Rotating Larkin-Ovchinnikov-Fulde-Ferrell state of the two-dimensional ultracold Fermi superfluid gas: Reentrant behavior of the critical angular velocity

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A rotating ultracold S-wave superfluid Fermi gas is studied when the mismatch in chemical potentials $\delta\mu$ (and the population imbalance δn) corresponds to the Larkin-Ovchinnikov-Fulde-Ferrell state in the vicinity of the Lifshitz critical point. It is shown that under these conditions the critical angular velocity Ω_{c2} in two-dimensional systems is an oscillating function of temperature and δn giving rise to reentrant superfluid phases. The reason for this behavior is the population by Cooper pairs of the Landau levels ($n \geq 1$) above the lowest one ($n=0$).

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I. INTRODUCTION

The Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) phase with a spatial variation in the superconducting order parameter $\Delta(\mathbf{r})$ can be realized in superconductors in external magnetic field (h). The Zeeman interaction between spins of electrons and the external field h produces unequal populations of the spin-up and spin-down electrons. This, in turn, implies a mismatch in chemical potentials $\delta\mu = h \equiv (\mu_\uparrow - \mu_\downarrow)/2$ of spin-up and spin-down components [1,2]. In ultracold fermionic superfluids the pairing occurs between atoms in two different hyperfine states (a,b), which can be loaded into a trap with unequal numbers $n_a \neq n_b$. Due to the rather large relaxation time for transitions between the different hyperfine states, the numbers of both species n_a and n_b can be considered as fixed to a good approximation. Since the particle losses are negligible, the population imbalance $\delta n = n_a - n_b$ is fixed as well. The latter implies also that the chemical potentials are mismatched; this mismatch is characterized by the quantity $\delta\mu = (\mu_a - \mu_b)/2$. The possibility to fix δn means that the physical realization of the LOFF phase in a superfluid ultracold Fermi gas is more favorable than in metallic superconductors. In metallic S-wave superconductors in an external magnetic field the genuine LOFF state is masked by the orbital effects due to the Lorenz force term in the Hamiltonian described by the minimal coupling prescription $\hat{\mathbf{p}} \to \hat{\mathbf{p}} - e\mathbf{A}$, where $\hat{\mathbf{p}}$ and e are the electron momentum and charge and $\mathbf{A}$ is the vector potential. To eliminate the orbital effects one needs specially designed materials such as a clean quasi-two-dimensional (2D) superconductor which is exposed to a magnetic field applied parallel to the conduction planes. Another example are the heterogeneous structures, where the LOFF phase can be realized in a much broader range of parameters than in bulk superconductors or superfluids. Specifically, in metallic superconductor-ferromagnet-superconductor (SFS) weak links the LOFF phase has already been realized within the ferromagnetic link [3], which are promising for small-scale applications, such as the superconducting quantum interference devices (SQUIDs), quantum computing, etc. The complications mentioned above do not arise in superfluid two-component ultracold Fermi gases, where δn is controlled in experiments, and orbital effects are absent due to the neutrality of gases. However, the experimental realization of the LOFF phase in superfluid ultracold Fermi gases is a unresolved issue. These systems are commonly placed in magnetic traps that create an inhomogeneous confining single-particle potential which plays the role of a container that keeps the ultracold gas in equilibrium. In such an inhomogeneous system a nonzero population imbalance leads to a phase separation into a homogeneous superfluid (S) and normal (N) fluid. This leaves a rather narrow spatial window between S and N phases where the LOFF phase can be realized [4]. The optical lattices may offer even more favorable conditions for the realization [5] and applications [6] of the LOFF phase in ultracold gases. In Ref. [6] a heterogeneous SNS weak link is proposed, where N is an ultracold Fermi gas in the normal state with the population imbalance $\delta n_N \neq 0$, while the superfluid banks S are from superfluid ultracold gas (of the same atoms), but without population imbalance, i.e., $\delta n_S = 0$. It turns out that in such a case, depending on parameters of the system, one can in principle realize a π-weak link (a type of Josephson contact with the phase difference $\varphi = \pi$ in the ground state), which when placed in a superfluid ring produces spontaneous mass flow—the so-called π-SQUID [6].

