Dynamics of a periodically modulated optical parametric oscillator near lasing threshold

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We present analytical investigation of the nonlinear dynamics of a degenerate optical parametric oscillator with periodic modulation of transverse refraction index. By a proper choice of the injected external field that must compensate for losses and match with the modulation period, nonlinear optical cavities can exhibit dissipative Bloch waves which are attracting solutions of nonequilibrium system. This allows us to propose method of experimental visualization of the band structure of the cavity medium. Using multiple-scale expansion near the leasing threshold, we obtain the equation of evolution of the small-amplitude envelop of the signal field which appears strongly affected by the periodic modulation of the refractive index. We discuss the physical meaning of the obtained equation.

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I. INTRODUCTION

Spatiotemporal complexity in spatially extended systems is nowadays a growing research activity in fields as different as physics [1], hydrodynamics [2], chemistry [3], and biology [4]. Nonlinear optics, in particular, represents a fruitful area of the research activity [5]. More importantly, among the solutions emerging from this complexity, dissipative structures (DSs) including patterned or localized states are promising candidate for future applications in information technology [6] and image processing [7,8].

Nonlinear optical cavities are highly suitable devices since they constitute nonequilibrium dissipative systems where spatial DSs spontaneously bifurcate from ground states (trivial or inhomogeneous stationary solutions) via many types of instabilities (Turing, Hopf, or Benjamin-Feir instabilities) leading to a variety of pattern formation [9,10]. This demonstrates the importance of pattern forming instabilities and their interactions when studying the complex spatiotemporal dynamics or the formation of optical DSs in spatially extended systems. This subject has been abundantly discussed in a number of overviews [11], in particular in the review paper [12] that reports very recent progress on transverse instabilities in nonlinear optics.

From experimental point of view, the observation of DSs was reported in a variety of nonlinear optical system devices [13]. However, the link between theoretical predictions and the exact experimental conditions for the occurrence of DS is still a subject of intense research activity [14]. An overview of the state of the art in the formation and the characterization of DSs in various fields of natural science such as biology, chemistry, plant ecology, mathematics, optics, and laser physics can be found in the very recent Focus Issue on DSs in extended systems [15].

Among the possible devices, optical parametric oscillators (OPOs) have recently appeared as one of the most promising system, not only for the richness in their nonlinear dynamics but also for their potential applications [16], including low noise measurements and detection [17] and information technologies [18]. Optical parametric oscillators are formed by a cavity where we have inserted (see Fig. 1) a nonlinear quadratic crystal. The high sensitivity of these crystals to quadratic nonlinearity ($\chi^{(2)}$) constitutes the basis of the parametric amplification leading to a rich variety of temporal and/or spatial dynamics. Optical solitons are the most studied owing to their ability to confine light power which makes them promising candidates for future technological applications. In the excellent review paper [19], an interesting and well

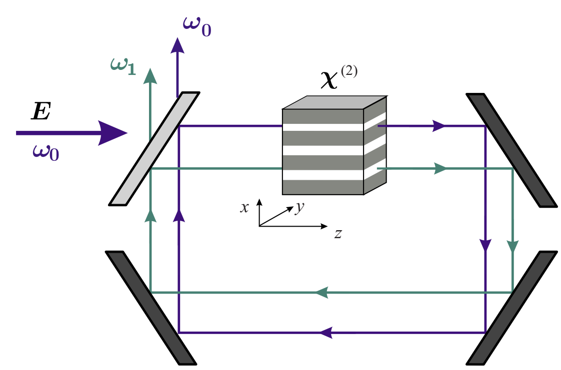

FIG. 1. (Color online) Schematic sketch of the experimental setup. Quadratic nonlinear crystal is inserted inside the cavity formed of four mirrors. The three black mirrors has a perfect reflectivity ($R=1$ and $T=0$). The two resonant pump and signal fields at frequencies ω_0 and ω_1, respectively, are transmitted by the first (gray) mirror. E is the incident optical pump.

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documented study of the main investigations and realizations in this subject is reported. An inspired discussion of open problems in this research activity is also addressed by the authors in their conclusion including the crucial role that photonic crystals may bring to the development of these materials. In fact, nowadays, the recent and considerable progress in photonic crystals opened the possibility to control the formation and the dynamics of DSs in optical systems via periodic modulations in the parameters [20].

The purpose of this paper is to report on the impact of transverse periodic modulations of the refractive index in the nonlinear dynamics of a degenerate OPO. In contrast with quadratic gap solitons in a periodic device with a much bigger spatial extension than the period of the structure, here we study the dynamics of DS having a spatial period of the order of the medium. This constitutes the main difference with the previous studies. In the previous work [21], we described a method, based on Bloch waves approach, of constructing spatially localized solutions in optical parametric amplifiers (OPAs). This approach has also been revealed fruitful for identifying the periodic patterns in OPOs under periodic modulations. The latter are nonlinear solutions that require specific conditions on the incident pump to exist. Here, we first show the role of the signal detuning in the emergence, the inhibition, and the coexistence of dissipative Bloch waves. A comparison to the case when periodic modulations are absent is also given to highlight the crucial role of spatial modulations in the selection of Bloch waves at the linear stage. Second, we extend the analytical analysis to nonlinear regimes so that our study is mainly focused on the description of the nonlinear dynamics of DSs. More precisely, we report on an extension of order parametric description to take into account the presence of spatial periodic modulations of the refractive index. In contrast to previous studies, we discuss an unforeseen effect of periodic modulations in the dynamics of degenerate OPOs which does not rely on linear approach. Here, we show how periodic modulations, that is a linear phenomenon, actually modify the intrinsic nonlinearity of the system and hence the main characteristics of DSs. Indeed, our analytical investigations reveal that the cubic term (saturation term), in the amplitude equation describing the near threshold behavior of the OPO, shows an original dependence on both the modulation and the dissipation. This demonstrates the utmost importance of spatial periodic modulations in the nonlinear properties of DSs emitted by the OPOs.

