

## Area law for the entropy of low-energy states

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It is often observed in the ground state of quantum lattice systems with local interactions that the entropy of a large region is proportional to its surface area. In some cases, this area law is corrected with a logarithmic factor. This contrasts with the fact that in almost all states of the Hilbert space, the entropy of a region is proportional to its volume. This paper shows that low-energy states have (at most) an area law with the logarithmic correction, provided two conditions hold: (i) the state has sufficient decay of correlations and (ii) the number of eigenstates with vanishing energy density is not exponential in the volume. These two conditions are satisfied by many relevant systems. The central idea of the argument is that energy fluctuations inside a region can be observed by measuring the exterior and a superficial shell of the region.

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### I. INTRODUCTION

Entropy quantifies the uncertainty about the state of a physical system. A bipartite system in a pure state has zero entropy, but the reduced state of one subsystem may have positive entropy. This is due to quantum correlations between the two subsystems, the entanglement. In fact, this entropy quantifies the entanglement in the sense of quantum information theory [1].

In classical physics, the entropy of a region inside a spatially extended system at finite temperature is proportional to the volume of the region—entropy is an extensive quantity. At zero temperature, it is small and independent of the region. In quantum physics, at finite temperature, the entropy of a region is also proportional to the volume. But it has been observed in several models that, at zero temperature, the entropy of a region is proportional to its surface area [2–10]. In some models of critical free fermions the entropy scales as the area times the logarithm of the volume [11,12]. This has been presented as a violation of the area law, although the dimensionality of the scaling is still that of the area. A celebrated proof shows that any one-dimensional system with finite-range interactions and an energy gap above the ground state obeys a strict area law [13].

The original motivation for this problem is the analogy with black-hole physics, where the thermodynamic entropy is proportional to the surface area of the event horizon [2,3,14]. The second motivation is to guide the development of efficient methods for simulating quantum systems with classical computers. The number of parameters needed for specifying an arbitrary pure state of an  $N$ -partite system is exponential in  $N$ . If the state is not entangled, the number of parameters is proportional to  $N$ . Hence, there seems to be a correspondence between entanglement and complexity. In one spatial dimension, the relation between entropy and the complexity of simulating a system is well understood [5,15,16]. The third motivation is to understand the kind of states that arise in quantum many-body systems with strong interactions. Almost all states in the Hilbert space obey a volume law for the entropy [17]. Hence, area laws tell a lot about the multipartite entanglement structure. At a finer level, the specific form of an area law tells additional infor-

mation about the system: the logarithmic correction is a signature of criticality [4,5,8,11,12]; and the appearance of a negative constant is a signature of topological order [18]. For further overview of the topics around area laws see the review article cited [19].

### II. RESULTS AND SUMMARY

Consider an arbitrary Hamiltonian  $H$  with finite-range interactions in an  $s$ -dimensional lattice. The eigenstates have a well-defined global energy, but inside a region  $\mathcal{X}$  of the lattice the energy may fluctuate. (The nomenclature of Fig. 1 is followed.) In Sec. III it is proven that these fluctuations can be observed by measuring the exterior of the region and a superficial shell inside the region, that is,  $\bar{\mathcal{X}} \cup S$ . In Sec. IV a condition is imposed to the ground state: if the operator  $X$  has support on the region  $\mathcal{R}$  which is separated from the support of the operator  $Y$  by a distance  $l$ , then the connected correlation function decays at least as

$$|\langle XY \rangle - \langle X \rangle \langle Y \rangle| \leq (l - \xi \ln |\mathcal{R}|)^{-s}, \quad (1)$$

where  $\xi$  is a constant. This implies that energy fluctuations inside the region  $\mathcal{X}$  cannot be observed in its bulk, namely,  $\mathcal{R}$ . This provides a characterization for the approximate support of the global ground state inside the region  $\mathcal{R}$ . In Sec. V a condition on the density of states is assumed: if  $H_{\mathcal{X}}$  is the

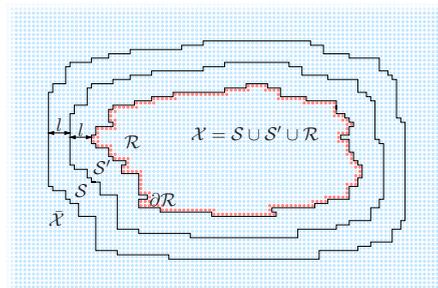


FIG. 1. (Color online)  $\mathcal{R}$  is the chosen region where the entropy is estimated; the sites belonging to its boundary  $\partial\mathcal{R}$  are darker;  $S$  and  $S'$  are two superficial shells with thickness  $l$  outside  $\mathcal{R}$ ;  $\bar{\mathcal{X}} = S \cup S' \cup \mathcal{R}$  is the extended region;  $\bar{\mathcal{X}}$  is the exterior of  $\mathcal{X}$ .

sub-Hamiltonian with all terms of  $H$  whose support is fully contained in  $\mathcal{X}$ , then the number of eigenvalues lower than  $e$  is bounded by

$$\Omega_{\mathcal{X}}(e) \leq (\tau|\mathcal{X}|)^{\gamma(e-e_0)+\eta|\partial\mathcal{X}|}, \quad (2)$$

where  $e_0$  is the lowest eigenvalue and  $\tau, \gamma, \eta$  some constants independent of  $\mathcal{X}$ . This condition is only assumed for  $e \sim |\partial\mathcal{X}|$ . This implies an upper bound on the dimension of the above-defined support subspace. This is used to bound the Von Neumann entropy for the reduction in the global ground state in the region  $\mathcal{R}$ ,

$$S(\rho_{\mathcal{R}}) = \text{tr}(-\rho_{\mathcal{R}} \ln \rho_{\mathcal{R}}) \leq \text{const} \times |\partial\mathcal{R}| \ln |\mathcal{R}|. \quad (3)$$

Section VI contains a simpler proof for the area law (3) without assuming (1), but assuming (2) for all the range of  $e$ . In Sec. VII the above results for the ground state are generalized to other low-energy states (not necessarily eigenstates). Section VIII contains the conclusions.

