a CV weighted graph state by applying LC operation.

The following proof contains three steps. First we will prove that a $2n\times n$ generator matrix $\Theta=[\begin{smallmatrix}Z\\X\end{smallmatrix}]$ of a stabilizer can be transformed to

$$\Theta=\begin{bmatrix}R_z^1 & S_z^1\\ R_z^2 & S_z^2\\ R_x^1 & 0\\ R_x^2 & 0\end{bmatrix}$$

by elementary column transformation, where S_z^2 and R_x^1 are invertible matrices. Second we will prove that Fourier transformation can transform matrix Θ into $\Theta'=[\begin{smallmatrix}\bar{Z}\\\bar{X}\end{smallmatrix}]$, where $\bar{X}$ is an $n\times n$ invertible matrix. Finally, the stabilizer state can be transformed into graph state by elementary column transformation and phase operation.

For a stabilizer with generator matrix $\Theta=[\begin{smallmatrix}Z\\X\end{smallmatrix}]$, we transform this generator matrix into the following form:

$$\Theta=\begin{bmatrix}R_z & S_z\\ R_x & 0\end{bmatrix}$$

by performing elementary column transformation on Θ. Here, assumed that R_x is a full rank $n\times k$ matrix and $k=\text{rank}\,X$; the blocks R_z and S_z have dimensions $n\times k$ and $n\times(n-k)$, respectively. Next, as R_x has rank k, it has an invertible $k\times k$ submatrix. We assume that the first k rows of R_x consist of the invertible $k\times k$ submatrix without loss of generality, i.e.,

$$R_x=\begin{bmatrix}R_x^1\\ R_x^2\end{bmatrix},$$

where the upper $k\times k$-block R_x^1 is invertible and R_x^2 has dimensions $(n-k)\times k$. Partitioning S_z similarly into a $k\times(n-k)$-block S_z^1 and a $(n-k)\times(n-k)$-block S_z^2, i.e.,

$$S_z=\begin{bmatrix}S_z^1\\ S_z^2\end{bmatrix}.$$

Then we will prove that S_z^2 is an invertible matrix. Remembering that $\Theta^TP\Theta=0$, it results in $S_z^TR_x=0$. Suppose that there exists $(n-k)$-dimensional real column vector x such that $(S_z^2)^Tx=0$; then the n-dimensional real column vector $v=(0,\ldots,0,x^T)^T$ satisfies that $S_z^Tv=0$ and, therefore, $v=R_xy$ for some k-dimensional real column vector. This last equation reads

$$\begin{bmatrix}0\\ x\end{bmatrix}=\begin{bmatrix}R_x^1\\ R_x^2\end{bmatrix}y=\begin{bmatrix}R_x^1y\\ R_x^2y\end{bmatrix}.$$

Since that R_x^1 is invertible, $R_x^1y=0$ implies that $y=0$, yielding that $x=R_x^2y=0$. This proves that S_z^2 is invertible. So, up to now we have proven that

$$\Theta=\begin{bmatrix}R_z^1 & S_z^1\\ R_z^2 & S_z^2\\ R_x^1 & 0\\ R_x^2 & 0\end{bmatrix}$$

in which S_z^2 and R_x^1 are invertible matrices by elementary column transformation.

In the next step, we perform Fourier transformation

$$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

on the modes $k+1,\ldots,n$. It is easy to verify that this operation yields a new generator matrix $\Theta'=[\begin{smallmatrix}\bar{Z}\\\bar{X}\end{smallmatrix}]$, where

$$\bar{Z}=\begin{bmatrix}R_z^1 & S_z^1\\ R_x^2 & 0\end{bmatrix}$$

and

$$\bar{X}=\begin{bmatrix}R_x^1 & 0\\ -R_z^2 & -S_z^2\end{bmatrix}.$$

Matrix $\bar{X}$ is invertible because R_x^1 and S_z^2 are invertible matrices, respectively.

Last, performing column transformation $\bar{X}^{-1}$ on Θ', we will get

$$\Theta''=\begin{bmatrix}\bar{Z}\bar{X}^{-1}\\ I\end{bmatrix}.$$

$\bar{Z}\bar{X}^{-1}$ is symmetrical because $\Theta''^TP\Theta''=0$. Then perform phase operator

$$P(-\Lambda_m)=\begin{bmatrix}1 & -\Lambda_m\\ 0 & 1\end{bmatrix},$$

where Λ_m is the mth diagonal elements of $\bar{Z}\bar{X}^{-1}$, to make the diagonal of the upper-block all zeroes. This final generator matrix represents a stabilizer of a weighted graph state, since its upper-block is symmetrical with a zero diagonal and its lower-block is an n-order identity matrix.