The most striking feature revealed by our investigations is the control of the emerging dissipative Bloch waves via the signal detuning which is directly related to the energy characterizing the band-gap structures. More precisely, variations in the signal detuning (which are easily accessible experimentally) *make it possible selective excitation of Bloch waves belonging to different bands.* Moreover, in sharp contrast to what happens in a homogeneous medium, in our case by increasing the signal detuning value, windows of stability can be opened: they correspond to band gaps when no Bloch waves exist. This offers a possibility to "visualize" the band-gap structure by using an optical cavity. Above threshold, an order parametric description of the nonlinear dynamics of dissipative Bloch waves is obtained. It shows that saturation effects (nonlinearities) limit the linear amplifications leading to the waves with a fixed amplitude. As a consequence, a dissipative Bloch wave appears as an attracting solution (attractor) that is asymptotically reached from a large domain of initial conditions, in contrast with conservative Bloch waves that depend on a particular initial condition (energy conservation).

The paper is organized as follows. In Sec. II, we recall the degenerate OPO governing equations, including spatial periodic modulations and diffraction. Linear analysis of these equations is performed as an eigenvalue problem to obtain the pump threshold for the existence of Bloch waves eigenfunctions and their properties. An approach, based on multiple-scale expansion, allows us to investigate the nonlinear evolution of DSs in the band-gap region of the degenerate OPO. The underlying role of spatial modulations on the formation and characteristics of DSs is discussed in Sec. III together with concluding remarks.

II. MODEL

We consider a degenerate OPO that results from the nonlinear interaction between two resonant optical fields. This process takes place in an optical cavity filled by a $\chi^{(2)}$ (quadratic) material. An external coherent beam E at the frequency ω_0 is coupled into the cavity where it undergoes down-conversion process: one photon with frequency ω_0 is absorbed and two photons with frequencies $\omega_1=\omega_0/2$ are emitted (see Fig. 1). In the mean-field approximation [22], the space-time evolution of the intracavity fields in presence of a periodically modulated refractive index and diffraction is described by the following partial differential equations [21]:

$$i\partial_\tau A_0 = (\Delta_0 - i\gamma)A_0 + \frac{1}{2}\mathcal{L}_0 A_0 - iA_1^2 + iE(x), \qquad (1a)$$

$$i\partial_\tau A_1 = (\Delta_1 - i)A_1 + \mathcal{L}_1 A_1 + 2iA_0\bar{A}_1, \qquad (1b)$$

where A_0 and A_1 are the normalized slowly varying envelops of the pump and signal fields, respectively. The parameters $\Delta_{0,1}$ and γ are the detunings and the ratio of the cavity decay rate of the pump to the one of the signal, respectively. E is the normalized incident pump and an overbar stands for the complex conjugation. The linear operators $\mathcal{L}_{0,1}$ are defined as follows:

$$\mathcal{L}_j \equiv -\frac{d^2}{dx^2} + \frac{1}{4^j}V_j(x), \quad j=0,1, \qquad (2)$$

where $V_j(x)$ that account for spatial periodic modulations in the refractive indices are real periodic functions, which are considered having the same period as the pump, A_0, and signal, A_1, fields [21]. We thus assume that $V_j(x+\pi)=V_j(x)$ (the period can be made π by proper renormalization of the coordinates).

We will consider a finite system, having the length 2ℓ, such that $x\in[-\ell,\ell]$, which is however much larger than the lattice period π: $2\ell=N\pi$, where $N\gg 1$ is a number of periods (as a matter of fact, we will be interested in the limit $\ell\to\infty$).

Then the most convenient, and also experimentally feasible, boundary conditions are the cyclic ones, i.e., we impose $A_{0,1}(x)=A_{0,1}(x+2\ell)$.

We also consider the eigenvalue problems

$$\mathcal{L}_j\varphi_{\nu,q}^{(j)}(x)=\mathcal{E}_{\nu,q}^{(j)}\varphi_{\nu,q}^{(j)}(x). \tag{3}$$

Here, $\varphi_{\nu,q}^{(j)}(x)$ are the Bloch waves, with the index ν standing for the number of the allowed band and q designating the wave vector in the first Brillouin zone (BZ), $q\in[-1,1]$. The Bloch functions are considered to be normalized

$$\int_{-\ell}^{\ell}\bar{\varphi}_{\nu,q}^{(j)}(x)\varphi_{\nu',q'}^{(j)}(x)dx=\delta_{\nu\nu'}\delta_{qq'}. \tag{4}$$