In this work we consider a 2D superfluid system with the LOFF phase in a two-component ultracold Fermi gas under rotation. The rotation of superfluid gases plays a role that is similar to the orbital effect in metallic superconductors. This fact opens up the possibility to investigate in ultracold superfluid Fermi gases the interplay of "orbital" and "spin" effects. In contrast to metallic superconductors these quantities can be varied in ultracold gases independently. Below we investigate the behavior of the critical angular velocity Ω_{c2} for the transition from the superfluid to the normal state in the vicinity of the Lifshitz critical point (with the temperature T^* and the chemical potential mismatch $\delta\mu^*$ in the $T-\delta\mu$ plane) in the LOFF state. We show that Ω_{c2} is an *oscillatory function of temperature or population imbalance*. These oscillations arise due to the occupation of (higher) Landau levels ($n \geq 1$) by Cooper pairs. The latter effect is realized if the quasiclassical condition $\hbar\Omega_{c2} < \pi(k_B T_c)^2/E_F$ is realized, where T_c is the critical temperature of superfluid phase transition and E_F is the Fermi energy. We refer to this effect as the *quasiclassical oscillation effect*. In the opposite case, more precisely when $\hbar\Omega_{c2} > T_c$, the quantity Ω_{c2} oscillates due to the population of the Landau levels in the normal state (the normal-state Landau quantization) [7,8]. This effect, which can be referred to as the quantum oscillation effect, is negligible near the point $(T^*, \delta\mu^*)$ and will not be studied here.

II. ROTATING THE LOFF PHASE

We consider a rotating Fermi gas with the angular velocity vector directed along the z axis, $\mathbf{\Omega}=\Omega\hat{\mathbf{z}}$. The ultracold gas is placed in a magnetic trap with the potential $V_t(\mathbf{r}) = M[\omega^2\mathbf{r}^2+\omega_z^2 z^2]/2$, where ω and ω_z are the trapping frequencies and M is the atomic mass. The BCS Hamiltonian in the rotating coordinate system but expressed via coordinates of the inertial laboratory system is given by

$$\hat{H} = \sum_{i=a,b}\int d^d x \hat{\psi}_i^\dagger [h_0(\hat{\mathbf{p}},\mathbf{r},\hat{\mathbf{L}}) - \mu_i]\hat{\psi}_i - g\int d^d x \hat{\psi}_a^\dagger(x)\hat{\psi}_b^\dagger(x)\hat{\psi}_b(x)\hat{\psi}_a(x), \quad (1)$$

where the summation is over the fermionic species, the integration involves the dimension of the space d, $\hat{\psi}_a^\dagger(x)$ and $\hat{\psi}_a(x)$ are the fermionic creation and annihilation operators, g is the coupling constant, and h_0 is the single-particle Hamiltonian, which will be specified below. The chemical potentials of the species are $\mu_{a,b}=\mu\pm\delta\mu$, where $\mu=(\mu_a+\mu_a)/2$ and $\delta\mu=(\mu_a-\mu_b)/2$ are the average and the mismatch in chemical potentials, respectively. In the following we specify the quantities μ and $\delta\mu$. Since the self-consistent equation for the superconducting (superfluid) order parameter are the same for fixed $\delta\mu$ and δn the obtained results are applicable also to systems with fixed δn. The single-particle Hamiltonian $h_0(\hat{\mathbf{p}},\mathbf{r},\hat{\mathbf{L}})$ in Eq. (1) is given by $h_0(\hat{\mathbf{p}},\mathbf{r},\hat{L}_z) = (\hat{\mathbf{p}}^2/2M)+V_t(\mathbf{r})-\mathbf{\Omega}\cdot\hat{\mathbf{L}}$, where $\hat{\mathbf{p}}=-i\hbar\nabla$ is the linear and $\hat{\mathbf{L}}=\hat{\mathbf{r}}\times\hat{\mathbf{p}}$ is the orbital momentum. $h_0(\hat{\mathbf{p}},\mathbf{r},\hat{\mathbf{L}})$ reads