This proof is the counterpart to the proof of M. Van den Nest [6], who showed that the DV stabilizer state is equivalent to DV graph state under LC operations. With some modification, the proof of M. Van den Nest can be used to prove the equivalence between CV stabilizer state and graph state, just as we have shown above.

Here we explain the definition of CV stabilizer state again. A n-mode CV stabilizer state is the eigenstate with eigenvalue 1 of the CV Pauli operators represented by the set $\{\theta_i, i=1,\ldots,n\}$ and Θ. The matrix Θ $2n\times n$ is full-ranked matrix, it satisfies $\Theta^T P\Theta=0$. A stabilizer state is also the eigenstate of the operators in an Ablien subgroup of Pauli group [15], meaning $\Theta^T P\Theta=0$. Here the fact that the stabilizer state is unique can be proved. If there exist two states defined by $\{\theta_i\}$ and Θ, according to Secs. III and IV, one can find the same LC operations that transform the two states into the same weighted graph states, which means the two initial states are the same states. So the quantum state defined by $\{\theta_i\}$ and Θ is unique.

V. MATRIX EQUATION TO DETERMINE THE EQUIVALENCE BETWEEN STABILIZER STATES UNDER LC OPERATIONS

In Sec. III, we have shown that two stabilizer states Θ_1 and Θ_2 are LC equivalent if and only if there exist a LC operation Q and a $2n$-order invertible matrix N such that $\Theta_2=Q\Theta_1 N$. In Sec. IV it is proved that any stabilizer state is equivalent to a weighted graph state under LC operations, so we can investigate the equivalence between two stabilizer states under LC operations by studying the equivalence between weighted graph states under LC operations.

Now let Θ_1 and Θ_2 be the generator matrices of two graph states, namely,

$$\Theta_1=\begin{bmatrix}G_1\\ I\end{bmatrix}$$

and

$$\Theta_2=\begin{bmatrix}G_2\\ I\end{bmatrix},$$

where G_1 and G_2 are the adjacent matrices of the two graph states respectively. If Θ_1 and Θ_2 are LC equivalent, using Eq. (9), we get that $\Theta_2=Q\Theta_1 N$. Transiting the matrices on both sides of the equation, we get $N^T\Theta_1^T Q^T=\Theta_2^T$. Right multiplied by $P\Theta_2$, obtaining $N^T\Theta_1^T Q^T P\Theta_2=\Theta_2^T P\Theta_2$. Noticing that $\Theta_2^T P\Theta_2=0$ and N is invertible, the above equation can be reduced to $\Theta_1^T Q^T P\Theta_2=0$. Suppose that $Q=[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}]$, it is simple to calculate that

$$G_1A+B-G_1CG_2-DG_2=0,$$

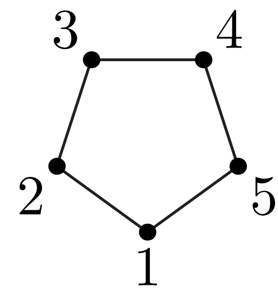

FIG. 1. Number assigning rule of the nodes in the graphs.

$$AD-BC=I, \tag{17}$$

where A, B, C, and D are diagonal matrices.

Thus, by calculating the above equations, we can get LC operations that transform Θ_1 to Θ_2, and if the equations above have no solution, the two weighted graph states are not LC equivalent. To determine whether the two stabilizer states are LC equivalent, one can first transform the stabilizer states into graph states by the LC operations proposed in Sec. IV, then apply Eq. (17) to determine whether the two corresponding graph states are LC equivalent or not. If the two corresponding graph states are LC equivalent, the two stabilizer states are also LC equivalent. Otherwise, the two stabilizer states are not LC equivalent.

VI. APPLICATION TO FIVE-MODE UNWEIGHTED GRAPH STATE

Unweighted graph state is a kind of graph state with the same interaction coefficients on all edges. For DV unweighted graph states, which are equivalent to DV stabilizer states under LC operations, the LC operations on the quantum states have been translated to local complementation of the corresponding graphs [6]. However, for CV unweighted graph states, the case becomes more complicated. We will show latter that for CV unweighted graph states, LC operations in the five-mode unweighted graph states can no longer be implemented by the local complementation on the corresponding graphs. We will also apply the LC equivalence criterion in Sec. V to five-mode unweighted graph states and give the different graph classes that are not LC equivalent for five-mode graph states. In this paper, each vertex of each graph is assigned a number according to the following rule, which is illustrated as Fig. 1, i.e., the vertex beneath the graph is 1, and 2, 3, 4, and 5 clockwise from 1.

