Dissipative soliton molecules with independently evolving or flipping phases in mode-locked fiber lasers

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We numerically demonstrate the existence of a discrete family of robust dissipative soliton bound state solutions (soliton molecules) in a mode-locked fiber laser with an *instantaneous* saturable absorber in the *normal* dispersion domain. For a certain domain of the small-signal gain, we obtain a robust first-level bound state with almost constant separation where the phase of the two pulses evolves independently. Moreover, their phase difference can evolve either periodically or chaotically depending on the small-signal gain. Interestingly, higher level bound states exhibit a fundamentally different dynamics. They represent oscillating solutions with a phase difference alternating between zero and π. We identify the crucial role of the linear gain saturation for the existence of these robust molecules independently of their level.

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I. INTRODUCTION

Two bound solitary pulses, so-called bound states (BSs) or soliton molecules, attracted a great deal of interest in the field of lasers and optical communications. Although a first multisoliton solution to the nonlinear Schrödinger equation was already derived as early as 1971 [1], it is still not thoroughly investigated from both theoretical and experimental sides. Here we consider scalar BS solutions consisting of two pulses and characterized by the peak-to-peak separation ρ and the phase difference ϕ between both pulses. In optical dissipative systems with Kerr nonlinearity, such as mode-locked fiber lasers [2–4] or fiber communication lines [5], such BS solutions have been mostly investigated in the framework of a distributed model by using the complex cubic-quintic Ginzburg-Landau equation (CQGLE) [6]. Essentially, three different types of BSs with *invariant* phase difference have been discovered, in-phase ($\phi=0$) and out-of-phase ($\phi=\pi$) stationary solutions, which are unstable as well as stable BS solutions with a $\pi/2$ phase difference. In-phase and out-of-phase unstable BSs have been observed in both normal [7,8] and anomalous [7,9] dispersion domains while stable $\pi/2$ BSs have only been obtained for anomalous dispersion [10]. However, recently we have shown that stable BSs in the normal dispersion domain may exist provided that the noninstantaneous recovery time of semiconductor saturable absorbers (SAs) required for mode locking is taken into account in a more realistic lumped model [8]. As expected in the instantaneous SA limit all in-phase and out-of-phase BSs became unstable. However, there was a slight discrepancy in this picture to be observed in Fig. 15 in Ref. [8] where it can be recognized that the linear gain region of stable BSs does not shrink to zero for vanishing SA recovery time. It is one aim of this contribution to shed some light upon this issue.

Compared to the scalar case, vector BSs behave generally different. For example, out-of-phase vector BSs are stable solutions in mode-locked birefringent fiber lasers at normal average dispersion [11].

Previous numerical investigations demonstrated that at fixed system parameters a discrete family of BS solutions or so-called BS levels with equidistantly increasing peak separation may exist rather than a single BS [8,12]. This discrete family of BSs had been previously analytically predicted [13] and then numerically demonstrated in the framework of the CQGLE and the lumped fiber laser model in the normal dispersion regime [8]. A discrete family of BSs has also been found in another laser model where the Kerr nonlinearity is disregarded and saturable instantaneous gain or loss has been assumed [14]. In the vectorial case different levels of BSs have been obtained as well [15].

Another kind of robust (stable) BS solution in the anomalous dispersion regime has been identified by Soto-Crespo and Akhmediev [16]. They used as a laser model a modified complex Ginzburg-Landau equation where both finite SA recovery effects and gain saturation were taken into account. This solution is essentially different to all other known BS solutions in lasers in that the phases at the peaks of the individual pulses evolve independently from each other and the pulse amplitudes are different. Thus, propagation mapped onto phase plane is characterized by circles rather than by fixed points [8,10] or limit cycles for slightly detuned parameters from the stationary case [17]. Because of this independent evolution of the phases in the following we will term these solutions as bound states with independently evolving phase.

In the present contribution we numerically investigate the BS formation in the lumped model of the all-normal dispersion fiber laser [3] with an instantaneous SA as, e.g., a nonlinear polarization rotator. Based on our previous work [8] we concentrate on the parameter domain where the region of stable BSs does not shrink to zero for vanishing SA recovery time. This may indicate that stable BS solutions may exist for instantaneously responding SAs. To date scalar BS solutions have not been reported for this case in the *normal dispersion domain* [4].

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In Sec. II of this paper we briefly present the equations used for modeling the fiber ring laser in the lumped approach where all individual elements in the cavity are modeled by separate equations and the pulse are sequentially propagated. We use an instantaneous SA which describes a wide range of passive mode-locking mechanisms as nonlinear polarization rotation, Kerr-lens mode locking or additive pulse mode locking.

In Sec. III we present the BS solutions in the normal dispersion domain. These BSs are robust two-pulse molecules with periodic or chaotic evolution of the peak amplitudes and the phase difference between the pulses. Simultaneously the pulse separation oscillates; however, the amplitude of these oscillations is very small. In the phase plane the BS trajectories are expressed as oscillations superimposed on the circle or chaotic circling.