A. Linearized problem and the lasing threshold

We are interested in the dynamics near the lasing threshold, which in the case at hand is intimately related to the shape of the periodic potential [24]. The simplest situation arises in the case when the threshold pump field is a constant (similar to the case of the homogeneous medium [23]), which we will designate as $A_0^{(st)}$. This however can be achieved only subject to proper choice of the external field $E(x)$, as this was suggested in [21]. More specifically, the constant pump signal cannot be arbitrary, but must be chosen to compensate the losses. This occurs when

$$E(x)=A_0^{(st)}[\gamma+i(\Delta_0+V_0(x)/2)]. \tag{5}$$

In order to obtain the specific value of $A_0^{(st)}$, we follow the standard procedure and consider the stability of the system (II) linearized about the threshold value. It is convenient to introduce the operator L,

$$L=L_0+L_1, \tag{6}$$

with

$$L_0=\mathrm{diag}\left(\frac{1}{2}\mathcal{L}_0+\Delta_0-i\gamma,0,0\right), \tag{7}$$

$$L_1=\begin{pmatrix}0 & 0 & 0\\ 0 & \mathcal{L}_1+\Delta_1-i & 2iA_0^{(st)}\\ 0 & 2iA_0^{(st)} & -\mathcal{L}_1-\Delta_1-i\end{pmatrix}, \tag{8}$$

and the Hill operators $\mathcal{L}_{0,1}$ defined in Eq. (2).

As far as L has the block structure, the operators L_0 and L_1 act in orthogonal spaces which we denote as $S_0\subset S$ and $S_1\subset S$: $S_0\perp S_1$, where $S=S_0\oplus S_1$ is the domain space of the operator L. Hence, looking for the envelops of the pump and the signal fields in the form $A_0\sim\exp(-i\lambda t)$ and $A_1\sim\exp(-i\mu t)$, we can consider two independent eigenvalue problems for the operators $L_{0,1}$,

$$L_0\Psi=\lambda\Psi,\quad \Psi\in S_0, \tag{9a}$$

$$L_1\Phi=\mu\Phi,\quad \Phi\in S_1, \tag{9b}$$

where Ψ and Φ are orthogonal and normalized eigenfunctions of the operators L_0 and L_1, respectively, and λ and μ are the corresponding eigenvalues. The scalar product of two vectors in S, $F=(f_1,f_2,f_3)^T\varphi(x)$ and $G=(g_1,g_2,g_3)^T\psi(x)$, is defined as

$$(F,G)=(\bar{f}_1g_1+\bar{f}_2g_2+\bar{f}_3g_3)\int_{-\ell/2}^{\ell/2}\bar{\varphi}(x)\psi(x)dx. \tag{10}$$

The eigenvalue problem (9a) allows for the immediate solution in the form

$$\Psi_{\mathbf{q}}=(\varphi_{\mathbf{q}}^{(0)},0,0)^T,\quad \lambda_{\mathbf{q}}=\frac{1}{2}\mathcal{E}_{\mathbf{q}}^{(0)}+\Delta_0-i\gamma, \tag{11}$$

where $\varphi_{\mathbf{q}}^{(0)}$ represent Bloch waves and $\mathcal{E}_{\mathbf{q}}^{(0)}$ is an eigenvalue of the operator $\mathcal{L}_0$ [see Eq. (3)] and we have introduced the notation $\pm\mathbf{q}\equiv(\nu,\pm q)$.

We look for the solution of the eigenvalue problem (9b) in the form $\Phi_{\mathbf{q}}=(0,a,b)^T\varphi_{\mathbf{q}}^{(1)}$, where a and b are complex coefficients. Substituting this ansatz in Eq. (9b) and using Eq. (3), the eigenvalue problem (9b) can be rewritten in the form

$$[(\mathcal{E}_{\mathbf{q}}^{(1)}+\Delta_1)-(\mu+i)]a+2iA_0^{(st)}b=0,$$

$$2iA_0^{(st)}a-[(\mathcal{E}_{\mathbf{q}}^{(1)}+\Delta_1)+(\mu+i)]b=0. \tag{12}$$

For this system to have a nontrivial solution, it is required that

$$(\mathcal{E}_{\mathbf{q}}^{(1)}+\Delta_1)^2=\mu^2+2i\mu+4(A_0^{(st)})^2-1, \tag{13}$$

which readily gives the two branches of the spectrum μ,

$$\mu=\mu_{\mathbf{q}}^{(\pm)}=-i\pm\kappa_{\mathbf{q}}\sqrt{|(\mathcal{E}_{\mathbf{q}}^{(1)}+\Delta_1)^2-4(A_0^{(st)})^2|}. \tag{14}$$

Here, we use the notation

$$\kappa_{\mathbf{q}}=\begin{cases} i & \text{for } |\mathcal{E}_{\mathbf{q}}^{(1)}+\Delta_1|\le 2A_0^{(st)}\\ 1 & \text{for } |\mathcal{E}_{\mathbf{q}}^{(1)}+\Delta_1|> 2A_0^{(st)}.\end{cases} \tag{15}$$

Since $\mathcal{E}_{\mathbf{q}}^{(1)}$ is an even function of the wave vector, we have that $\mu_{\mathbf{q}}^{(\pm)}=\mu_{-\mathbf{q}}^{(\pm)}$.

It is relevant to notice that possibility of rather complete investigation of non-Hermitian eigenvalue problem (9b) is a nontrivial fact, although having known analogs. In particular, it has been shown in [25] that the eigenvalue problem emerging in the stability analysis of the parametrically driven solitons can be reduced to a Hermitian one using a nonlinear eigenvalue transformation.