### III. LOCALITY AND ENERGY FLUCTUATIONS

#### A. Local interactions

Consider a system with one particle at each site  $x \in \mathcal{L}$  of a finite  $s$ -dimensional cubic lattice  $\mathcal{L} \subset \mathbb{Z}^s$ . The distance between two sites  $x, y \in \mathcal{L}$  is defined by

$$\mathcal{D}(x, y) = \max_{1 \leq i \leq s} |x_i - y_i|. \quad (4)$$

In the case of periodic boundary conditions or hybrids, this distance has to be modified with the appropriate identification of sites. Each particle  $x \in \mathcal{L}$  has associated a Hilbert space with finite dimension  $q$ .

The Hamiltonian of the system can be written as

$$H = \sum_{x \in \mathcal{L}} K_x, \quad (5)$$

where each term  $K_x$  can have nontrivial support on first neighbors ( $y \in \mathcal{L}$  such that  $\mathcal{D}(y, x) \leq 1$ ). There is a constant  $J$  which bounds the operator norm of all terms  $\|K_x\| \leq J$ . (The operator norm of a matrix is equal to its largest singular value.) Translational symmetry is not assumed, so each term  $K_x$  is arbitrary. The eigenstates and eigenvalues of  $H$  are denoted by

$$H|\Psi_n\rangle = E_n|\Psi_n\rangle, \quad (6)$$

where the index  $n=0, 1, \dots$  labels the eigenvalues in increasing order  $E_n \leq E_{n+1}$ .

Note that any Hamiltonian with finite-range interactions in a sufficiently regular lattice can be brought to the form of  $H$ , by coarse graining the lattice. Quantum field theories with local interactions can also be brought to the form of  $H$  by lattice regularization. In the case of bosons, a truncation in the local degrees of freedom is needed. In the case of fermions, a multidimensional Jordan-Wigner transformation [20] is needed.

#### B. Lieb-Robinson bound

The Hamiltonian  $H$  satisfies the premises for the Lieb-Robinson Bound [21,22]. Let  $X, Y$  be two operators acting

respectively on the regions  $\mathcal{X}, \mathcal{Y} \subset \mathcal{L}$ , with  $\|X\|, \|Y\| \leq 1$ . The distance between two regions is defined by

$$\mathcal{D}(\mathcal{X}, \mathcal{Y}) = \min_{x \in \mathcal{X}, y \in \mathcal{Y}} \mathcal{D}(x, y). \quad (7)$$

The time evolution of an operator in the Heisenberg picture is  $X(t) = e^{iHt} X e^{-iHt}$ . The Lieb-Robinson Bound states that

$$\| [X(t), Y] \| \leq 2|\mathcal{X}| \frac{(vt)^{|\mathcal{D}(\mathcal{X}, \mathcal{Y})/2|}}{[\mathcal{D}(\mathcal{X}, \mathcal{Y})/2]!}, \quad (8)$$

where  $v=2J5^s$ . When  $vt \ll \mathcal{D}(\mathcal{X}, \mathcal{Y})$  the two operators almost commute. In other words, the dynamics generated by  $H$  does not allow for the propagation of signals at speed much larger than  $v$ . A simple proof of the Lieb-Robinson bound (8) is provided in Appendix C.

#### C. Average for the energy fluctuations

For any region  $\mathcal{X} \subset \mathcal{L}$  and any integer  $l \geq 5$  define the exterior, the boundary and the superficial shell as

$$\bar{\mathcal{X}} = \mathcal{L} \setminus \mathcal{X} = \{x \in \mathcal{L} : x \notin \mathcal{X}\}, \quad (9)$$

$$\partial\mathcal{X} = \{x \in \mathcal{X} : \mathcal{D}(x, \bar{\mathcal{X}}) = 1\}, \quad (10)$$

$$S = \{x \in \mathcal{X} : \mathcal{D}(x, \bar{\mathcal{X}}) \leq l\}, \quad (11)$$

respectively (see Fig. 1). The Hamiltonian  $H_{\mathcal{X}}$  is defined as the sum of all terms  $K_x$  whose support is fully contained in  $\mathcal{X}$ . The eigenstates and eigenvalues of  $H_{\mathcal{X}}$  are denoted by

$$H_{\mathcal{X}}|\psi_n\rangle = e_n|\psi_n\rangle, \quad (12)$$

where the index  $n=0, 1, \dots$  labels the eigenvalues in increasing order  $e_n \leq e_{n+1}$ . The sum of all terms  $K_x$  which simultaneously act on  $\mathcal{X}$  and  $\bar{\mathcal{X}}$  is  $H_1 = H - H_{\mathcal{X}} - H_{\bar{\mathcal{X}}}$ , and has norm  $\|H_1\| \leq J3^s|\partial\mathcal{X}|$ . Without loss of generality it can be assumed that each  $K_x$  is positive semidefinite, which implies

$$\begin{aligned} \langle H_{\mathcal{X}} \rangle + \langle H_{\bar{\mathcal{X}}} \rangle &\leq \langle H_{\mathcal{X}} + H_1 + H_{\bar{\mathcal{X}}} \rangle \\ &\leq \text{tr}[(H_{\mathcal{X}} + H_1 + H_{\bar{\mathcal{X}}})(|\psi_0\rangle\langle\psi_0| \otimes \text{tr}_{\mathcal{X}}|\Psi_0\rangle\langle\Psi_0|)] \\ &\leq e_0 + J3^s|\partial\mathcal{X}| + \langle H_{\bar{\mathcal{X}}} \rangle, \end{aligned}$$

and

$$e_0 \leq \langle H_{\mathcal{X}} \rangle \leq e_0 + J3^s|\partial\mathcal{X}|. \quad (13)$$

This can be summarized as follows: The energy frustration of the global ground state  $|\Psi_0\rangle$  in a region  $\mathcal{X}$  is, at most, proportional to the boundary  $\partial\mathcal{X}$ .