Section IV is devoted to another type of robust BS solutions with larger peak separation not observed before numerically or experimentally in fiber lasers. These BSs are periodically oscillating solutions with a mutual dependence of separation and phase difference showing a characteristic behavior of the trajectories in the phase plane. Internal BS oscillations represent a periodic transition between two unstable stationary BS solutions with an invariant phase (second level in-phase and second level out-of-phase BSs, respectively). Therefore we term these solutions as flipping phase BSs.

Moreover, we found that gain saturation plays a crucial role for the existence of both types of robust BSs, namely, those with independently evolving and flipping phase. Thus, we derive the domains of existence of robust BSs as they depend on the saturation energy of the gain.

These results are generalized in Sec. V. For a fixed parameter set where both types of robust BSs coexist we consider a BS solution with even larger peak separation. The results obtained allow us to conclude that both types of robust BSs belong to a single discrete family of pulse pair solutions. It turns out that in this framework first-level BSs have an independently evolving phase difference because of the strong nonlinear interpulse influence interaction. Higher BS levels reveal an unlimited flipping phase evolution since the overlap between the pulses is sufficiently weak.

In order to keep close contact to realistic fiber laser systems and to facilitate the comparison with experimental results we use the characteristic experimental data in our numerical modeling.

II. MODEL

Our model is based on a simple scheme of a ring fiber laser which consists of a doped fiber, a SA, and an output coupler. This laser model allows including the dominant effects into the simulations and is still close to reality. In the framework of the lumped model the propagation through each element is treated separately.

The equation used to model the doped fiber is based on the nonlinear Schrödinger equation and is given in Refs. [4,8,18,19] (when the carrier optical frequency equals the dopant's atomic resonance frequency),

$$\frac{\partial U(z,t)}{\partial z} + \frac{i}{2}[\beta_2 + ig(z)T_1^2]\frac{\partial^2 U(z,t)}{\partial^2 t} = \frac{g(z)U(z,t)}{2} + i\gamma|U(z,t)|^2 U(z,t), \tag{1}$$

where $U(z,t)$ is the envelope of the pulse, z is the propagation coordinate, t is the retarded time, β_2 is the second-order dispersion (group velocity dispersion) coefficient, and γ represents the fiber nonlinearity. $g(z)$ is the saturable gain of the doped fiber and T_1 is the dipole relaxation time (inverse linewidth of the parabolic gain). Assuming that the conditions are close to stationarity, the gain can be approximated by [4,8]

$$g(z) = \frac{g_0}{1 + \int_{\text{pulses}} |U(z,t)|^2 dt / E_{\text{sat}}^{\text{gain}}}, \tag{2}$$

where g_0 is the small-signal gain, which characterizes the pumping level, and $E_{\text{sat}}^{\text{gain}}$ is the gain saturation energy.

To describe the SA we use the well-known transmission equation in the instantaneous response approximation (ideal SA) [8],

$$U_{\text{out}}(t) = U_{\text{in}}(t)\exp\left(-\frac{1}{2}\frac{\delta_0 \Delta z}{1 + |U_{\text{in}}(t)|^2/P_{\text{sat}}}\right), \tag{3}$$

where $\delta_0 \Delta z$ defines the modulation depth of the absorber and P_{sat} is the saturation power.

For the modeling we use the same parameters as in our previous work [8] which are closely related to experimental data [3,4]. The modulation depth of the absorber is 30% and the saturation power $P_{\text{sat}}=33.4$ W. The output loss amounts to 30%. For the doped fiber we assume $L_{\text{f}}=1$ m, $\beta_2=0.024$ ps^2 m^{-1}, $\gamma=0.005$ W^{-1} m^{-1}, $T_1=300$ fs, $g_0=0.75$ m^{-1}, and $E_{\text{sat}}^{\text{gain}}=1$ nJ. The pulse and the phase profiles were always recorded after the output coupler.

III. INDEPENDENTLY EVOLVING PHASE BOUND STATES

We start our considerations with the simplest case when the separation between the pulses is smallest and consequently the dynamics of their interaction is fastest. As initial conditions we use here two in-phase resting small amplitude pulses separated by a distance which is sufficiently large to form a first-level in-phase BS. Departing from the well-studied case $g_0=0.75$ m^{-1} we are decreasing the pumping level moving toward the region where the existence of stable BSs is expected due to the results presented in our previous paper [8] where the size of the calculated existence region of stable BSs did not shrink to zero for vanishing SA relaxation time (see Fig. 15 in [8]). In order to find the region of stable BSs we track changes in the interaction dynamics between two pulses for different magnitudes of the small-signal gain.

To visualize the evolution of both pulses we use a representation in the $[\rho \sin(\phi)$ and $\rho \cos(\phi)]$ phase plane where ρ and ϕ designate the peak separation and the phase difference between the pulses, respectively.