In our notations, the neutral stability curve is determined by the condition $\mathrm{Im}(\mu)=0$, which can be satisfied only for the branch $\mu_{\mathbf{q}}^{(+)}$ provided that $\kappa_{\mathbf{q}}=i$. Thus, the lasing threshold is determined from the condition

$$1-\sqrt{4(A_0^{(neutr)})^2-(\mathcal{E}_{\mathbf{q}}^{(1)}+\Delta_1)^2}=0,$$

which must be minimized with respect to $\mathbf{q}$. This immediately gives the lasing threshold

$$A_0^{(th)}=\frac{1}{2}, \tag{16}$$

which is achieved for the wave vectors $\pm q_0$ corresponding to the band ν_0 and satisfy the condition

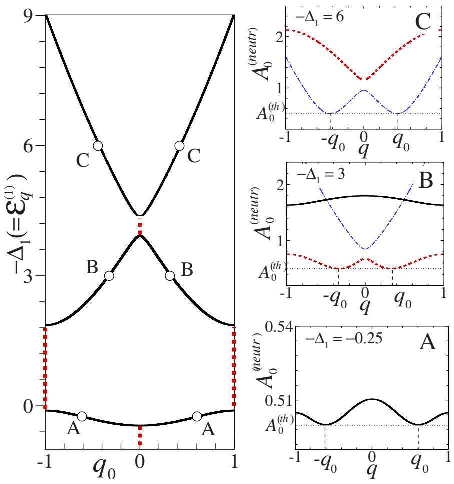

FIG. 2. (Color online) In the left panel, Δ_1 vs q_0 is shown in the reduced zone representation for the example $V_1(x)=-8\cos(2x)$. The stability windows corresponding to gaps are shown by thick dashed (red) lines. In the right panels, we show the neutral stability curves obtained in the (a) first, (b) second, and (c) third lowest bands of the dependence $-\Delta_1(q_0)$. The thick (black), dashed (red), and dashed-dotted (blue) lines correspond to the first, second, and third bands. The horizontal dotted lines indicate $A_0^{(th)}=1/2$.

$$
(\mathcal{E}^{(1)}_{\nu_0,q_0}+\Delta_1)\left.\frac{\partial\mathcal{E}^{(1)}_{\nu_0,q}}{\partial q}\right|_{q=q_0}=0. \tag{17}
$$

(In that case $\mu_{\mathbf{q}}^{(+)}=0$ and we deal with the mode $\Phi_{\mathbf{q}}^{(+)}$.)

In the case of the periodic medium, the neutral stability curve significantly differs from its homogeneous counterpart (it can be found, say, in [23]). First of all, taking into account that $\mathcal{E}_{\mathbf{q}}^{(1)}>-\max_x|V_1|$, one immediately concludes that solutions of Eq. (17) exist only if $\Delta_1>-\max_x|V_1|$. Moreover, no solution of Eq. (17) exists for a set of intervals, where $-\Delta_1$ matches any of the gaps of the spectrum $\mathcal{E}_{\mathbf{q}}^{(1)}$. Besides that, due to the band structure of the spectrum of Eq. (3), the dependence of Δ_1 on q_0 is nonmonotonic. This is illustrated in Fig. 2 where we present the dependence $-\Delta_1(q_0)$ as well as the neutral stability curves for the cases when condition (17) is satisfied inside the first (A), second (B), and third (C) bands of the spectrum of the operator $\mathcal{L}_1$ using the example of the potential $V_1(x)=V_1\cos(2x)$, where V_1 is a constant amplitude.

Since we are interested, in what follows, in the dynamics near threshold, it will be assumed that (16) is hold. Then a solution of Eq. (12) can be chosen in the form

$$
a_{\mathbf{q}}^{(\pm)}=e^{-i\pi/4\pm\kappa_{\mathbf{q}}\theta_{\mathbf{q}}},\quad b_{\mathbf{q}}^{(\pm)}=e^{i\pi/4\mp\kappa_{\mathbf{q}}\theta_{\mathbf{q}}}, \tag{18}
$$

where we have introduced the quantity $\theta_{\mathbf{q}}$ through the relation

$$
\cosh(2\kappa_{\mathbf{q}}\theta_{\mathbf{q}})=\mathcal{E}_{\mathbf{q}}^{(1)}+\Delta_1. \tag{19}
$$

Taking into account the definition (15), one ensures that $\theta_{\mathbf{q}}$ is real.

Assuring the correspondence of the lower and upper indexes in all formulas below, we finally arrive at the following form of the eigenfunctions of the operator L_1, $L_1\Phi_{\mathbf{q}}^{(\pm)}=\mu_{\mathbf{q}}^{(\pm)}\Phi_{\mathbf{q}}^{(\pm)}$,

$$
\Phi_{\mathbf{q}}^{(\pm)}=(0,a_{\mathbf{q}}^{(\pm)},b_{\mathbf{q}}^{(\pm)})^T\varphi_{\mathbf{q}}^{(1)}. \tag{20}
$$

Next we introduce the adjoint operators $L^\dagger=L_0^\dagger+L_1^\dagger$, where

$$
L_0^\dagger=\mathrm{diag}\left(\frac{1}{2}\mathcal{L}_0+\Delta_0+i\gamma,0,0\right), \tag{21}
$$

$$
L_1^\dagger=\begin{pmatrix}0 & 0 & 0\\ 0 & \mathcal{L}_1+\Delta_1+i & -2iA_0^{(st)}\\ 0 & -2iA_0^{(st)} & -\mathcal{L}_1-\Delta_1+i\end{pmatrix}, \tag{22}
$$

and consider the respective eigenvalue problems: $L_1^\dagger\tilde{\Phi}_{\mathbf{q}}^{(\pm)}=\bar{\mu}_{\mathbf{q}}^{(\pm)}\tilde{\Phi}_{\mathbf{q}}^{(\pm)}$. Subject to the condition (16), the eigenfunctions can be represented in the form