Applying the criterion proposed in Sec. V to every pair of five-mode unweighted graph states with different adjacent matrices whose number is 728, we get 28 different graph states that are not LC equivalent, whose graphs are shown in Fig. 2. It is obvious that some of the graphs in Fig. 2 are isomorphic, but they cannot be transformed into each other without exchanging modes of different vertices. Here, No. $M(N)$ represents that there are N graph states that are equivalent with graph state No. M under LC operations.

In the case of DV, DV graph states are LC equivalent if and only if the graphs of the graph states are equivalent under local complement operation [6]. The action of the local

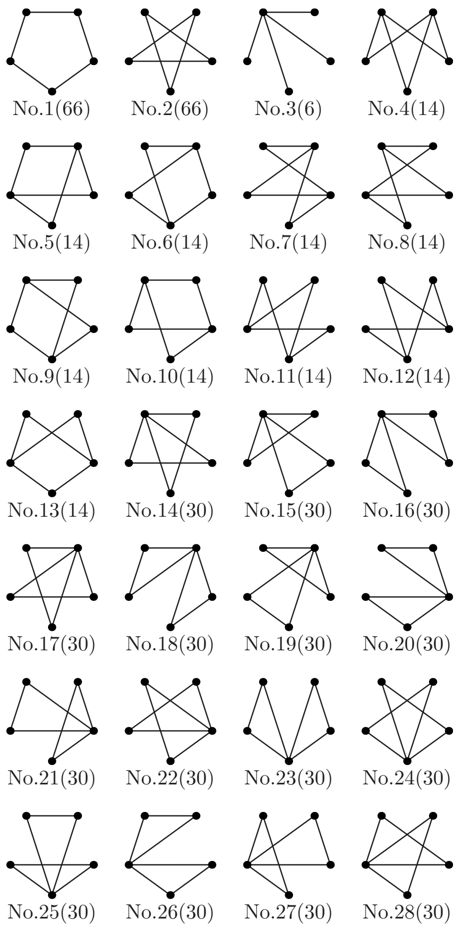

FIG. 2. Classification of five-mode unweighed graph states that are not equivalent to each other under local Clifford operators.

complement rule can be described as follows: letting $G=(V,E)$ be a graph and $a\in V$ be a vertex, the local complement of G for vertex a, denoted by $\lambda_a(G)$, is obtained by complementing the subgraph of G generated by the neighborhood N_a of a and leaving the rest of the graph unchanged. However, for the case of CV, this rule is no longer available. Considering the graphs in Fig. 3, by application of local complement on the node 4 of the left graph, we can get the right graph, but there does not exist LC operations that realize such transformation. The proof is not difficult. Applying Eq. (17) and solving this equation, one will find that no solution exists. Now we will show the LC operation among the graphs that are LC equivalent to the graph No. 4(14) in Fig. 2. The graphs are shown in Fig. 4. The LC operations are as follows:

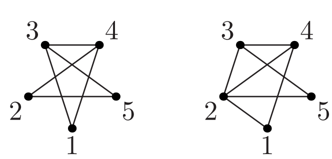

FIG. 3. Applying local complement on the node 4 of the graph on the left, one can get the graph on the right. However there does not exist local Clifford operations to implement such transformation.

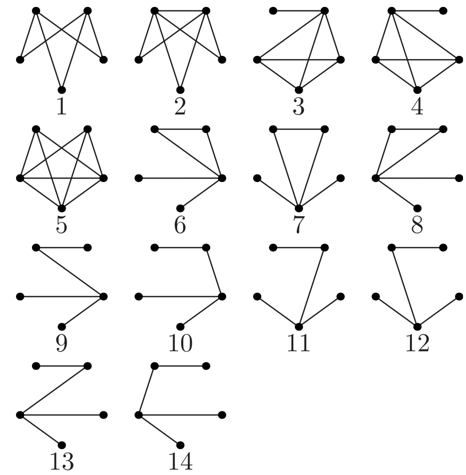

FIG. 4. Five-mode graphs that are LC equivalent to graph No. 4(14) in Fig. 2.