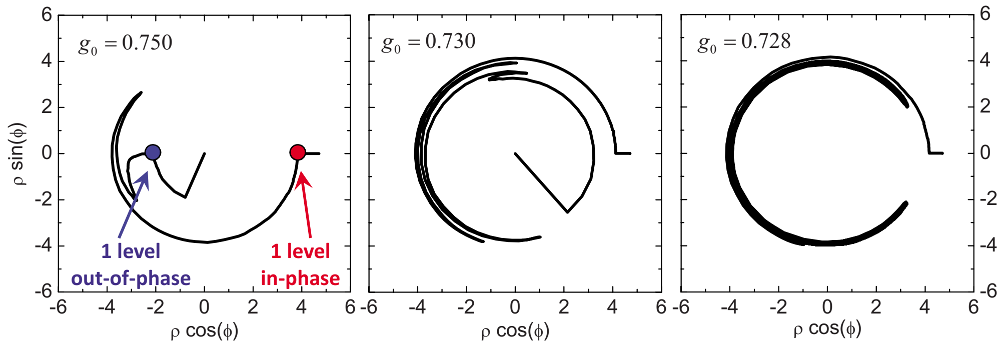

FIG. 1. (Color online) Evolution trajectory of two interacting pulses in the phase plane at different pumping levels (small-signal gain) showing the transition from an unstable BS to a robust (stable) one.

Figure 1 shows the gradual changes of the evolution trajectory in the phase plane for the transition from the region of unstable BS solutions (g_0=0.75 m^{-1}) to the region of robust (stable) BSs (g_0=0.728 m^{-1}). We can see that the small curl in the evolution trajectory at g_0=0.75 m^{-1} turns into strong oscillations for smaller pumping g_0=0.73 m^{-1} and then, at the critical value g_0=0.728 m^{-1}, degenerates into infinitely stable oscillations. In other words we obtain a robust (oscillating) BS solution. It takes about 3000 round-trips to reach the stationary BS evolution with an independently evolving phase. Its stability was checked by propagating it up to 100 000 round-trips. In dependence on the pumping level the phase difference can evolve periodically or chaotically. The chaotic behavior was verified following the route to chaos from the periodic solutions and will be discussed in detail elsewhere.

The example of a robust BS solution with periodic evolution of the phase difference at g_0=0.728 m^{-1} is presented in Fig. 2. In this case the phase difference oscillates with a relatively large amplitude of 0.8π and a period of 135 round-trips. Simultaneously the separation of the peaks also oscillates with the half of that period, however, with almost negligible oscillation amplitude.

Both peak amplitudes oscillate asymmetrically caused by the balance of gain and loss in the cavity due to gain saturation. Hence, the amplitude of the leading pulse is growing while the amplitude of the trailing pulse is decreasing keeping the intracavity power almost invariant. Moreover, one can see that there is an exact relation between the oscillations of the peak amplitude and those of the phase difference. Since the fiber is a Kerr medium the phase velocity of the pulses depends on the respective amplitude. Thus, if the amplitude of the leading pulse is bigger than the amplitude of the trailing pulse the difference of the phase velocities of leading and trailing pulses is negative, which corresponds to the decreasing part of the phase difference in the figure and vice versa. At the very moment when the amplitudes equal the phase difference keeps constant which is reflected by the turning points. Obviously, the oscillations of the amplitudes and of the phase difference are related to each other. Another slightly oscillating quantity in this system is the saturated gain [Eq. (2)] which is also shown in Fig. 2.

Next, we consider a robust BS with chaotically evolving phase for a reduced small-signal gain g_0=0.713 m^{-1} (see Fig. 3). This BS is a chaotically evolving solution with non-periodic internal dynamic. First of all the chaotic behavior is expressed in the evolution of the phase difference. It is evident that at the beginning the phase difference is decreasing during 400 round-trips and then after two oscillations starts growing (see Fig. 3). Because of such significant changes of the phase difference at almost constant peak separation the evolution trajectory in the phase plane looks like a circle. Because the phase dynamics and the respective peak amplitudes are connected they also evaluate chaotically (see Fig. 3). Hence, for the first 400 round-trips, when the phase difference decreases, the averaged peak amplitude of the leading pulse exceeds that of the trailing one. Also the gain is chaotically evolving. For some parameters a genuine circular trajectory appears, but usually it is just a chaotic or periodic evolution.