$$
\tilde{\Phi}_{\mathbf{q}}^{(\pm)}=(0,\tilde{a}_{\mathbf{q}}^{(\pm)},\tilde{b}_{\mathbf{q}}^{(\pm)})^T\varphi_{\mathbf{q}}^{(1)} \tag{23}
$$

with

$$
\tilde{a}_{\mathbf{q}}^{(\pm)}=\pm\frac{1}{2\sinh(2\bar{\kappa}_{\mathbf{q}}\theta_{\mathbf{q}})}e^{-i\pi/4\pm\bar{\kappa}_{\mathbf{q}}\theta_{\mathbf{q}}},
$$

$$
\tilde{b}_{\mathbf{q}}^{(\pm)}=\mp\frac{1}{2\sinh(2\bar{\kappa}_{\mathbf{q}}\theta_{\mathbf{q}})}e^{i\pi/4\mp\bar{\kappa}_{\mathbf{q}}\theta_{\mathbf{q}}}. \tag{24}
$$

The factors in the above expressions are chosen to ensure the orthogonality and normalization conditions

$$
(\tilde{\Phi}_{\mathbf{q}}^{(\pm)},\Phi_{\mathbf{q}'}^{(\pm)})=\delta_{\mathbf{q}\mathbf{q}'},\quad(\tilde{\Phi}_{\mathbf{q}}^{(\mp)},\Phi_{\mathbf{q}}^{(\pm)})=0. \tag{25}
$$

Thus, the domain space of the operator L can be spanned over the basis $\{\Psi_{\mathbf{q}}\}\cup\{\Phi_{\mathbf{q}}^{(-)}\}\cup\{\Phi_{\mathbf{q}}^{(+)}\}$ whose completeness follows from the fact that the sets of the Bloch waves are known to make up a complete basis and from the identity

$$
\int[\Phi_{\mathbf{q}}^{(+)}(\tilde{\Phi}_{\mathbf{q}'}^{(+)})^\dagger+\Phi_{\mathbf{q}}^{(-)}(\tilde{\Phi}_{\mathbf{q}'}^{(-)})^\dagger]dx=\delta_{\mathbf{q}\mathbf{q}'}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}.
$$

Respectively, the spectrum of the operator L is given by $\sigma=\{\lambda_{\mathbf{q}}\}\cup\{\mu_{\mathbf{q}}^{(-)}\}\cup\{\mu_{\mathbf{q}}^{(+)}\}$.

B. Equation for slowly varying amplitudes

Now turn to the study of the dynamics near the lasing threshold (16). We apply the multiple-scale expansion method explored earlier for dissipative systems (see, e.g., [23,26] and references therein) and properly modified for taking into account deep grating of the cavity medium [27,28] (we also notice that the similar approach was extensively used in the mean-filed theory of Bose-Einstein condensates in optical lattices [29]).

We introduce a formal small parameter ϵ, sets of slow temporal $t_j=\epsilon^j\tau$, and spatial $x_j=\epsilon^j x$ variables, which are regarded as independent, and look for a solution of Eq. (1) in the form

$$\mathbf{A}=\begin{pmatrix} A_0^{(st)} \\ 0 \\ 0 \end{pmatrix}+\epsilon\mathbf{A}^{(1)}+\epsilon^2\mathbf{A}^{(2)}+\cdots, \tag{26}$$

where $\mathbf{A}^{(j)}=(A_0^{(j)},A_1^{(j)},\bar{A}_1^{(j)})^T$ $(j=1,2,\ldots)$. We also assume that $V_j(x)=V_j(x_0)$, i.e., medium modulation is "rapid" compared to the modulation of the pump and signal waves whose scales are defined below.

Now we substitute the expansion (26) in Eq. (1) and gathering the terms of the ϵ^j order we obtain

$$-i\partial_{\tau_0}\mathbf{A}^{(j)}+L\mathbf{A}^{(j)}=\mathbf{F}^{(j)}. \tag{27}$$

For the three lowest orders, the right-hand sides of Eq. (27) are given by $\mathbf{F}^{(1)}=(0,0,0)^T$,

$$\mathbf{F}^{(2)}=i\partial_{\tau_1}\mathbf{A}^{(1)}+\mathcal{M}_1\mathbf{A}^{(1)}+i\begin{pmatrix} [A_1^{(1)}]^2 \\ -2A_0^{(1)}\bar{A}_1^{(1)} \\ -2\bar{A}_0^{(1)}A_1^{(1)} \end{pmatrix}, \tag{28}$$

$$\mathbf{F}^{(3)}=i\partial_{\tau_1}\mathbf{A}^{(2)}+i\partial_{\tau_2}\mathbf{A}^{(1)}+\mathcal{M}_1\mathbf{A}^{(2)}+\mathcal{M}_2\mathbf{A}^{(1)} +2i\begin{pmatrix} A_1^{(1)}A_1^{(2)} \\ -A_0^{(1)}\bar{A}_1^{(2)}-A_0^{(2)}\bar{A}_1^{(1)} \\ -\bar{A}_0^{(1)}A_1^{(2)}-\bar{A}_0^{(2)}A_1^{(1)} \end{pmatrix}, \tag{29}$$

with

$$\mathcal{M}_1=\partial_{x_0}\partial_{x_1}\,\mathrm{diag}(1,2,-2) \tag{30}$$

$$\mathcal{M}_2=\left(\frac{1}{2}\partial_{x_1}^2+\partial_{x_0}\partial_{x_2}\right)\mathrm{diag}(1,2,-2) -2i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{31}$$