#### D. Observation of energy fluctuations

For any value of  $e_{\text{cut}}$  define the operator

$$Q = \int_{-\infty}^{e_{\text{cut}}} d\omega \int \frac{dt}{2\pi} e^{-(\sigma^2/2)t} e^{i(E_0 - \omega)t} e^{iH_{\mathcal{X}}t} e^{-iHt}, \quad (14)$$

where  $\sigma = 10^4 v^2 / l$ . The action of  $Q$  onto the global ground state  $|\Psi_0\rangle$  implements an approximate projection onto the

subspace with energy lower than  $e_{\text{cut}}$  inside the region  $\mathcal{X}$ ,

$$Q|\Psi_0\rangle = \left[ \sum_n \int_{-\infty}^{e_{\text{cut}}-e_n} d\omega \frac{e^{-(\omega^2/2\sigma)}}{\sqrt{2\pi\sigma}} |\psi_n\rangle\langle\psi_n| \right] |\Psi_0\rangle. \quad (15)$$

This integral is the error function, which is a soft step function. In the limit where the softness parameter  $\sigma$  tends to zero, the operator inside the square brackets becomes a projector. The operator  $Q$  has nontrivial support on the whole lattice  $\mathcal{L}$ , but remarkably, it can be approximated by the operator

$$\tilde{Q} = \int_{-\infty}^{e_{\text{cut}}} d\omega \int \frac{dt}{2\pi} e^{-(\sigma t^2/2)} e^{i(E_0-\omega)t} e^{iH_S t} e^{-iH_{\bar{\mathcal{X}}\cup S} t}, \quad (16)$$

which has nontrivial support only in the region  $\bar{\mathcal{X}}\cup S$ . More quantitatively, the bound

$$\|Q - \tilde{Q}\| \leq |\mathcal{X}|^3 e^{-l} \quad (17)$$

is proven in Lemma 1 (Appendix A), using techniques similar to the ones in [13,22,23]. The fact that  $Q \approx \tilde{Q}$  is solely a consequence of the locality of interactions and can be understood as follows. According to the Lieb-Robinson bound (8), if  $t < l/v$ , any operator  $Y$  with support on  $\mathcal{X}\setminus S$  evolves to an operator  $Y(t)$  with approximate support on  $\mathcal{X}$ . Then  $e^{-iH_{\mathcal{X}} t} Y(t) e^{iH_{\mathcal{X}} t} \approx Y$ , or in other words, the unitary  $e^{iH_{\mathcal{X}} t} e^{-iH t}$  in Eq. (14) approximately acts like the identity inside  $\mathcal{X}\setminus S$ , or in other words  $e^{iH_{\mathcal{X}} t} e^{-iH t} \approx e^{iH_S t} e^{-iH_{\bar{\mathcal{X}}\cup S} t}$ , which justifies the definition (16).

The right-hand side of Eq. (16) is an average of unitaries; therefore  $\|\tilde{Q}\| \leq 1$ . Then, the operators  $|\tilde{Q}\rangle$  and  $(\mathbb{1} - |\tilde{Q}\rangle)$  define a two-outcome generalized measurement on  $\bar{\mathcal{X}}\cup S$ , which tells whether the energy inside  $\mathcal{X}$  is below or above  $e_{\text{cut}}$ , approximately.

Everything shown in this sec. for the ground state generalizes to all eigenstates. The action of  $Q$  onto  $|\Psi_n\rangle$  is

$$Q|\Psi_n\rangle = \left[ \sum_n \int_{-\infty}^{e'_{\text{cut}}-e_n} d\omega \frac{e^{-(\omega^2/2\sigma)}}{\sqrt{2\pi\sigma}} |\psi_n\rangle\langle\psi_n| \right] |\Psi_n\rangle, \quad (18)$$

where  $e'_{\text{cut}} = e_{\text{cut}} + E_n - E_0$ . Summarizing, for each eigenstate  $|\Psi_n\rangle$  there is an operator  $\tilde{Q}$  which approximately projects onto the subspace with energy  $e_n \leq e_{\text{cut}}$  inside the region  $\mathcal{X}$ , by only acting on the exterior and the shell  $\bar{\mathcal{X}}\cup S$ . The degree of approximation increases with  $l$ , the width of  $S$ . The larger  $l$  is, the closer  $Q$  and  $\tilde{Q}$  are, and the smaller the softness parameter  $\sigma$  is. The energy fluctuations of an eigenstate  $|\Psi_n\rangle$  inside a region  $\mathcal{X}$  can be observed by measuring the exterior and a superficial shell inside the region, that is,  $\bar{\mathcal{X}}\cup S$  (see Fig. 1).

#### IV. SUPPORT OF THE GROUND STATE INSIDE A REGION

##### A. Decay of correlations

It is usually the case that, when the system is in the ground state, the correlation between two observables acting on different sites decrease with the distance between the

sites. Let  $\Gamma$  be a function which upper-bounds the connected correlation function of any pair of operators  $X, Y$  acting respectively on the disjoint regions  $\mathcal{X}, \mathcal{Y} \subset \mathcal{L}$ , with  $|\mathcal{X}| \leq |\mathcal{Y}|$  and  $\|X\|, \|Y\| \leq 1$ ,

$$|\langle XY \rangle - \langle X \rangle \langle Y \rangle| \leq \Gamma(\mathcal{D}(\mathcal{X}, \mathcal{Y}), |\mathcal{X}|). \quad (19)$$

(The expectation of any operator  $X$  with the ground state is denoted by  $\langle X \rangle = \langle \Psi_0 | X | \Psi_0 \rangle$ .) For the argument of this paper, both, the decay with the distance  $\mathcal{D}(\mathcal{X}, \mathcal{Y})$  and the scaling with size of the support of the operators  $|\mathcal{X}|$ , are relevant. It is shown in [22] that any Hamiltonian  $H$  with an energy gap above the ground state  $\Delta = E_1 - E_0 > 0$  has

$$\Gamma(l, |\mathcal{X}|) = c_1 |\mathcal{X}| e^{-l/\xi}, \quad (20)$$

with correlation length  $\xi = 10v/\Delta$ . To prove the area law for the entropy the following condition is needed.

*Assumption 1.* The correlation functions for the ground state decay at least as

$$\Gamma(l, |\mathcal{X}|) = \frac{c_1}{(l - \xi \ln |\mathcal{X}|)^\nu}, \quad (21)$$

where  $c_1, \xi$  and  $\nu > s$  are constants.

Note that both, Eqs. (20) and (21), have the same relative scaling of  $l$  and  $|\mathcal{X}|$ , but assumption (21) is weaker than Eq. (20). Although the decay (21) is polynomial in  $l$ , it is not the correlation function of a critical Hamiltonian, where one expects  $\Gamma \sim (|\mathcal{X}|^{1/s}/l)^\nu$ . Unfortunately, the argument of this paper does not give an area law with such scaling in  $|\mathcal{X}|$ .