$$h_0(\hat{\mathbf{p}},\mathbf{r},\hat{L}_z) = \frac{1}{2M}(\hat{\mathbf{p}} - M\mathbf{V}_\Omega)^2 + \frac{1}{2}M(\omega^2-\Omega^2)r^2 + \frac{1}{2}M\omega_z^2 z^2, \quad (2)$$

where $\mathbf{V}_\Omega=\Omega\hat{\mathbf{z}}\times\mathbf{r}$ is the velocity due to the rotation. In the following we consider a 2D gas which is realized for the pancakelike trap with $\omega_z \gg \omega$. It is also assumed that $\omega \gtrsim \Omega$ in order to keep the gas stable and quasihomogeneous. When necessary, the system can be additionally placed in a periodic optical-lattice potential $V_{op}(\mathbf{r})$, which changes the atomic-particle spectrum from the parabolic one $(\mathbf{p}^2/2M)$ to the tight-binding-like $\epsilon(\hat{\mathbf{p}})$, which creates more favorable conditions for the realization of the LOFF phase [5]. We study here the system with the parabolic band which is near the second-order transition line between the normal (unpaired) and the LOFF phase in the $T-\delta\mu$ plane. This case is realized for $T\leq T^*$ and $\delta\mu \geq \delta\mu^*$ [9]. It will be shown that near this line the *critical angular velocity* Ω_{c2} (for the transition from the normal to the superfluid state) is small and the effect of rotation can be accounted for in the quasiclassical approximation which means that the Landau quantization of the single-particle levels in the normal Fermi gas is negligible. At this transition line the equation for the order parameter can be linearized

$$\frac{\Delta(\mathbf{r})}{g} = \int d^2 r_1 K_0(\mathbf{r}-\mathbf{r}_1)e^{-i(\mathbf{r}-\mathbf{r}_1)(\hat{\mathbf{p}}-2M\mathbf{V}_\Omega)}\Delta(\mathbf{r}), \quad (3)$$

where the kernel is $K_0(\mathbf{r}-\mathbf{r}_1)=T\Sigma_{\omega_n}G_a(\omega_n,\mathbf{r}-\mathbf{r}_1)G_b(-\omega_n,\mathbf{r}-\mathbf{r}_1)$. Here, $G_{a/b}(\omega_n,\mathbf{r}-\mathbf{r}_1)$ are the single-particle Green's functions and $\hbar\omega_m \equiv \eta_m = \pi k_B T(2m+1)$ are the fermionic Matsubara frequencies. The quasiclassical approximation in Eq. (3) is valid if the phase change $\delta\varphi$ due to the rotation is small on the characteristic scales [in $K_0(\mathbf{r}-\mathbf{r}_1)$] of the order of the superconducting coherence length ξ_0. The estimated phase change is

$$\delta\varphi = 2M\int_{\mathbf{r}}^{\mathbf{r}+\xi_0}\mathbf{V}_\Omega(\mathbf{l})d\mathbf{l} \sim \frac{2\hbar\Omega E_F}{(k_B T_c)^2}, \quad (4)$$

and the quasiclassical condition $\delta\varphi \ll 2\pi$ gives $\hbar\Omega \ll \pi(k_B T_c)^2/E_F$, where E_F is the Fermi energy. When Ω is large enough so that the cyclotron radius $R_\Omega \sim v_F/\Omega$ of the atomic orbit is smaller than ξ_0, i.e., when $\hbar\Omega \gtrsim k_B T_c$, it is necessary to take into account the Landau quantization of atomic motion in the normal state. Consider next the Fourier image of the kernel $K_0(\mathbf{q}) = k_B T\Sigma_m\int d^2 p G_a(\omega_m,\mathbf{p})G_b(-\omega_m,\mathbf{p}+\mathbf{q})$. In the quasihomogeneous case, when the effect of the potential $M(\omega^2-\Omega^2)r^2/2$ on the single-particle spectrum is small, the Green's functions $G_{a,b}$ are given by $G_{a,b}(\omega_m,\mathbf{p})=[i\eta_m-\xi(\mathbf{p})\pm\delta\mu]^{-1}$, where $\xi(\mathbf{p})=(\mathbf{p}^2/2M)-\mu$. After the integration over the energy ξ and by assuming a circular Fermi surface one obtains