In the ϵ^1 order, Eq. (27) is satisfied by the ansatz representing two counterpropagating Bloch waves

$$\mathbf{A}^{(1)}=\mathcal{A}\Phi_{\mathbf{q}_0}^{(+)}+\bar{\mathcal{A}}\Phi_{-\mathbf{q}_0}^{(+)}=\begin{pmatrix} 0 \\ a_{\mathbf{q}_0}^{(+)} \\ b_{\mathbf{q}_0}^{(+)} \end{pmatrix}(\mathcal{A}\varphi_{\mathbf{q}_0}^{(1)}+\bar{\mathcal{A}}\varphi_{-\mathbf{q}_0}^{(1)}), \tag{32}$$

where the complex amplitude $\mathcal{A}$ depends on the slow variables among which we indicate only the most rapid ones, e.g., $\mathcal{A}(x_1,\tau_1)\equiv\mathcal{A}(x_1,x_2,\ldots;\tau_1,\tau_2,\ldots)$ and we used the properties $a_{\mathbf{q}}^{(\pm)}=a_{-\mathbf{q}}^{(\pm)}$ and $b_{\mathbf{q}}^{(\pm)}=b_{-\mathbf{q}}^{(\pm)}$.

According to Eqs. (14) and (16) now $\kappa_{\mathbf{q}_0}=i$ and $\theta_{\mathbf{q}_0}=\pi/4$ which results in $a_{\pm\mathbf{q}_0}^{(+)}=b_{\pm\mathbf{q}_0}^{(+)}=1$ with $\mu_{\pm\mathbf{q}_0}^{(+)}=0$ and $a_{\pm\mathbf{q}_0}^{(-)}=-b_{\pm\mathbf{q}_0}^{(-)}=-i$ with $\mu_{\pm\mathbf{q}_0}^{(-)}=-2i$. Using these coefficients, we recast the ansatz (32) in the form

$$\mathbf{A}^{(1)}=(\mathcal{A}\varphi_{\mathbf{q}_0}^{(1)}+\bar{\mathcal{A}}\varphi_{-\mathbf{q}_0}^{(1)})(0,1,1)^T. \tag{33}$$

The dependence of the envelope on the slow variables is found from the solvability conditions at the three lowest orders of ϵ, i.e., from the conditions of the orthogonality of $\mathbf{F}^{(2)}$ and $\mathbf{F}^{(3)}$ to the null space of the adjoin operator $L^{\dagger}$. In other words, the requirements (they are also known as the Fredholm alternative)

$$(\tilde{\Phi},\mathbf{F}^{(j)})=0, \quad j=2,3 \tag{34}$$

must be satisfied.

Now we consider the second-order term $\mathbf{A}^{(2)}$ in the form

$$\mathbf{A}^{(2)}=\mathcal{C}_+\Phi_{\mathbf{q}_0}^{(-)}+\mathcal{C}_-\Phi_{-\mathbf{q}_0}^{(-)}+\sum_{\mathbf{q}}\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(0)}\Psi_{\mathbf{q}} +\sum_{\mathbf{q}\neq\mathbf{q}_0}\{\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(+)}\Phi_{\mathbf{q}}^{(+)}+\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(-)}\Phi_{\mathbf{q}}^{(-)}\}, \tag{35}$$

where the coefficients $\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(0)}$ and $\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(\pm)}$ are the functions of the slow variables. We also have required $\mathbf{A}^{(2)}$ to be orthogonal to the leading order: $(\mathbf{A}^{(1)},\mathbf{A}^{(2)})=0$.

For $\mu_{\mathbf{q}_0}^{(+)}=0$, the equation at the second order in ϵ is $L\mathbf{A}^{(2)}=\mathbf{F}^{(2)}$. Using the explicit form (33), the orthogonality of the spaces S_0 and S_1, as well as the representation (6), we get

$$L_0\sum_{\mathbf{q}}\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(0)}\Psi_{\mathbf{q}}=i\begin{pmatrix} (\mathcal{A}\varphi_{\mathbf{q}_0}^{(1)}+\bar{\mathcal{A}}\varphi_{-\mathbf{q}_0}^{(1)})^2 \\ 0 \\ 0 \end{pmatrix}, \tag{36}$$

$$L_1\Big[\sum_{\mathbf{q}\neq\mathbf{q}_0}(\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(+)}\Phi_{\mathbf{q}}^{(+)}+\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(-)}\Phi_{\mathbf{q}}^{(-)})+\mathcal{C}_+\Phi_{\mathbf{q}_0}^{(-)}+\mathcal{C}_-\Phi_{-\mathbf{q}_0}^{(-)}\Big] =(i\partial_{\tau_1}+\mathcal{M}_1)(\mathcal{A}\varphi_{\mathbf{q}_0}^{(1)}+\bar{\mathcal{A}}\varphi_{-\mathbf{q}_0}^{(1)})\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{37}$$