##### B. Energy fluctuations inside a region cannot be observed in its bulk

For any region  $\mathcal{R} \subset \mathcal{L}$  and any integer  $l \geq 5$  define the extended region as

$$\mathcal{X} = \{x \in \mathcal{L} : \mathcal{D}(x, \mathcal{R}) \leq 2l\}, \quad (22)$$

which redefines Eqs. (9)–(11) (see Fig. 1). The region  $\mathcal{R}$  can be considered the bulk of  $\mathcal{X}$ .

Suppose the existence of an operator  $Z$  with support in  $\mathcal{R}$  such that

$$Z|\Psi_0\rangle \approx \sum_{n: e_n \leq e_{\text{cut}}} |\psi_n\rangle\langle\psi_n| \Psi_0\rangle.$$

This operator acts onto the ground state in a similar way as  $\tilde{Q}$  does; then the two operators are correlated,

$$\langle Z\tilde{Q} \rangle \approx \langle Z \rangle \approx \langle \tilde{Q} \rangle,$$

and their corresponding supports are separated by a distance  $l$ . For the right choice of  $e_{\text{cut}}$  and large enough  $l$  the existence of  $Z$  is in contradiction with Assumption 1; therefore the energy fluctuations of the global ground state inside a region  $\mathcal{X}$  cannot be observed in the bulk of the region, that is  $\mathcal{R}$ .

In the following subsection, a quantitative example of this fact is given.

##### C. Characterization of the support

In what follows, the assignation

$$e_{\text{cut}} = 2J3^s|\partial\mathcal{X}| + e_0 + 20v \quad (23)$$

is assumed in the definitions of  $Q$  and  $\tilde{Q}$  [Eqs. (14) and (16)].

*Definition of  $P$ .* For each eigenstate  $|\psi_n\rangle$  of  $H_{\mathcal{X}}$  with  $e_n \leq e_{\text{cut}} + 20v$  consider the Schmidt decomposition [1]  $|\psi_n\rangle = \sum_i \mu_n^i |\alpha_n^i\rangle \otimes |\beta_n^i\rangle$  with respect to the partition  $|\alpha_n^i\rangle \in \mathcal{H}_{\mathcal{R}}$  and  $|\beta_n^i\rangle \in \mathcal{H}_{\mathcal{X}\setminus\mathcal{R}}$ . Define  $P$  as the projector onto the subspace of  $\mathcal{H}_{\mathcal{R}}$  generated by all vectors  $|\alpha_n^i\rangle$  defined above, symbolically

$$P = \text{supp}_{\mathcal{R}}\{|\psi_n\rangle; e_n \leq 2J3^s|\partial\mathcal{X}| + e_0 + 40v\}. \quad (24)$$

Let  $P^\perp = \mathbb{1} - P$  be the projector onto the complementary subspace. Lemma 3 (Appendix A) shows that the assignment (23) implies

$$\langle \tilde{Q} \rangle \geq \frac{1}{2} - 2|\mathcal{X}|^3 e^{-l}, \quad (25)$$

$$\langle P^\perp \tilde{Q} \rangle \leq 2|\mathcal{X}|^3 e^{-l}. \quad (26)$$

Recalling that the respective supports of  $P^\perp$  and  $\tilde{Q}$  are separated by a distance  $l$ , one can invoke the decay of correlations (19) without specifying the function  $\Gamma$ ,

$$\langle P^\perp \rangle \langle \tilde{Q} \rangle - \langle P^\perp \tilde{Q} \rangle \leq \Gamma(l, |\mathcal{R}|). \quad (27)$$

The combination of Eqs. (25)–(27) gives

$$\langle P \rangle \geq 1 - 4\Gamma(l, |\mathcal{R}|), \quad (28)$$

for sufficiently large  $l$ , where  $1/2 \geq \Gamma(l, |\mathcal{R}|) \geq 6|\mathcal{X}|^3 e^{-l}$  holds. Concluding, the support of the global ground state inside  $\mathcal{R}$  is contained in the subspace characterized by  $P$ , up to some small weight (28).

#### D. Renormalization-group scheme

The projector  $P$  defined above allows for certifiably generating a low-energy effective theory for  $H$ : the Hamiltonian terms  $K_x$  inside  $\mathcal{R}$  can be renormalized as

$$K_x \xrightarrow{\text{RG}} PK_x P. \quad (29)$$

The whole lattice can be divided in similar regions, and the transformation (29) performed in each of them. The fidelity between the effective and the original ground states can be bounded with Eq. (28), and increased by enlarging  $l$ . As explained in Sec. VI, one can also obtain arbitrarily good fidelities for any low-energy state.

### V. ENTANGLEMENT IN THE GROUND STATE

#### A. Energy spectrum

In the previous section, a subspace which approximately contains the support of the ground state inside a region has been characterized. In order to bound its dimension, an additional assumption is needed: if the boundary conditions of the Hamiltonian are left open, the number of eigenstates with vanishing energy-density must not be exponential in the volume.

*Assumption 2.* There are constants  $c_2, \tau, \gamma, \eta$  such that, for any region  $\mathcal{X}$  and energy

$$e = 2J3^s|\partial\mathcal{X}| + e_0 + 40v, \quad (30)$$

the number of eigenvalues of  $H_{\mathcal{X}}$  lower than  $e$  satisfies

$$\Omega_{\mathcal{X}}(e) = \max\{n: e_n \leq e\} \leq c_2(\tau|\mathcal{X}|)^{\gamma(e-e_0)+\eta|\partial\mathcal{X}|}. \quad (31)$$

The area law is nontrivial when applied to regions  $\mathcal{R}$  such that  $|\partial\mathcal{R}| \ll |\mathcal{R}|$ , or equivalently  $|\partial\mathcal{X}| \ll |\mathcal{X}|$ . In this case, the eigenstates with energy proportional to the boundary  $|\partial\mathcal{X}|$  (30) have vanishing energy density  $e_n/|\mathcal{X}|$ . According to [23], Assumption 2 holds for many systems that have an energy gap above the ground state. There are known Hamiltonians which violate Assumption 2 and have a gap, but when the boundary conditions are opened there appears a degeneracy for the ground state which is exponential in the volume [23]. Massive free bosons and fermions satisfy Assumption 2. Contrary, massless free fermions violate it as  $\Omega \sim \exp\sqrt{(e-e_0)|\mathcal{X}|^{1/s}}$ .