$$K_0(\mathbf{q},\eta_m>0) = iN(0)\int_0^{2\pi}\frac{d\varphi}{2(i\eta_m+\delta\mu)+qv_F\cos\varphi}, \quad (5)$$

where v_F is the Fermi velocity and $N(0)(\sim 1/E_F)$ is the density of states at the Fermi surface. The integration in Eq. (5) is straightforward (the contour integral runs over the contour $|z|=1$ with $z=\exp\{i\varphi\}$), and we obtain

$$K_0(q) = \mathrm{Re}\left(\sum_m \frac{\pi N(0)k_B T}{\sqrt{(\eta_m+i\delta\mu)^2+(\hbar q v_F/2)^2}}\right). \quad (6)$$

Since we consider the problem near the Lifshitz critical point $(T^*,\delta\mu^*)$ the wave vector of the LOFF phase is small, i.e., $q \ll \xi_0^{-1}$, and it is sufficient to carry out a small q expansion. Thus, we approximate $K_0(\mathbf{q}) \approx K_0(0)+K_2 q^2+K_4 q^4$, with $K_0(0)=N(0)[\lambda^{-1}-\tau(t,\delta\bar{\mu})]$, where the dimensionless coupling constant $\lambda=N(0)g$ and $\tau(t,\delta\bar{\mu})$ is defined in Eq. (9). For further calculations we define a dimensionless quantity $X(q)=(\hbar q)^2/2ME_c$, with $E_c=(k_B T_{c0})^2/E_F$, which can be also expressed via the coherence length $\xi_0=\hbar v_F/(1.76\pi)k_B T_{c0}$, i.e., one has $X(q)=(1.76\pi\xi_0 q)^2$. It gives $K_0(\mathbf{q}) \approx K_0(0)+k_2 X+k_4 X^2$, where

$$k_2 = \frac{t^{-2}}{(4\pi)^2}\mathrm{Re}\ \psi^{(2)}\left(\frac{1}{2}+i\frac{\delta\bar{\mu}}{2\pi t}\right), \quad (7)$$

$$k_4 = -\frac{t^{-4}}{4(4\pi)^4}\mathrm{Re}\ \psi^{(4)}\left(\frac{1}{2} + i\frac{\delta\bar{\mu}}{2\pi t}\right), \quad (8)$$

and

$$\tau(t,\delta\bar{\mu}) = \ln t + \mathrm{Re}\ \psi\left(\frac{1}{2} + i\frac{\delta\bar{\mu}}{2\pi t}\right) - \psi\left(\frac{1}{2}\right). \quad (9)$$

Here, $t=(T/T_{c0})$ and $\delta\bar{\mu}=\delta\mu/T_{c0}$, where T_{c0} is the critical temperature of superfluid phase transition when the number densities of the two species are equal, i.e., $\delta n=0$, and $\psi(x)=d\ln\Gamma(x)/dx$ is digamma function and $\psi^{(n)}(x)=d^n\psi(x)/dx^n$. In case of $\Omega=0$ one obtains from Eq. (3) the equation defining the transition line $(t_Q,\delta\bar{\mu})$ between the normal and the LOFF phase

$$[\tau_L(t_Q,\delta\bar{\mu}) - k_2X(Q) - k_4X^2(Q)] = 0, \quad (10)$$

where $\Delta(r)=\Delta_q\exp[\mathbf{Q}\cdot\mathbf{r}]\neq 0$ and $X(Q)\equiv(\hbar Q)^2/2ME_c=k_2/2|k_4|$. The magnitude of the LOFF wave vector Q_L is determined by the maximum of the function $k_2X(Q)+k_4X^2(Q)$, i.e., by maximizing the LOFF critical temperature $t_L(Q,\delta\bar{\mu})$. The direction of the wave vector Q_L is chosen spontaneously. At the Lifshitz point where $t^*=0.56$ and $\delta\bar{\mu}^*=1.04$ one obtains $\tau_L(t^*,\delta\bar{\mu}^*)=0$ and $k_2(t^*,\delta\bar{\mu}^*)=0$, where $Q(t^*,\delta\mu^*)=0$. The LOFF phase is realized for $t<t^*$ and $\delta\bar{\mu}>\delta\mu^*$, therefore near the Lifshits point one has $Q(t,\delta\bar{\mu})\ll\xi_0^{-1}$, which justifies the small-q expansion of the kernel $K_0(\mathbf{q})$ above.