These results show that independently evolving phase BSs can be obtained in the normal dispersion domain. For anomalous dispersion similar robust BS solutions with circular trajectories have already been discovered by Soto-Crespo

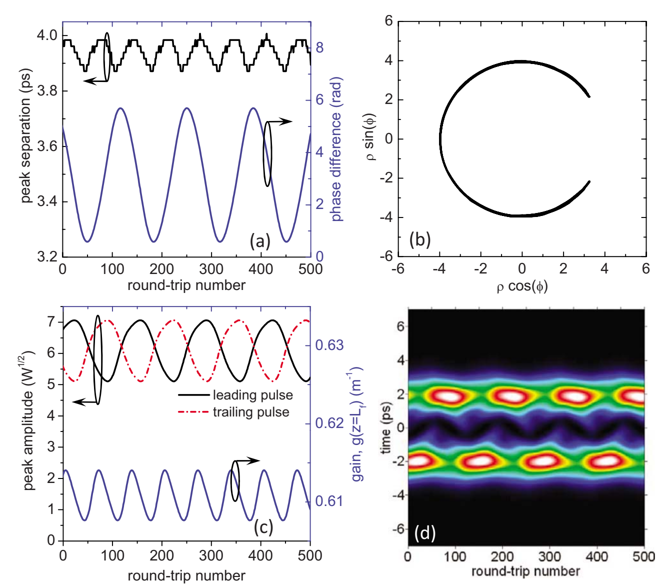

FIG. 2. (Color online) Evolution of the robust BS solution with an independently (periodically) evolving phase at g_0=0.728 m^{-1}: (a) peak separation and phase difference between pulses as a function of round-trips, (b) evolution trajectory in the phase plane, (c) peak amplitude of leading and trailing pulses and magnitude of the gain in dependence on the round-trips, and (d) BS evolution.

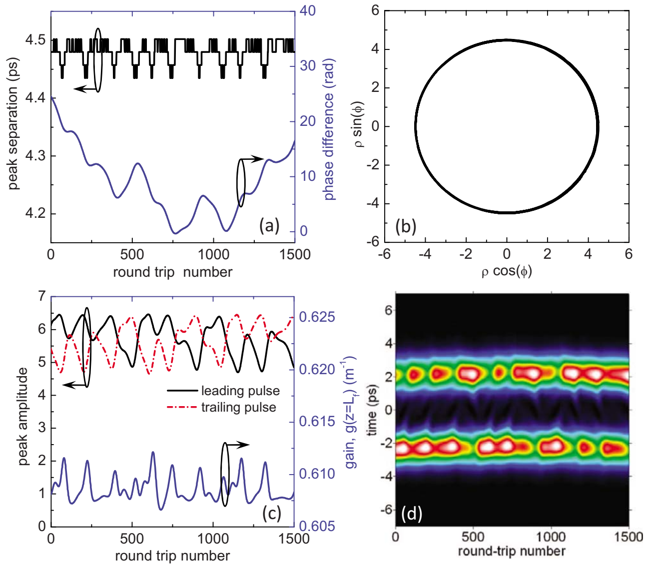

FIG. 3. (Color online) Evolution of a robust BS solution with independently (chaotically) evolving phase at g_0=0.713 m^{-1}: (a) peak separation and phase difference as a function of round-trips, (b) evolution trajectory in the phase plane, (c) peak amplitude of leading and trailing pulses and saturated magnitude of the gain in dependence on the round-trips, and (d) BS evolution.

and Akhmediev [16]. Nevertheless besides the opposite dispersion regime there is another significant difference between both models. In the previous paper a distributed model, i.e., a modified Ginzburg-Landau equation with temporal relaxation effects in an absorber included, has been used. In the present work we demonstrated that these robust BS solutions may exist without any noninstantaneous SA effects. It increases the variety of potential lasers with different mode-locking schemes where such solutions can be experimentally obtained.

From the results presented we can conclude that these BSs do not form due to a constructive interference between the pulses, as in out-of-phase or in-phase solutions [8]. In the present case the pulses are bound due to the balance of saturated gain and nonlinear loss at a certain pulse separation. In other words, the nature of these BSs is nonlinearly dissipative. Moreover, the gain saturation plays a crucial role for the existence of robust BSs with independently evolving phase. This saturation phenomenon manifests itself in a periodic or chaotic evolution of the small-signal gain (see Figs. 2 and 3).

In order to demonstrate the relevance of gain saturation we look at the existence region of the independently evolving phase BS solutions in dependence on $E_{\text{sat}}^{\text{gain}}$. First of all we consider the equation for the saturated gain [Eq. (2)] in two limiting cases. The first limit is $\int_{\text{pulses}}|U(z,t)|^2dt/E_{\text{sat}}^{\text{gain}} \gg 1$ where the unity in the denominator can be neglected and Eq. (2) can be written as

$$g(z) = \frac{g_0 E_{\text{sat}}^{\text{gain}}}{\int_{\text{pulses}} |U(z,t)|^2 dt}. \tag{4}$$

This expression clearly indicates that changes of the saturation energy lead to a trivial rescaling of g_0.