These equations are readily solved by projecting them on the respective eigenfunctions of the adjoint operators to obtain

$$\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(0)}=\frac{i}{\lambda_{\mathbf{q}}}\Big[2|\mathcal{A}|^2\int_{-\ell}^{\ell}\bar{\varphi}_{\mathbf{q}}^{(0)}|\varphi_{\mathbf{q}_0}^{(1)}|^2dx_0+\mathcal{A}^2\int_{-\ell}^{\ell}\bar{\varphi}_{\mathbf{q}}^{(0)}(\varphi_{\mathbf{q}_0}^{(1)})^2dx_0 +\bar{\mathcal{A}}^2\int_{-\ell}^{\ell}\bar{\varphi}_{\mathbf{q}}^{(0)}(\varphi_{-\mathbf{q}_0}^{(1)})^2dx_0\Big], \tag{38a}$$

$$\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(\pm)}=2\frac{\bar{a}_{\mathbf{q}}^{(\pm)}-\bar{b}_{\mathbf{q}}^{(\pm)}}{\mu_{\mathbf{q}}^{(\pm)}}\Big(\partial_{x_1}\mathcal{A}\int_{-\ell}^{\ell}\bar{\varphi}_{\mathbf{q}}^{(1)}\partial_{x_0}\varphi_{\mathbf{q}_0}^{(1)}dx_0 +\partial_{x_1}\bar{\mathcal{A}}\int_{-\ell}^{\ell}\bar{\varphi}_{\mathbf{q}}^{(1)}\partial_{x_0}\varphi_{-\mathbf{q}_0}^{(1)}dx_0\Big), \tag{38b}$$

and

$$\mathcal{C}_{\pm}=\mp\frac{i}{2}v_{\mathbf{q}_0}^{(1)}\partial_{x_1}\mathcal{A},$$

where

$$v_{\mathbf{q}_0}^{(1)}=2i\int_{-\ell}^{\ell}\bar{\varphi}_{\mathbf{q}_0}^{(1)}\partial_{x_0}\varphi_{\mathbf{q}_0}^{(1)}dx_0 \tag{39}$$

is the group velocity of the signal wave with the Bloch wave vector q_0 in the band ν_0.

An important feature revealed by the formulas (38) becomes evident in the limit of $\gamma=0$. Then if the resonant condition $-\Delta_0=\mathcal{E}_{\mathbf{q}}$ is satisfied, the approximation breaks down. This has two consequences: (i) the dissipation is necessary to attenuate the resonant energy transfer from the idler to pump wave and (ii) alternatively the resonances can be avoided by choosing the detuning Δ_0 to match one of the gaps of the spectrum of the pump wave: in this last case, $\lambda_{\mathbf{q}}\neq 0$ even for $\gamma=0$.

Let us now take into account the Bloch functions represented as $\varphi_{\mathbf{q}}^{(j)}(x)=p_{\mathbf{q}}^{(j)}(x)e^{-iqx}$, where $p_{\mathbf{q}}^{(j)}(x)$ is a periodic function. Introducing the identity $\sum_{m=1}^{N}e^{-i\pi mq}=N\delta_{qQ}$, where $Q=2n$, $(n=0,\pm1,\pm2,)$ is the vector of the reciprocal lattice, together with $\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(\pm)}$ for $\mathbf{q}\neq\mathbf{q}_0$, we obtain that $\int_{-\ell}^{\ell}\bar{\varphi}_{\mathbf{q}}^{(1)}\partial_{x_0}\varphi_{\mathbf{q}_0}^{(1)}dx_0\propto\delta_{\mathbf{q}\mathbf{q}_0}$ and thus $\mathcal{B}_{\mathbf{q}\mathbf{q}_0}^{(\pm)}=0$. Next we project Eq. (37) over $\tilde{\Phi}_{\pm\mathbf{q}_0}^{(+)}$ and get the solvability condition $\partial_{\tau_1}A^{(1)}=0$, meaning that the envelop solution $\mathcal{A}$ does not depend on τ_1.

Finally, applying the Fredholm alternative theorem to Eq. (34) at the third order in ϵ and using $\int_{-\ell}^{\ell}\bar{\varphi}_{\mathbf{q}}^{(0)}(\varphi_{\pm\mathbf{q}_0}^{(1)})^2dx_0\propto\delta_{\mathbf{q},\pm2\mathbf{q}_0+Q}$, we obtain the following amplitude equation for the evolution of the slowly varying amplitude envelope $\mathcal{A}$,

$$\partial_{\tau_2}\mathcal{A}-\frac{1}{2}[v_{\mathbf{q}_0}^{(1)}]^2\partial_{x_1}^2\mathcal{A}-2\mathcal{A}+\chi\mathcal{A}|\mathcal{A}|^2=0, \tag{40}$$

where

$$\chi=\sum_{\mathbf{q}}\frac{\gamma}{\left(\Delta_0+\frac{1}{2}\mathcal{E}_{\mathbf{q}}^{(0)}\right)^2+\gamma^2}\left(4\left|\int_{-\ell}^{\ell}\varphi_{\mathbf{q}}^{(0)}|\varphi_{\mathbf{q}_0}^{(1)}|^2dx_0\right|^2+\left|\int_{-\ell}^{\ell}\varphi_{\mathbf{q}}^{(0)}(\varphi_{\mathbf{q}_0}^{(1)})^2dx_0\right|^2+\left|\int_{-\ell}^{\ell}\bar{\varphi}_{\mathbf{q}}^{(0)}(\varphi_{\mathbf{q}_0}^{(1)})^2dx_0\right|^2\right) \tag{41}$$

is the nonlinear coefficient, which allows further simplification. Indeed, taking into account that q and $\pm q_0$ are in the first BZ and assuming, without lost of generality, q_0 is positive [i.e., $q_0\in(0,1)$ and $q\in(-1,1)$], one can perform summation with respect to q,

$$\chi=\gamma N^2\sum_{\nu}\left[\frac{4S_\nu^0}{\left(\Delta_0+\frac{1}{2}\mathcal{E}_{\nu,0}^{(0)}\right)^2+\gamma^2}+\frac{S_\nu^{(+)}+S_\nu^{(-)}}{\left(\Delta_0+\frac{1}{2}\mathcal{E}_{\nu,2q_0-Q}^{(0)}\right)^2+\gamma^2}\right], \tag{42}$$