In the case of a 2D rotating system, with $\Omega\neq 0$ and $\mathbf{V}_\Omega=\mathbf{\Omega}\times\mathbf{r}$, there is an upper critical angular velocity $\Omega_{c2}(t,\delta\bar{\mu})$ below which there is a nucleation of superfluidity in the form of quantized vortices. The linear Ginzburg-Landau (GL) equation for $\Delta^{(lin)}(\mathbf{r})$ on the second-order normal-state LOFF transition reads as

$$\left[\tau(t,\delta\bar{\mu}) - \left(\frac{k_2}{E_c}\right)\hat{H} - \left(\frac{k_4}{E_c^2}\right)\hat{H}^2\right]\Delta^{(lin)}(\mathbf{r}) = 0, \quad (11)$$

where $\hat{H}\equiv(\hat{\mathbf{p}}-2M\mathbf{V}_\Omega)^2/2M$ is the Hamiltonian for the harmonic oscillator in the isotropic Coulomb gauge. To obtain $\Omega_{c2}(t,\delta\bar{\mu})$ we need a solution of the eigenvalue problem $\hat{H}\Delta_n^{(lin)}(\mathbf{r})=\epsilon_n\Delta_n^{(lin)}(\mathbf{r})$, where $\Delta_n^{(lin)}(\mathbf{r})$ is highly degenerated and depends on the used gauge. For instance, by assuming a rotationally infinite 2D system, such as disk, a solution of the eigenvalue problem is given by $\Delta_{n,rot}^{(lin)}(\mathbf{r})\sim z^n\exp\{-(|z|/2l_0)^2\}$, with $z=x+iy$ and the *magnetic length* $l_0=\sqrt{\hbar/4M\Omega_{c2}}$, while the eigenvalues are $\epsilon_n=4\hbar\Omega_{c2}(n+1/2)$, $n=0,1,2\ldots$ Note that the chosen linear solutions $\Delta_{n,rot}^{(lin)}(\mathbf{r})$ are also eigenstates of the zth component of the angular momentum operator $\hat{L}_z=-i\hbar(x\partial_y-y\partial_x)\equiv\hbar(z\partial_z-z^*\partial_{z^*})$, i.e., $\hat{L}_z\Delta_{n,rot}^{(lin)}(\mathbf{r})=\hbar n\Delta_{n,rot}^{(lin)}(\mathbf{r})$. However, the eigenvalues of Eq. (11) do not depend on the degeneracy and are given by

$$\tau(t,\delta\bar{\mu}) - \left(\frac{k_2}{E_c}\right)\epsilon_n - \left(\frac{k_4}{E_c^2}\right)\epsilon_n^2 = 0. \quad (12)$$