The factor  $(\tau|\mathcal{X}|)^{\gamma(e-e_0)}$  in Eq. (31) can be understood with the following example. Consider the Hamiltonian

$$H_{\mathcal{X}} = \sum_{x \in \mathcal{X}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_x,$$

where the subindex  $x$  specifies in which site the matrix acts. The energy  $e \in \{0, 1, \dots, |\mathcal{X}|\}$  counts the number of local excitations; hence the degeneracy is the binomial of  $|\mathcal{X}|$  over  $e$ , which can be upper bounded by  $|\mathcal{X}|^e$ . The constant factor  $(\tau|\mathcal{X}|)^{\eta|\partial\mathcal{X}|}$  in Eq. (31) is introduced because some Hamiltonians with open boundary conditions have a degeneracy (or approximate degeneracy) which is exponential in the size of the boundary.

Consider again the Schmidt decomposition of each eigenstate  $|\psi_n\rangle$  with respect to the partition  $\mathcal{R}$  and  $\mathcal{X}\setminus\mathcal{R}$  (definition of  $P$ ). The dimension of the Hilbert space  $\mathcal{H}_{\mathcal{X}\setminus\mathcal{R}}$  is  $q^{|\mathcal{X}\setminus\mathcal{R}|}$ ; therefore the support of each  $|\psi_n\rangle$  on  $\mathcal{R}$  has at most dimension  $q^{|\mathcal{X}\setminus\mathcal{R}|}$ . This and Assumption 2 provide a bound for the rank of the projector  $P$ ,

$$\text{rank } P \leq q^{|\mathcal{X}\setminus\mathcal{R}|} c_2(\tau|\mathcal{X}|)^{[\partial\mathcal{X}(\gamma 2J3^s + \eta) + \gamma 40v]}. \quad (32)$$

#### B. Entropy of an arbitrary region

Consider a region  $\mathcal{R} \subset \mathcal{L}$  being a completely arbitrary subset of the lattice. It need not be convex, full-dimensional, nor connected. For any site  $x$

$$|\{y \in \mathcal{L}: D(y, x) \leq 2l\}| \leq (5l)^s,$$

$$|\{y \in \mathcal{L}: D(y, x) = 2l\}| \leq 2s(5l)^{s-1},$$

which imply

$$|\mathcal{X}| \leq |\mathcal{R}|(5l)^s,$$

$$|\partial\mathcal{X}| \leq |\partial\mathcal{R}|2s(5l)^{s-1},$$

$$|\mathcal{X}/\mathcal{R}| \leq |\partial\mathcal{R}|(5l)^s. \quad (33)$$

Let  $\rho_{\mathcal{R}} = \text{tr}_{\mathcal{L} \setminus \mathcal{R}} |\Psi_0\rangle\langle\Psi_0|$  be the reduction in the ground state in  $\mathcal{R}$ , and  $\lambda_1 \leq \lambda_2 \leq \dots$  its eigenvalues in decreasing order. Assumptions 1 and 2 imply Eqs. (21), (28), and (32), which impose the following constraints on the eigenvalues: for any integer  $l \geq 5$ ,

$$\sum_{k=1}^{\Theta(l)} \lambda_k \geq 1 - \theta(l),$$

$$\theta(l) = \frac{4c_1}{(l - \xi \ln|\mathcal{R}|)^\nu},$$

$$\ln \Theta(l) = |\partial\mathcal{R}| 2s(5l)^{s-1} (\gamma 2J3^s + \eta) \ln[\tau|\mathcal{R}|(5l)^s] + |\partial\mathcal{R}|(5l)^s \ln q + O(\ln|\mathcal{R}|). \quad (34)$$

Now one can find the probability distribution  $\lambda_k$  which maximizes the entropy  $(-\sum \lambda_k \ln \lambda_k)$  given the above constraints. This is done in Appendix B with the following result.

*Result 1.* The entropy of the reduction in the ground state inside an arbitrary region  $\mathcal{R}$  satisfies

$$S(\rho_{\mathcal{R}}) \leq |\partial\mathcal{R}|(10\xi \ln|\mathcal{R}|)^s \left[ \frac{s}{\xi} (\gamma J3^s + \eta) + \ln q \right] + O(|\partial\mathcal{R}|(\ln|\mathcal{R}|)^{s-1}). \quad (35)$$

### C. Entropy of a cubic region

Consider the case where the chosen region is a hypercube  $\mathcal{R} = \{x \in \mathcal{L} : 0 \leq x_i \leq L\}$ . One can proceed as before, but the bounds analogous to Eq. (33) are smaller, implying a smaller bound for the entropy. All this is worked out in Appendix B.

*Result 2.* The entropy of the reduction in the ground state inside a cubic region  $\mathcal{R}$  satisfies

$$S(\rho_{\mathcal{R}}) \leq |\partial\mathcal{R}| \ln|\mathcal{R}| (\gamma 2J3^s + \eta + 4\xi \ln q) + O(|\partial\mathcal{R}|). \quad (36)$$

It is expected that the entropy of any full-dimensional convex region  $\mathcal{R}$  obeys the same scaling (36).

## VI. SIMPLER PROOF FOR THE AREA LAW

An area law can be easily proven without Assumption 1, if Assumption 2 is extended to all values of the energy  $e$ , not only the ones satisfying Eq. (30). Let  $\mathcal{R}$  be the region where the entropy is estimated, and  $H_{\mathcal{R}}$  the sum of all terms of the Hamiltonian (5) which are fully contained in  $\mathcal{R}$ . Following the conventions of this paper, the eigenstates and eigenvalues are denoted by  $H_{\mathcal{R}}|\psi_n\rangle = e_n|\psi_n\rangle$ , where  $e_0 \leq e_1 \leq \dots$ . The strong version of Assumption 2 tells that all the eigenvalues  $e_n$  satisfy

$$n \leq c_2 (\tau|\mathcal{R}|)^{\gamma(e_n - e_0) + \eta|\partial\mathcal{R}|}. \quad (37)$$

The global ground state can be written as

$$|\Psi_0\rangle = \sum_k \sqrt{\mu_n} |\psi_n\rangle \otimes |\varphi_n\rangle, \quad (38)$$

where the coefficients  $\mu_n$  are non-negative and add up to one. It is shown in [1] that the entanglement entropy of  $|\Psi_0\rangle$  is upper bounded by the entropy of the  $\mu$  coefficients

$$S(\rho_{\mathcal{R}}) \leq - \sum_n \mu_n \ln \mu_n. \quad (39)$$

Locality implies Eq. (13), which can be written as

$$\sum_n \mu_n e_n \leq e_0 + J3^s |\partial\mathcal{R}|. \quad (40)$$

Maximizing the right-hand side of Eq. (39) over the probability distribution  $\mu_n$  and the numbers  $e_n$  subjected to the constrains (37) and (40) gives

$$S(\rho_{\mathcal{R}}) \leq \text{const} |\partial\mathcal{R}| \ln|\mathcal{R}|, \quad (41)$$

the area law.