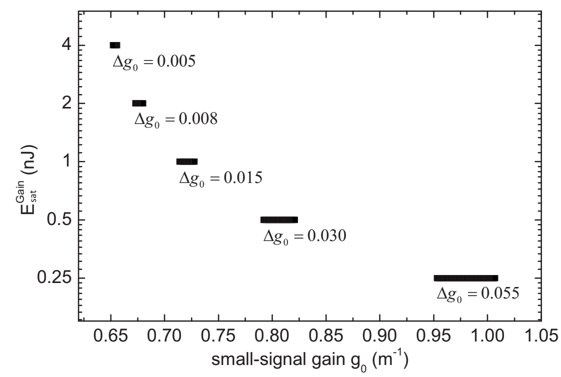

FIG. 4. Existence region of robust BS solutions with independently evolving phase for different $E_{\text{sat}}^{\text{gain}}$ in dependence on the small-signal gain. Δg_0 denotes the width of the regions. The BS energy at the fiber termination amounts to ~0.19 nJ which corresponds to the unsaturated limit of the gain medium in all considered cases.

In the second limit when $\int_{\text{pulses}}|U(z,t)|^2dt/E_{\text{sat}}^{\text{gain}} < 1$ Eq. (2) can be expanded in a Taylor series as

$$g(z) = g_0 + g_0\frac{\int_{\text{pulses}} |U(z,t)|^2 dt}{E_{\text{sat}}^{\text{gain}}} + g_0\left(\frac{\int_{\text{pulses}} |U(z,t)|^2 dt}{E_{\text{sat}}^{\text{gain}}}\right)^2 + \cdots. \tag{5}$$

In this case changes of the saturation energy $E_{\text{sat}}^{\text{gain}}$ do not lead to any rescaling and represent just the saturation properties of the gain medium while g_0 defines the small-signal gain in the system.

We consider the existence domain of independently evolving phase BSs in dependence on the gain saturation energy and a variable small-signal gain in the region where the condition $\int_{\text{pulses}}|U(z,t)|^2dt/E_{\text{sat}}^{\text{gain}} \gg 1$ is not satisfied in order to avoid a trivial rescaling of g_0. As initial conditions for the simulations we used two in-phase resting pulses separated by 4.7 ps being sufficient for the formation of a BS solution with independently evolving phases. The existence region of robust BSs is calculated for five values of $E_{\text{sat}}^{\text{gain}}$ which are successively changing by a factor of 2. The second variable parameter is the small-signal gain g_0 being changed by steps of 0.001 m^{-1}. For each point we consider an evolution of 40 000 round-trips. The calculated region of robust BS solutions with independently evolving phase is presented in Fig. 4.

From Fig. 4 we can see that for the largest saturation energy $E_{\text{sat}}^{\text{gain}}$=4 the existence region is smallest (Δg_0 =0.005 m^{-1}). Moreover, the value Δg_0 is growing with decreasing $E_{\text{sat}}^{\text{gain}}$. In other words, the closer the system is to the unsaturated limit [$\int_{\text{pulses}}|U(z,t)|^2dt/E_{\text{sat}}^{\text{gain}} \to 0$] the smaller the existence region of oscillating BSs is and vice versa, the better the saturated nature of the gain is expressed, the larger the existence region of these BSs is. Anyway, it should be noted that, if the limit $\int_{\text{pulses}}|U(z,t)|^2dt/E_{\text{sat}}^{\text{gain}} \gg 1$ is reached, a further decrease of $E_{\text{sat}}^{\text{gain}}$ leads to a trivial rescaling of the small-signal gain parameter. Obviously, the saturated nature

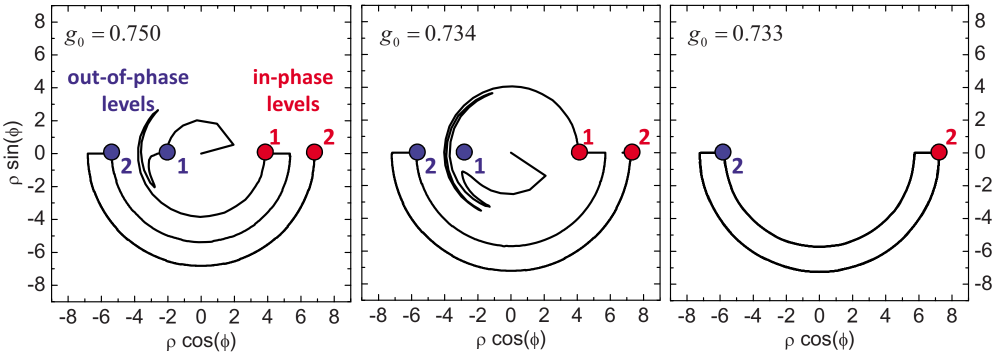

FIG. 5. (Color online) Evolution trajectory of two interacting pulses in the phase plane for different small-signal gain; transition from the well-studied region of unstable BS solutions to a robust (stable) one.

of the gain plays a key role for the existence of the robust independently evolving phase BSs.