Note that the solutions with $n=1,2\ldots$ correspond to a multiply quantized vortex in the sense that the angular momentum $(\hat{L}_z)$ per Cooper pair is finite, i.e., $n\geq 1$. In that respect, the case with $n=0$, which corresponds to the standard linear Abrikosov solution without mismatch $(\delta\mu=0)$, is peculiar since its vorticity is zero. However, the above terminology is conditional since it is based on the solution of the linear eigenvalue problem and in order to find the realistic vortex state one should solve the *nonlinear self-consistent equation* for the order parameter which minimizes the Gibbs free energy. In that case even if the linear solution has zero vorticity, as it is the case with the standard Abrikosov solution with $n=0$, the solution of the nonlinear GL equation [which contains terms up to the third power in $\Delta(\mathbf{r})$] has the form of the triangular vortex lattice, where at the regular lattice points the superconducting order parameter is zero, i.e., $\Delta(\mathbf{r}_v)=0$. Moreover, in this nonlinear solution the phase change around each of these points is $\delta\varphi=2\pi$, which means that there is a single vortex in the unit cell each with the vorticity one and with one elementary flux quant Φ_0 per unit cell. In that respect, the calculation of the real structure of $\Delta(\mathbf{r})$ (of the complex vortex lattice) in the LOFF state for $\Omega<\Omega_{c2}$ and near the Lifshitz point point $(T^*,\delta\mu^*)$ is more complex since it is based on the nonlinear GL equation up to the fifth power in $\Delta(\mathbf{r})$ [10]. For instance, in metallic superconductors various type of solutions are possible differing by the number of flux quanta in the unit cell. For instance, by assuming one flux quanta per unit cell in the case $n=1$ various vortex lattices, from the linear chain up to the triangular one, can be realized with the number of zeros of $\Delta(\mathbf{r})$ (each with the vorticity one) in the unit cell which are larger than the number of flux quanta [11]—see extended discussion in the concluding section (ii).

On the second-order transition line the solution of the eigenvalue problem posed by Eq. (12) gives the explicit expression $\Omega_{c2,\pm}^{(n)}$ for given n

$$\frac{\Omega_{c2,\pm}^{(n)}(t,\delta\bar{\mu})}{\omega_c} = \frac{1}{n+\frac{1}{2}}\frac{k_2\pm\sqrt{k_2^2-4\tau|k_4|}}{8|k_4|}, \quad (13)$$

where $\omega_c=E_c/\hbar$. It is seen that for the *fixed value* of $\delta\bar{\mu}>\delta\bar{\mu}^*$ (or for fixed δn) one has two branches $\Omega_{c2,+}^{(n)}>\Omega_{c2,-}^{(n)}$ which are functions of the temperature $t<t^*$. They meet each other on the line $k_2^2(t,\delta\bar{\mu})-4\tau(t,\delta\bar{\mu})|k_4(t,\delta\bar{\mu})|=0$, which is, in fact, the LOFF line $t_L(t_Q,\delta\bar{\mu})$ given by Eq. (10). The lower line for the nth level intersects the upper line for the $(n+1)$th level at the point $\Omega_{c2,-}^{(n)}(t_n)=\Omega_{c2,+}^{(n+1)}(t_n)$. This means that in the temperature interval $t_n<t<t_L$ (for fixed $\delta\bar{\mu}$) the normal state is realized for $\Omega_{c2,+}^{(n+1)}(t)<\Omega<\Omega_{c2,-}^{(n)}(t)$. Therefore, a cascade of oscillatory (interchanging) normal and LOFF-vortex-lattice reentrant transitions will arise, i.e., a sequence of transitions from the normal ultracold Fermi gas to a superfluid gas featuring exotic vortex lattices with one or more vortex quanta per unit cell or, possibly, multiquantized vortices. This reentrant behavior is clearly seen in the phase diagram $\Omega_{c2}(t)$ shown in Fig. 1.

We would like to stress two points. First, from Eq. (13) it is seen that $\Omega_{c2}^{(n)}(t)$ tends to zero for $n\to\infty$ where the quantum mechanical quantized solution $\Delta_n(\mathbf{r})$ tends to the

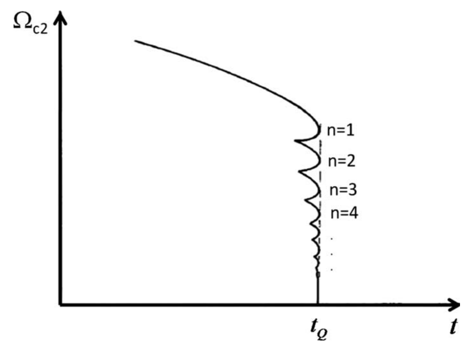

FIG. 1. Schematic figure of the oscillation of the critical angular velocity $\Omega_{c2}(t)$ as function of the relative temperature $t=T/T_{co}$ for fixed mismatch in chemical potentials $\delta\bar{\mu}$ (or the population imbalance δn). $t_Q=T_Q/T_{co}$ with T_Q—the LOFF critical temperature and T_{co}—the bare critical temperature.