## VII. ENTANGLEMENT IN EXCITED STATES

The entanglement properties of excited states have also been studied. In Refs. [24–26] the motivation was to study the robustness of the area law for the entropy of black holes. They show that in systems of free bosons, the entropy of some low-energy excited states scales at most like the area. In [27] the entropy scaling in one-dimensional spin systems is analyzed. They show that some excited states have entropy proportional to the volume, but low-energy states obey an area law. All this work is for integrable systems. In what follows, we address the general case.

Sometimes, low-lying excited states  $|\Psi_n\rangle$  have correlation functions similar to the ones of the ground state. The single-mode ansatz for excitations with momentum  $k$  is

$$|\Psi_k^{\text{sm}}\rangle \propto \sum_x e^{ix \cdot k} Z_x |\Psi_0\rangle, \quad (42)$$

where  $Z_x$  is an operator acting on site  $x$  such that  $\langle Z_x \rangle = 0$ . If  $X, Y$  have support on finite regions and the volume of the system  $|\mathcal{L}|$  tends to infinite, then the correlation function (19) for the state (42) is the same as for  $|\Psi_0\rangle$ . The same happens to excited states containing a small number of single-mode excitations. Examples of single-mode excitations are: spin waves, free bosons, and free fermions. In this section it is shown that such excited states obey an area law similar to the one for the ground state. Actually, this is done with a bit more generality.

Consider an arbitrary superposition of eigenstates with bounded energy

$$|\Phi\rangle = \sum_{n: E_n \leq E_m} \mu_n |\Psi_n\rangle. \quad (43)$$

In this case, the correct assignation for  $e_{\text{cut}}$  in the definitions of  $Q, \tilde{Q}$ , and  $P$  (14, 16, 24) is

$$e_{\text{cut}} = 2J3^s |\partial\mathcal{X}| + e_0 + 20v + E_m - E_0. \quad (44)$$

Applying Assumption 1 to the state (43), the arguments follow exactly as for the ground state. Repeating the calculation

of the entropy for a cubic region  $\mathcal{R}$ , and keeping track of the term proportional to  $(E_m - E_0)$  one obtains

$$S(\text{tr}_{\mathcal{L}\mathcal{R}}|\Phi\rangle\langle\Phi|) \leq |\partial\mathcal{R}|\ln|\mathcal{R}|(\gamma 2J3^s + \eta + 4\xi \ln q) + (E_m - E_0)(\ln \tau|\mathcal{R}|)^{1-\nu} \frac{\gamma c_1 2^{\nu+3}}{\nu \xi} + O(|\partial\mathcal{R}|). \tag{45}$$

**VIII. CONCLUSIONS**

It is shown that ground states and low-energy states obey an area law for the entropy, provided two conditions hold: (i) the state has a sufficient decay of correlations, and (ii) the number of eigenstates with vanishing energy-density is not exponential in the volume of the system.

A universal property for local Hamiltonians is also here established. The energy fluctuations of eigenstates inside an arbitrary region can be observed by measuring the exterior and a superficial shell of the region. This extends to any pure state that can be written as a superposition of eigenstates with similar energy.

Some thermodynamic quantities at finite temperature only depend on the density of states. Examples are: free energy, (global) entropy, heat capacity, etc. This paper establishes a relation between these thermodynamic quantities and ground-state entanglement.

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**APPENDIX A: PROOFS**

*Lemma 1.* Let  $Q, \tilde{Q}$  be the operators defined in Eqs. (14) and (16), then

$$\|Q - \tilde{Q}\| \leq |\mathcal{X}|^3 e^{-l}. \tag{A1}$$

*Proof.* First, express  $Q$  and  $\tilde{Q}$  with a single integral, by using the identity

$$\int_{-\infty}^e d\omega \int \frac{dt}{2\pi} e^{-(\sigma^2/2)} e^{-i\omega t} = \int \frac{dt}{2\pi} \frac{e^{-(\sigma^2/2)}}{0^+ - it} e^{-iet}.$$

Second, define the operators

$$H_0 = H_{S \cup S'} - H_S - H_{S'},$$

$$H_1 = H - H_{\mathcal{X}} - H_{\bar{\mathcal{X}}},$$

which respectively act on the regions  $\mathcal{H}_0, \mathcal{H}_1 \subset \mathcal{L}$ . Note that  $D(\mathcal{H}_0, \mathcal{H}_1) = l - 4$ ,  $|\mathcal{H}_0| \leq |\mathcal{X}|$ ,  $|\mathcal{H}_1| \leq 5^s |\mathcal{X}|$ , and

$$\|e^{iH_{\mathcal{X}}t} e^{-iHt} - e^{iH_S t} e^{-iH_{S \cup S'} t}\| = \|e^{i(H-H_1)t} e^{-iHt} - e^{i(H-H_1-H_0)t} e^{-i(H-H_0)t}\|.$$

These, the triangular inequality, Lemma 2, and the Lieb-Robinson bound (8), give

$$\begin{aligned} \|Q - \tilde{Q}\| &\leq 2 \int_0^{t_0} dt \frac{1}{2\pi t} \|e^{iH_{\mathcal{X}}t} e^{-iHt} - e^{iH_S t} e^{-iH_{S \cup S'} t}\| \\ &\quad + 2 \int_{t_0}^{\infty} dt \frac{e^{-(\sigma^2/2)}}{2\pi t} - 2 \\ &\leq \frac{2|\mathcal{X}|^3 J^2 5^s}{\pi} \int_0^{t_0} dt \frac{1}{t} \int_0^t dt_2 \int_0^{t_2} dt_1 \frac{(vt_1)^{\lfloor l/2 - 2 \rfloor}}{\lfloor l/2 - 2 \rfloor!} \\ &\quad + \frac{4}{t_0 \sqrt{\sigma}} e^{-(\sigma_0^2/2)} \\ &\leq \frac{|\mathcal{X}|^3}{\pi 5^s (l-1)} \frac{(vt_0)^{\lfloor l/2 \rfloor}}{\lfloor l/2 \rfloor!} + \frac{4}{t_0 \sqrt{\sigma}} e^{-(\sigma_0^2/2)}. \end{aligned}$$