Eventually, we demonstrate that $E_{\rm sat}^{\rm gain}$ plays a crucial role for the solution stability and must not be used as a scaling factor in numerical simulations.

IV. FLIPPING PHASE BOUND STATES

In a next step we increase the initial separation between pulses up to 6.7 ps in order to investigate the existence of stable higher level BSs. The laser parameters are defined in Sec. II. Similarly to Sec. III we are searching for stable BS solutions by gradually decreasing the small-signal gain. To track changes of the evolution dynamics we again use the phase plane. Starting at g_0=0.75 m^{-1} we obtain the well-studied trajectory of the two interacting pulses [8] (see Fig. 5). The pulses approach each other upon propagation and consecutively form a second level in-phase BS, a second level out-of-phase BS, a first-level in-phase BS, and a first-level out-of-phase BS solution and finally fuse. If the linear gain is reduced, however, still larger than the critical value of g_0=0.733 m^{-1} no significant changes occur in the evolution trajectories in the phase plane except for an increasing amplitude of the oscillations during the transition from the first level of in-phase BS to the first level of out-of-phase BS, which has been already discussed in Sec. III.

When the critical value is reached the interaction between two pulses suddenly changes from gradual fusion to infinitely stable oscillations in consecutively passing through the unstable stationary BS solutions; thus the pulses form an oscillating robust (stable) BS solution with a larger average separation. From Fig. 5 it is obvious that the oscillation of the stable BS is a periodic flipping between two unstable stationary invariant phase BS solutions (second level in-phase BS and second level out-of-phase BS). Therefore we call these solutions flipping phase BSs. Unfortunately, the current model does not permit to calculate the unstable stationary solutions at g_0=0.733 m^{-1}, but they can be clearly identified via the transition from the region of the unstable BS solution (g_0=0.734 m^{-1}) to the robust one (g_0=0.733 m^{-1}). These robust BSs can exhibit two kinds of interaction dynamics between the pulses in dependence on the small-signal gain. Below we consider two examples of robust BSs with flipping phase, namely, for g_0=0.73 m^{-1} and g_0=0.71 m^{-1} which represent two types of possible dynamics. In both cases the evolution is shown after 500 000 round-trips in order to reach stationary conditions.

Figure 6 shows the evolution of the flipping phase BS solution at g_0=0.73 m^{-1}. This is a periodically oscillating BS with a period of ~126 000 round-trips. Peak separation and phase difference are oscillating. In contrast to the independently evolving phase BSs considered in Sec. III, there is a definite dependence between ϕ and ρ, which is proven by the evolution trajectory [see Fig. 6(b)]. The evolution trajectory consists of two half circles where each of them corresponds to a specific unstable invariant phase BS (second level in-phase BS and second level out-of-phase BS). These robust BSs with flipping phase oscillate between two unstable two-pulse solutions. The flipping nature can be explicitly seen in looking at the phase evolution, changing abruptly from zero to π and vice versa.

In the transition regions the peak amplitudes of the leading and the trailing pulses differ considerably. The saturated

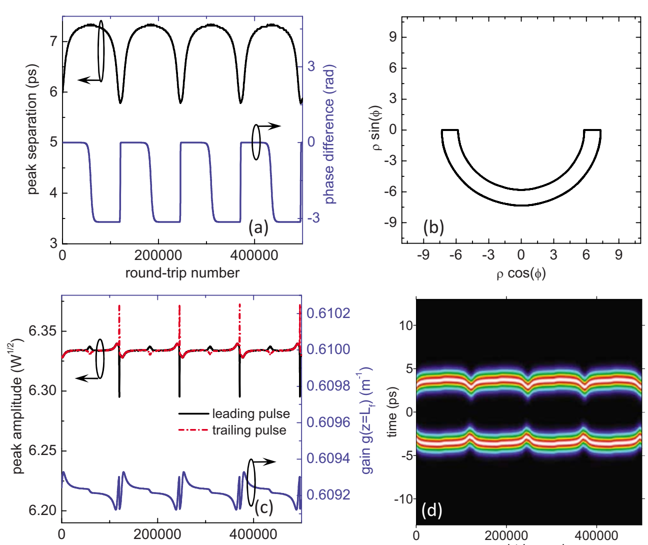

FIG. 6. (Color online) Evolution of the robust BS solution with flipping phase at g_0=0.73 m^{-1}: (a) peak separation and phase difference between pulses as a function of round-trips, (b) evolution trajectory in phase plane, (c) peak amplitudes of leading and trailing pulses and saturated magnitude of the gain in dependence on the round-trip number, and (d) BS evolution.