LOFF solution $\Delta(\mathbf{r})\sim\sin\mathbf{Q}_L\cdot\mathbf{r}$ (or $\exp\{\mathbf{Q}_L\cdot\mathbf{r}i\}$) with $Q_L^2=C\lim_{n\to\infty}\Omega_{c2}^{(n)}(t)n$, $C=8M/\hbar$. This means that for small angular velocity $\Omega\to 0$ the optimum quantum number for producing the vortex lattice must be very large, i.e., $n\to\infty$. Therefore, the LOFF state is recovered from the vortex state. Second, the phase diagram in Fig. 1 is also generic for the case when the temperature $t<t^*$ is fixed but the mismatch in chemical potentials $\delta\bar{\mu}>\delta\bar{\mu}^*$ (or the population imbalance δn) is varied. In that case the variables t and $\delta\bar{\mu}$ change the roles but the oscillatory effect is the same. Again one has $\Omega_{c2,+}^{(n)}(\delta\bar{\mu})>\Omega_{c2,-}^{(n)}(\delta\bar{\mu})$, and these two curves meet at the LOFF transition line $\delta\bar{\mu}_Q(t)$. In the intervals $\Omega_{c2,+}^{(n+1)}(\delta\bar{\mu})<\Omega<\Omega_{c2,-}^{(n)}(\delta\bar{\mu})$ one has again the cascade of normal to the LOFF-vortex-lattice reentrant transitions.

III. DISCUSSION AND CONCLUSIONS

(i) Our study shows that it is possible to realize a rotating LOFF state in superfluid ultracold Fermi gases in a quasi-2D magnetic trap if the trap frequency ω is adapted to be around (but slightly larger than) the angular velocity $\Omega_{c2}(T,\delta\mu_r)$ given by Eq. (13), i.e., $\omega>\Omega_{c2}$. The latter condition allows, first, the stability of the ultracold gas and second it gives rise to a rather small inhomogeneity in the quasiparticle spectrum. This means that the kernel $K_0(\mathbf{r})$ can be calculated with the help of the Green's functions for the quasihomogeneous system. The realization of the LOFF phase is even more favorable if the ultracold gas is placed in a 2D optical lattice since in cases where the van Hove singularities near (or at) the Fermi surface exist they can favor the realization of the LOFF state in a much broader region of the phase diagram $(\delta n,T)$ [5]. In the case of the rotating LOFF superfluid in an optical lattice there is always an additional weak electric trap, which can be tuned (if necessary by an additional magnetic trap) to fulfill the condition $\omega\gtrsim\Omega_{c2}$ in such a way that the centrifugal potential due to the rotation is compensated.