Putting  $t_0 = \lfloor l/2 \rfloor / (e^3 v)$  and using Stirling's approximation

$$\frac{(vt_0)^{\lfloor l/2 \rfloor}}{\lfloor l/2 \rfloor!} \leq e^{\lfloor l/2 \rfloor \ln(evt_0/\lfloor l/2 \rfloor)} \leq e^{l-1}.$$

Putting  $\sigma = 10^4 v^2 / l \geq 2lt_0^{-2}$  one obtains Eq. (A1). ■

*Lemma 2.* Let  $H, X, Y$  be Hermitian matrices and  $t > 0$ , then

$$\|e^{i(H-X)t} e^{-iHt} - e^{i(H-X-Y)t} e^{-i(H-Y)t}\| \leq \int_0^t dt_2 \int_0^{t_2} dt_1 \| [X(t_1), Y] \|, \tag{A2}$$

where  $X(t) = e^{iHt} X e^{-iHt}$ .

*Proof.* If  $f(t)$  is a differentiable function with  $f(0) = 0$  then  $f(t) = \int_0^t dt_1 f'(t_1)$ . This implies the following two equalities. The following two inequalities are a consequence of the triangular inequality for the operator norm,

$$\begin{aligned} &\|e^{i(H-X-Y)t} e^{-i(H-Y)t} e^{iHt} e^{-i(H-X)t} - \mathbb{1}\| \\ &= \left\| \int_0^t dt_2 e^{i(H-X-Y)t_2} [-iX e^{-i(H-Y)t_2} e^{iHt_2} + e^{-i(H-Y)t_2} e^{iHt_2} iX] e^{-i(H-X)t_2} \right\| \\ &\leq \int_0^t dt_2 \| -X + e^{-i(H-Y)t_2} e^{iHt_2} X e^{-iHt_2} e^{i(H-Y)t_2} \| \\ &= \int_0^t dt_2 \left\| \int_0^{t_2} dt_1 e^{-i(H-Y)t_1} [Y, X(t_1)] e^{i(H-Y)t_1} \right\| \\ &\leq \int_0^t dt_2 \int_0^{t_2} dt_1 \| [X(t_1), Y] \| \end{aligned}$$

■  
*Lemma 3.* The operator  $\tilde{Q}$  defined in Eq. (16) with  $e_{\text{cut}} = 2J3^s |\partial\mathcal{X}| + e_0 + 20v$  and the projector  $P^\perp$  defined by Eq. (24) satisfy

$$\langle \tilde{Q} \rangle \geq \frac{1}{2} - 2|\mathcal{X}|^3 e^{-l}, \quad (\text{A3})$$

$$\langle P^\perp \tilde{Q} \rangle \leq 2|\mathcal{X}|^3 e^{-l}. \quad (\text{A4})$$

*Proof.* The positive operator

$$M = \sum_n \int_{-\infty}^{e_{\text{cut}} - \epsilon_n} d\omega (2\pi\sigma)^{-1/2} e^{-(\omega^2/2\sigma)} |\psi_n\rangle \langle \psi_n| \quad (\text{A5})$$

allows for writing equality (15) as

$$Q|\Psi_0\rangle = M|\Psi_0\rangle. \quad (\text{A6})$$

The two projectors

$$M_\pm = \sum_{n: e_n \leq e_{\text{cut}} \pm \delta} |\psi_n\rangle \langle \psi_n|, \quad (\text{A7})$$

with  $\delta = 20v$ , satisfy

$$M_- - e^{-l} \mathbb{1} \leq M \leq M_+ + e^{-l} \mathbb{1}, \quad (\text{A8})$$

where we have used that  $e^{-\delta^2/2\sigma} \leq e^{-l}$ . The positivity of  $M$  and the second inequality in Eq. (A8) imply

$$M^2 \leq (1 + 2e^{-l})M_+ + e^{-2l}. \quad (\text{A9})$$

A worst-case estimation gives

$$\langle H_{\mathcal{X}} \rangle \geq \langle M_- \rangle e_0 + (\mathbb{1} - M_-)(e_{\text{cut}} - \delta). \quad (\text{A10})$$

Performing the assignation  $e_{\text{cut}} = 2J3^s |\partial\mathcal{X}| + e_0 + \delta$  in Eq. (A10) and using bound (13) one obtains  $\langle M_- \rangle \geq 1/2$ . The combinations of Eqs. (17), (A6), and (A8) gives Eq. (A3).

Using Lemma 1 and Eq. (A6), the Cauchy-Schwarz inequality, bound (A9), and the definition of  $M_+$  and  $P^\perp$ , one obtains respectively the following chain of inequalities:

$$\begin{aligned} \langle P^\perp \tilde{Q} \rangle &\leq \langle P^\perp M \rangle + |\mathcal{X}|^3 e^{-l} \leq \langle P^\perp \rangle^{1/2} \langle P^\perp M^2 P^\perp \rangle^{1/2} + |\mathcal{X}|^3 e^{-l} \\ &\leq [(1 + 2e^{-l}) \langle P^\perp M_+ P^\perp \rangle + e^{-2l}]^{1/2} + |\mathcal{X}|^3 e^{-l} \\ &\leq 2|\mathcal{X}|^3 e^{-l}, \end{aligned} \quad (\text{A11})$$

which is Eq. (A4).  $\blacksquare$

## APPENDIX B: CALCULATION OF THE ENTROPY

### 1. Entropy of an arbitrary region

Consider the probability distribution defined by

$$\begin{aligned} p_k &= \frac{1 - \theta(l_0)}{\Theta(l_0)} \quad \text{for } 1 \leq k \leq \Theta(l_0), \\ p_k &= \frac{\theta(l-1) - \theta(l)}{\Theta(l) - \Theta(l-1)} \quad \text{for } \Theta(l-1) + 1 \leq k \leq \Theta(l), \end{aligned} \quad (\text{B1})$$

for every integer  $l \geq l_0 = 2\xi \ln|\mathcal{R}|$ , and

$$\theta(l) = \frac{4c_1}{(l - \xi \ln|\mathcal{R}|)^\nu},$$

$$\begin{aligned} \ln \Theta(l) &= |\partial\mathcal{R}| 2s(5l)^{s-1} (\gamma 2J3^s + \eta) \ln[\tau|\mathcal{R}|(5l)^s] + |\partial\mathcal{R}| \\ &\quad \times (5l)^s \ln q + O(\ln|\mathcal{R}|). \end{aligned} \quad (\text{B2})$$