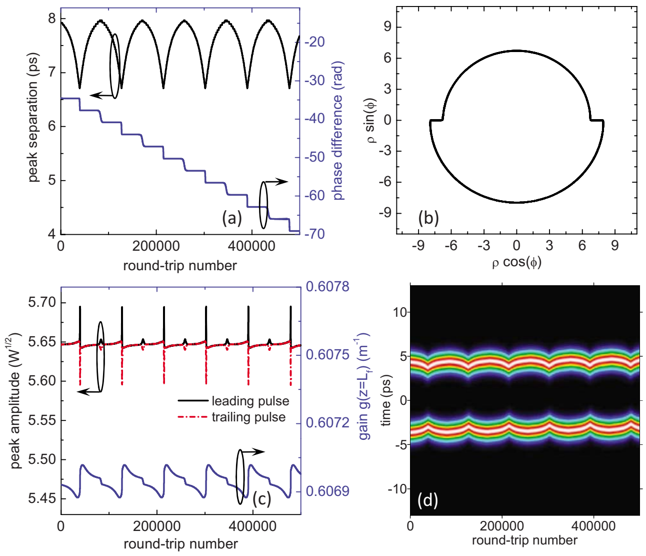

FIG. 7. (Color online) Evolution of the robust BS solution with flipping phase at g_0=0.71 m^{-1} with different internal dynamics: (a) peak separation and phase difference as a function of round-trips, (b) evolution trajectory in phase plane, (c) peak amplitudes of leading and trailing pulses and saturated magnitude of the gain in dependence on the round-trips, and (d) BS evolution.

value of the gain also oscillates with the same period.

A different evolution dynamics of the robust BS with flipping phase occurs for a further reduced linear gain. For the same initial conditions and g_0=0.71 m^{-1} the stationary dynamics of the oscillating BS is presented in Fig. 7. Compared to the previous case there are two differences. First, the oscillation period is shorter and equals about 88 000 round-trips. Second, the phase difference between the pulses is infinitely increasing, which leads to a clockwise circulation in the phase plane. Nevertheless the evolution trajectory in the phase plane also consists of two half circles which correspond to the two different unstable invariant phase BSs. The phase difference between the pulses is endlessly flipping from an in-phase to an out-of-phase state.

Evidently, these BSs with flipping phase have a fundamental difference to the BSs with independently evolving phase. This is mainly caused by the different interaction between the pulses. For the independently evolving phase BSs there is no definite dependence between peak separation ρ and phase difference ϕ. The phase difference evolves periodically or chaotically whereas the separation remains almost constant. On the contrary, the flipping phase BSs are characterized by a mutually dependent oscillation of separation and phase difference. This dependence is a consequence of interference phenomena between the pulses [8] and can be clearly recognized in looking at the evolution trajectory in the phase plane.

Moreover, the existence of flipping phase BSs also depends on nonlinear dissipative effects where gain saturation plays a crucial role. To demonstrate this we consider the existence region of robust BSs with flipping phase in dependency on the saturation energy. In order to avoid trivial renormalization we change the saturation energy in the region where the condition $\int_{\text{pulses}}|U(z,t)|^2 dt/E_{\text{sat}}^{\text{gain}} \gg 1$ is not satisfied. The existence region of the BSs is calculated for five different values of $E_{\text{sat}}^{\text{gain}}$ and a small-signal gain which is varied by steps of 0.001 m^{-1}. For each parameter set propagation over 400 000 round-trips has been considered.

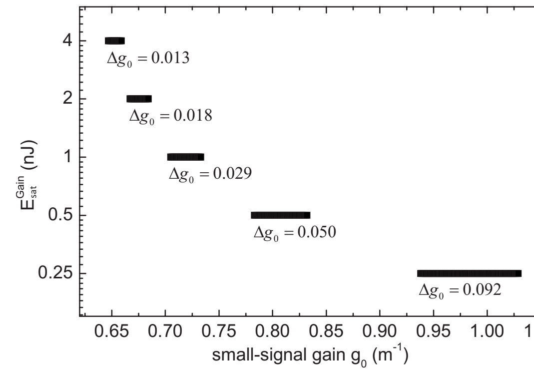

FIG. 8. Existence region of robust BS solutions with flipping phase for different $E_{\text{sat}}^{\text{gain}}$ in dependence on the small-signal gain. The width of the region is denoted by Δg_0. The BS energy at the fiber termination amounts to ~0.2 nJ corresponding to the unsaturated limit of the gain medium in all considered cases.

The calculated map of robust flipping phase BS solutions is shown in Fig. 8. It is evident that the width of the existence region strongly depends on the gain saturation characteristic. For the largest saturation energy the existence region is smallest or in other words, the less the gain saturates the smaller the existence region of the robust BS and vice versa.

Hence gain saturation characteristic plays a crucial role for the existence of both independent evolving phase and flipping phase BS solutions. Therefore these solutions behave similarly as can be seen in Figs. 4 and 7. On the other side, they have principal differences in the interaction dynamics of pulses. In order to build a more general picture and to identify the physical difference between independently evolving phase and switching phase BS solutions we consider the BS formation with even higher peak separation.