(ii) In Fig. 1 the curves $\Omega_{c2}(X)$ (with $X=T$ or $\delta\mu$) represent second-order transition lines (for different n), which correspond to the solutions $\Delta_n^{(lin)}(\mathbf{r},p)$ of the linear Ginzburg-Landau equation with $n=0,1,2,\ldots$. As we briefly explained after Eq. (12) the solution with $n=0,1,2,\ldots$, which are degenerate (p is the degeneracy parameter), correspond to zero, one, two, etc. vorticities, i.e., to multiple-quantized vortices. However, the solution of the nonlinear GL equation can change this conditional classification since it is known that the vortex-lattice solution $\Delta(\mathbf{r})$ for the Abrikosov lattice becomes zero at the regular lattice points, $\Delta(\mathbf{r}_v)=0$, and with the phase change around each of these points is $\delta\varphi=2\pi$—single vorticity, in spite the fact that it is the linear superposition of degenerate zero vorticity solutions with $n=0$, i.e., $\Delta_{n=0}^{(lin)}(\mathbf{r},p)$. In the rotating LOFF state the situation is more complex since for a given parameter $X<X_Q$ and by lowering Ω the system goes from the nth to $(n+1)$th Landau level and in the region with fixed n one should solve the nonlinear Ginzburg-Landau equation by including terms up to $|\Delta(r)|^6$ in the free energy [Eqs. (10) and (11)]. In metallic superconductors the solution for Δ is searched [11] analogously to the Abrikosov ansatz [12] in the form of superposition of the wave functions characterizing the nth Landau level (which are degenerated and characterized by the degeneracy parameter p) $\Delta(\mathbf{r})=\Sigma_p C_p\Delta_n^{(lin)}(\mathbf{r},p)$. The real structure of the order parameter $\Delta(\mathbf{r})$ in the rotating LOFF state results from the *competition of two lengths*, the magnetic one $l_0=\sqrt{\hbar/4M\Omega}$ and the LOFF length $L=2\pi/Q_L$. One expects that for smaller value of l_0, i.e., for small values of n and near T_Q, the vortex chains are realized with the LOFF stripes [at which $\Delta(\mathbf{r})=0$] between, while for very large l_0 (for very large n and small Ω) one expects that the structure is dominated by the LOFF structure. Indeed, by assuming that one flux quantum per unit cell is realized (note that this is an assumption only, which has not been proven rigorously) in Ref. [11] several exotic vortex lattices were found to minimize the free energy depending on the choice of the parameters. It turns out that under these assumptions and, for example, in the case where $n=1$, there emerges a vortex lattice in form of chains which are separated by the LOFF state with depressed order parameter. For some parameters one can obtain quadratic (or triangular) lattice which contains one flux quantum per unit cell accompanied by three vorticity $(+2\pi,-2\pi,+2\pi)$, i.e., the order parameter has three zeros in the unit cell. An intriguing question is in which parameter region is possible to have more than one (if at all) flux quanta per unit cell? This is a matter for future research.

(iii) Let us estimate the temperature interval $\delta t_n=(T_L-T_n)/T_{c0}$, where the oscillation of Ω_{c2} are pronounced. From the condition $\Omega_{c2,-}^{(n)}(t_n)=\Omega_{c2,+}^{(n+1)}(t_n)$ [see discussion after Eq. (13)] and in the case of the superfluid Li^6 where $E_F\sim 2\ \mu K$ and $k_BT_c\sim 10^{-1}E_F$ one obtains $\delta t_1\sim 10^{-1}-10^{-2}$. In that case the order of magnitude of $\Omega_{c2}^{(n)}$ near the Lifshitz point is $\hbar\Omega_{c2}^{(n)}\sim(10^{-1}-10^{-2})(k_BT_c)^2/E_F$, i.e., one has $\Omega_{c2}^{(n)}<(30-300)\ s^{-1}$. These values of $\Omega_{c2}^{(n)}$ are in the range of the experimentally reached rotational frequencies which can be realized by the stirring method. The latter method is usually used for creating vortices [13]. Finally, we stress that the whole analysis was carried out in the framework of the BCS weak coupling theory. We anticipate that our analysis can be extended toward the Bose-Einstein condensate limit, and it should remain (qualitatively) valid at stronger couplings but on the BCS side of the phase diagram. This expectation is justified by the studies of the vortex state in metallic superconductors, for which the fermionic excitations in the vortex cores are realized also in the strong-coupling limit but still on the BCS side of the phase diagram.

In conclusion, we have studied a rotating two-component superfluid ultracold Fermi gas with a mismatch in chemical potentials of the species (or with population imbalance), which is in the LOFF phase and near the Lifshitz point. We have shown that the critical angular velocity Ω_{c2} in two-dimensional systems is an oscillatory function of temperature or the population imbalance $\delta n(=n_a - n_b)$. This effect gives rise to a cascade of reentrant superfluid transitions with the superfluid featuring multiple-quantized circulation quanta. The reason for this oscillatory effect is the population of the higher Landau levels $(n \geq 1)$ by Cooper pairs in the LOFF state. The obtained results on the critical angular velocity Ω_{c2} in the LOFF state might be of interest also for rotating 2D metallic superconductors in the parallel magnetic field and for color superconductors in quark matter, where in the two-flavor color-superconducting quark matter the LOFF phase may compete with the gluonic condensate [14].

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