This distribution is uniform in blocks of the maximum size that constraints (34) allow. Then, it is the distribution satisfying (34) with maximum entropy. The upper bound on the entropy of  $p_k$  gets simplified by using the substitutions  $\Theta(l) - \Theta(l-1) \leq \Theta(l)$  and

$$\theta(l-1) - \theta(l) \leq \frac{c_1 2^{\nu+3}}{(l - \xi \ln|\mathcal{R}|)^{\nu+1}}. \quad (\text{B3})$$

Using this, one obtains

$$\begin{aligned} -\sum_k p_k \ln p_k &\leq |\partial\mathcal{R}| (10\xi \ln|\mathcal{R}|)^s \left[ \frac{s}{\xi} (\gamma J3^s + \eta) + \ln q \right] \\ &\quad + O(|\partial\mathcal{R}| (\ln|\mathcal{R}|)^{s-1}). \end{aligned} \quad (\text{B4})$$

### 2. Entropy of a cubic region

Consider the case where the chosen region is an hypercube  $\mathcal{R} = \{x \in \mathcal{L} : 0 \leq x_i \leq L\}$ . It is easy to calculate

$$|\mathcal{R}| = L^s,$$

$$|\partial\mathcal{R}| = 2sL^{s-1}.$$

Following definitions (22) and (10) one obtains

$$|\mathcal{X}| = |\mathcal{R}|(1 + 4l/L)^s,$$

$$|\partial\mathcal{X}| = |\partial\mathcal{R}|(1 + 4l/L)^{s-1},$$

$$|\mathcal{X} \setminus \mathcal{R}| \leq |\partial\mathcal{X}| 2l.$$

Consider the probability distribution (B1) with  $\theta(l)$  given in Eq. (B2) but  $\Theta(l)$  defined as

$$\begin{aligned} \ln \Theta(l) &= |\partial\mathcal{R}| (1 + 4l/L)^{s-1} (\gamma 2J3^s + \eta) \ln[\tau|\mathcal{R}|(1 + 4l/L)^s] \\ &\quad + |\partial\mathcal{R}| (1 + 4l/L)^{s-1} 2l \ln q + O(\ln|\mathcal{R}|). \end{aligned}$$

Using the same tricks as above one obtains the following upper bound for the entropy of  $p_k$ ,

$$-\sum p_k \ln p_k \leq |\partial\mathcal{R}| \ln|\mathcal{R}| (\gamma 2J3^s + \eta + 4\xi \ln q) + O(|\partial\mathcal{R}|).$$

### APPENDIX C: THE LIEB-ROBINSON BOUND

Let  $X, Y$  be two operators with support on the regions  $\mathcal{X}, \mathcal{Y}$ , respectively, and  $L = \mathcal{D}(\mathcal{X}, \mathcal{Y})$ . Let  $Z$  be an arbitrary operator and  $F(t) = [X(t), Z]$ , where  $X(t) = e^{iHt} X e^{-iHt}$  and  $H$  is the Hamiltonian (5). Using the Jacobi identity  $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$  twice one obtains

$$\begin{aligned} \partial_t F(t) &= i[[H, X(t)], Z] = -i[F(t), H] - i\left[[Z, \sum_{x \in \mathcal{Z}} K_x], X(t)\right] \\ &= i[A, F(t)] + i \sum_{x \in \mathcal{Z}} [Z, [X(t), K_x]], \end{aligned} \quad (\text{C1})$$

where  $\mathcal{Z} = \{x : [K_x, Z] \neq 0\}$  and  $A = \sum_{x \in \mathcal{L} \setminus \mathcal{Z}} K_x$ . The above is equivalent to

$$\partial_t(e^{-iAt}F(t)e^{iAt}) = i \sum_{x \in \mathcal{Z}} e^{-iAt}[Z, [X(t), K_x]]e^{iAt},$$

which can be integrated

$$e^{-iAt}F(t)e^{iAt} = F(0) + i \sum_{x \in \mathcal{Z}} \int_0^t dt_1 e^{-iAt_1}[Z, [X(t_1), K_x]]e^{iAt_1}. \tag{C2}$$

The triangular inequality for the operator norm gives

$$\|[X(t), Z]\| \leq \|[X(0), Z]\| + 2\|Z\| \sum_{x \in \mathcal{Z}} \int_0^t dt_1 \|[X(t_1), K_x]\|. \tag{C3}$$

Define  $g_x(t) = \|[X(t), K_x]\|$  and use Eq. (C3) with  $Z=K_x$  to obtain

$$g_x(t) \leq g_x(0) + 2J \sum_{x': \mathcal{D}(x, x') \leq 2} \int_0^t dt_1 g_{x'}(t_1).$$

If  $r = \mathcal{D}(x, \mathcal{X}) \geq 2$  then  $g_x(0) = 0$ . The above can be iterated  $n = \lfloor (r-1)/2 \rfloor$  times

$$g_x(t) \leq v^n \max_{x': \mathcal{D}(x, x') \leq 2n} \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 g_{x'}(t_1),$$

where  $v = 2J5^s$  and  $|\{x': \mathcal{D}(x, x') \leq 2\}| = 5^s$ . For any  $x'$  the bound  $g_{x'}(0) \leq 2J\|X\|$  holds, then

$$g_x(t) \leq 2J\|X\| \frac{(vt)^{\lfloor (r-1)/2 \rfloor}}{[\lfloor (r-1)/2 \rfloor]!}. \tag{C4}$$

Note that  $|\{x: [K_x, X] \neq 0\}| \leq 5^s |\mathcal{X}|$ . This and the bound (C4) can be substituted in Eq. (C3) with  $Z=Y$ , giving

$$\|[X(t), Y]\| \leq 2|\mathcal{X}|\|X\|\|Y\| \frac{(vt)^{\lfloor L/2 \rfloor}}{[\lfloor L/2 \rfloor]!}.$$

This is the Lieb-Robinson bound.

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