$1\rightarrow 2$: $A=\mathrm{diag}(1,1,1,1,1)$, $B=\mathrm{diag}(0,0,-1,-1,0)$, $C=\mathrm{diag}(-1,0,0,0,0)$, $D=\mathrm{diag}(1,1,1,1,1)$,

$1\rightarrow 3$: $A=\mathrm{diag}(1,1,-1,1,0)$, $B=\mathrm{diag}(-1,-1,0,1,-1)$, $C=\mathrm{diag}(0,0,0,-1,1)$, $D=\mathrm{diag}(1,1,-1,0,1)$,

$1\rightarrow 4$: $A=\mathrm{diag}(1,1,1,-1,0)$, $B=\mathrm{diag}(-1,-1,1,0,-1)$, $C=\mathrm{diag}(0,0,-1,0,1)$, $D=\mathrm{diag}(1,1,0,-1,1)$,

$1\rightarrow 5$: $A=\mathrm{diag}(1,1,1,1,1)$, $B=\mathrm{diag}(-1,-1,0,0,-1)$, $C=\mathrm{diag}(0,0,0,-1,0)$, $D=\mathrm{diag}(1,1,1,1,1)$,

$1\rightarrow 6$: $A=\mathrm{diag}(-1,-1,1,0,1)$, $B=\mathrm{diag}(0,0,-1,-1,1)$, $C=\mathrm{diag}(0,0,0,1,-1)$, $D=\mathrm{diag}(-1,-1,1,1,0)$,

$1\rightarrow 7$: $A=\mathrm{diag}(1,-1,1,0,-1)$, $B=\mathrm{diag}(1,0,-1,-1,0)$, $C=\mathrm{diag}(-1,0,0,1,0)$, $D=\mathrm{diag}(0,-1,1,1,-1)$,

$1\rightarrow 8$: $A=\mathrm{diag}(-1,1,1,0,-1)$, $B=\mathrm{diag}(0,1,-1,-1,0)$, $C=\mathrm{diag}(0,-1,0,1,0)$, $D=\mathrm{diag}(-1,0,1,1,-1)$,

$1\rightarrow 9$: $A=\mathrm{diag}(1,1,0,-1,0)$, $B=\mathrm{diag}(0,0,1,0,-1)$, $C=\mathrm{diag}(0,0,-1,0,1)$, $D=\mathrm{diag}(1,1,0,-1,0)$,

$1\rightarrow 10$: $A=\mathrm{diag}(1,1,-1,0,0)$, $B=\mathrm{diag}(0,0,0,1,-1)$, $C=\mathrm{diag}(0,0,0,-1,1)$, $D=\mathrm{diag}(1,1,-1,0,0)$,

$1 \to 11$: $A = \mathrm{diag}(0,-1,1,0,-1)$,
$B = \mathrm{diag}(1,0,0,-1,0)$, $C = \mathrm{diag}(-1,0,0,1,0)$,
$D = \mathrm{diag}(0,-1,1,0,-1)$,

$1 \to 12$: $A = \mathrm{diag}(0,-1,0,1,-1)$,
$B = \mathrm{diag}(1,0,-1,0,0)$, $C = \mathrm{diag}(-1,0,1,0,0)$,
$D = \mathrm{diag}(0,-1,0,1,-1)$,

$1 \to 13$: $A = \mathrm{diag}(-1,0,1,0,-1)$,
$B = \mathrm{diag}(0,1,0,-1,0)$, $C = \mathrm{diag}(0,-1,0,1,0)$,
$D = \mathrm{diag}(-1,0,1,0,-1)$,

$1 \to 14$: $A = \mathrm{diag}(-1,0,0,1,-1)$,
$B = \mathrm{diag}(0,1,-1,0,0)$, $C = \mathrm{diag}(0,-1,1,0,0)$,
$D = \mathrm{diag}(-1,0,0,1,-1)$.

The local operations are represented by $Q=\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, with A, B, C, and D illustrated above, respectively. In the above equations, we do not take the phase into account because the phase can be adjusted to any value by Eq. (16). It is emphasized that anyone of the operations above is not unique, meaning that there are a number of different LC operations that can implement the same transformation.

VII. CONCLUSION

In this paper, we map the n-mode Pauli operator into $2n$-dimension real column vector, and map n-mode local Clifford operator U into $2n\times 2n$ matrix Q. Continuous variable stabilizer state is described by its generator matrix Θ. We prove the fact that any stabilizer state could be reduced to the corresponding weighed graph state under local unitary operation in the Clifford group. A matrix equation, which is used to determine whether two stabilizer states are equivalent or not under local unitary operation in the Clifford group, is proposed. Then we demonstrate that our theory could be able to correctly and quickly find the equivalent classes of five-mode unweighed graph states. For one class of five-mode unweighed graph states that are equivalent each other under LC operations, we apply the proposed method to solve the problem that how to find the corresponding LC operations between two five-mode graph states.

In addition, we also illustrate that local complement rule which governs discrete variable graph states is no longer available for CV graph states. The reason may be as follows. Under DV condition, the stabilizer states are equivalent to unweighed graphs, but under CV condition the stabilizer states are equivalent to weighted graphs.

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