V. HIGHER LEVEL BOUND STATE AND GENERALIZATION

To understand the nature of both of these robust BS solutions, namely, the independently evolving phase and flipping phase BSs, we numerically consider the interaction dynamics of the two pulses in dependence on the initial peak separation of up to 8.6 ps. All laser parameters are defined in Sec. II and g_0=0.725 m^{-1} for the small-signal gain has been chosen, where both types of robust BSs coexist. As initial conditions we use the in-phase composition of two resting small amplitude pulses.

The BS formation and their stationary evolution in dependence on the initial peak separation are shown in Fig. 9. We obtain three levels of robust BSs. In agreement with our previous work [8], the formation and evolution dynamics strongly depend on the separation between pulses. For the smallest separation distance the pulses form a robust BS after 3000 round-trips. At the same time for the biggest separation distance the stationary dynamics of the robust BS is not yet

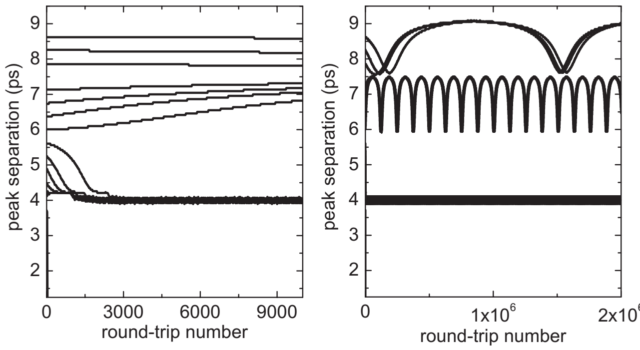

FIG. 9. Evolution of the peak separation for initially in-phase pulses with different initial separations at g_0=0.725 m^{-1}. Formation and stationary evolution of the three robust BS levels are displayed. The first level is a BS with independently evolving phase; second and third levels are BSs with flipping phase.

reached even after 2 000 000 round-trips. The period of BS oscillations also depends on the peak separation. Specifically, for the second level the period is ~100 000 round-trips and for the third one this figure is ~12 times larger while the amplitudes of the oscillations in both cases are the same. As already known this dependence is caused by the different overlaps between the pulses in a BS.

In the phase plane the evolution trajectories of the second and third level solutions are similar and correspond to the flipping phase BSs (see Fig. 10). The first-level robust BS solution is peculiar due to its independently evolving phase. Thus we may draw the conclusion that the BS solutions exhibit an independently evolving phase because of the strong overlap of the pulses and consequently the nonlinear interaction is significant. But if the distance between the pulses is big enough the phase difference will forever evolve from in-phase to out-of-phase state.

VI. CONCLUSION

In conclusion we numerically demonstrated the existence of a discrete family of robust BS solutions in a fiber laser with an instantaneous saturable absorber in the normal dispersion regime. First level robust BS solutions are characterized by an independently evolving phase caused by the strong nonlinear interaction between pulses. In dependence on the small-signal gain the phase difference can evolve periodically or chaotically at nearly constant peak separation. This results in slight oscillations on a circle or in a chaotic circling in the phase plane.

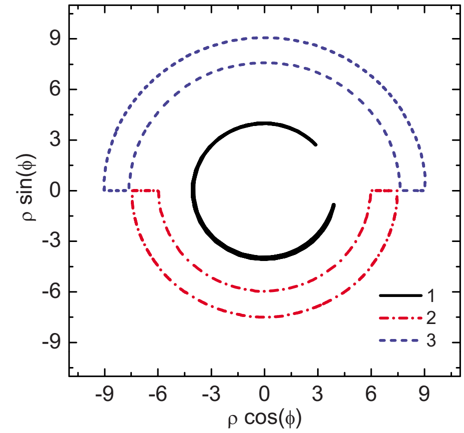

FIG. 10. (Color online) Evolution trajectories in the phase plane for the three robust BS levels at g_0=0.725 m^{-1}. The robust two-pulse solution with (1) independently evolving phase and (2) and (3) flipping phase.

The next level BS is a robust solution with larger peak separation which endlessly oscillates between two unstable invariant phase BSs. At the same time the phase difference is infinitely switching from in-phase to out-of-phase states. Moreover, in the second and third level BSs there is a dependence between separation and phase difference which is a consequence of interference phenomena between the pulses [8].

The existence domain of these robust BSs has also been investigated. It was verified that in both cases the gain saturation characteristic plays a crucial role for their existence. Namely, in the unsaturated limit the width of the existing region with regard to the small-signal gain approaches zero, whereas in the strongly saturated limit it exists for virtually any value of the small-signal gain.

The current investigations extend the number of potential laser systems where discrete families of robust (stable) BS solutions can be obtained. And we demonstrated that the internal dynamics of the different level BSs can have fundamental differences due to the significant nonlinear interaction of pulses.

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