where we have introduced the notations

$$S_\nu^0\equiv\left|\int_0^{\pi}p_{\nu,0}^{(0)}(x)|p_{\nu_0,q_0}^{(1)}(x)|^2dx\right|^2, \tag{43}$$

$$S_\nu^{(\pm)}\equiv\left|\int_0^{\pi}e^{\mp iQx}\bar{p}_{\nu,2q_0\mp Q}^{(0)}(x)[p_{\nu_0,q_0}^{(1)}(x)]^2dx\right|^2, \tag{44}$$

and

$$Q=\begin{cases}0, & \text{if } 0<q_0<1/2\\ 2, & \text{if } 1/2<q_0<1.\end{cases}$$

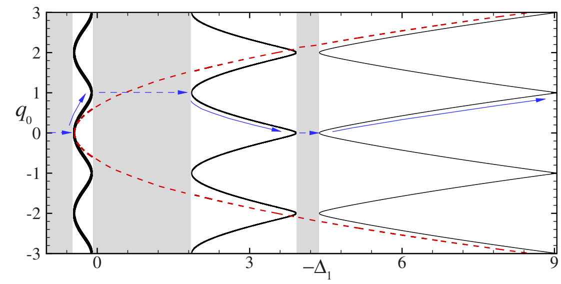

FIG. 3. (Color online) The neutral stability curve $q_0(-\Delta_1)$ in the extended zone representation for the periodic (solid black line) and homogeneous (dashed red line) media (the minimum of latter is shifted from zero to $\Delta_{\min}$ to make the comparison more evident) for the same data as in Fig. 2. The shadowed domains correspond to the stability windows. Blue arrows show evolution of the stability points as $-\Delta_1$ grows.

In the amplitude Eq. (40), the second term describes spatial evolution (diffusion) of the instability. It becomes zero, as the edges of the signal wave gap spectrum where $v_{\mathbf{q}_0}^{(1)}=0$. This is achieved at either center, $q_0=0$, or at the boundary, $q_0=\pm1$, of the BZ and for the frequencies corresponding to the band edges.

The last term in Eq. (40) is a third-order nonlinearity that describes nonlinear interaction among signal wave harmonics and ensures saturation of the linear amplification. Indeed, considering the signal as composed of the harmonics, interpreted below as quasiparticles, with the equal "decay rate" given by γ [see Eq. (7)], one observes close analogy between Eq. (42) and the known formula for the total inelastic scattering of a low-energy particle with a nucleus [30]. More precisely, Eq. (42) should be viewed as a sum of all scattering cross sections over all "quasiparticles" (i.e., pump-signal modes, in the case at hand) each of them proportional to the hopping integral $S_\nu^{(0)}$ and $S_\nu^{(\pm)}$ and to the dissipative factor determined by γ which corresponds to the quantum-mechanical probability of the decay of a nucleus: $\frac{2\gamma}{(\mathcal{E}_\nu^{(0)}+2\Delta_0)^2+4\gamma^2}$.

III. DISCUSSION AND CONCLUSION

Let us now discuss the obtained results vs those known for the standard case of the homogeneous medium. To this end in Fig. 3, we superimpose the neutral curves for the periodic medium in the extended zone representation and for its homogeneous counterpart.

In the homogeneous case, larger detunings correspond to more distant instability points in the Fourier space: $\pm q_0\to\infty$ as $-\Delta_1\to\infty$ (the dashed red curve). In the periodic medium, the initial growth of q_0 accompanies $-\Delta_1$ until the latter "collides" the edge of the BZ where $v_{\nu_0,q_0}^{(1)}=0$ (as shown in Fig. 3 by the blue arrows). This occurs at $|q_0|=1$. Further increase of $-\Delta_1$ makes it to fall in the gap of the spectrum (the middle shadowed domain) where excitation of linear modes is forbidden by the periodicity, which gives the origin to the lowest stability window. Having reached the upper boundary of the gap, $-\Delta_1$ enters the new (the second) band and instability appears. However, unlike in the homogeneous situation, now $|q_0|$ changes from 1 to zero until $-\Delta_1$ achieves the second lowest gap. Under further increase of $-\Delta_1$, the qualitative behavior is repeated for higher bands and gaps.

This gives an idea on how can be performed an experiment where the band structure is explicitly visualized. To this end, it is enough to arrange a set of beams with different optical paths, such that the respective detuning $-\Delta_1$ scans the desirable range of energies. Then, the instabilities of each of the beams in the far zone will be either observed at different points of the BZ or not observed if the respective detuning matches one of the stability windows.

To conclude, we have presented an analytical study of the occurrence and the dynamics of dissipative Bloch waves in degenerate optical parametric oscillators under the combined effects of diffraction and spatial periodic modulations of index refraction. We have showed that a reduced model in the form of amplitude equation can capture the weekly spatiotemporal dynamics of the system. It results that spatial modulations strongly affect the nonlinear dynamics of dissipative solutions. In contrast to Bloch waves in conservative systems, the obtained dissipative Bloch waves are attracting solutions that are stable in a large domain of the parameters. This opens the possibility to experimentally visualize the band structures explicitly. Although, we present our investigations in the context of optics, we believe that our result is generic for spatially extended systems with spatial periodic modulations and characterizes the key role of the spatial modulations in the nonlinear dynamics of such systems.

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