Cavity quantum optomechanics of ultracold atoms in an optical lattice: Normal-mode splitting

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We consider the dynamics of a movable mirror (cantilever) of a cavity coupled through radiation pressure to the light scattered from ultracold atoms in an optical lattice. Scattering from different atomic quantum states creates different quantum states of the scattered light, which can be distinguished by measurements of the displacement spectrum of the cantilever. We show that for large pump intensities the steady-state displacement of the cantilever shows bistable behavior. Due to atomic back action, the displacement spectrum of the cantilever is modified and depends on the position of the condensate in the Brillouin zone. We further analyze the occurrence of splitting of the normal mode into three modes due to mixing of the mechanical motion with the fluctuations of the cavity field and the fluctuations of the condensate with finite atomic two-body interaction.

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I. INTRODUCTION

In recent years two distinct subjects, optical microcavities and nanomechanical resonators have become entangled experimentally by underlying mechanism of optical, radiation-pressure forces. The coupling of mechanical and optical degrees of freedom via radiation pressure has been a subject of early research in the context of laser cooling [1–3] and gravitational-wave detectors [4]. Recently there has been a great surge of interest in the application of radiation forces to manipulate the center-of-mass motion of mechanical oscillators covering a huge range of scales from macroscopic mirrors in the laser interferometer gravitational wave observatory (LIGO) project [5,6] to nanomechanical cantilevers [7–12], vibrating microtoroids [13,14] membranes [15], and Bose-Einstein condensates [16,17]. The quantum optical properties of a mirror coupled via radiation pressure to a cavity field show interesting similarities to an intracavity Kerr-like interaction [18]. Recently, in the context of classical investigations of nonlinear regimes, the dynamical instability of a driven cavity having a movable mirror has been investigated [19]. Theoretical work has proposed to use the radiation-pressure coupling for quantum nondemolition measurements of the light field [20].

In the field of quantum degenerate gases, standard methods to observe quantum properties of ultracold atoms are based on destructive matter-wave interference between atoms released from traps [21]. Recently, a new approach was proposed, which is based on all optical measurements that conserve the number of atoms. It was shown that atomic quantum statistics can be mapped on transmission spectra of high-Q cavities, where atoms create a quantum refractive index. This was shown to be useful for studying phase transitions between Mott insulator and superfluid states since various phases show qualitatively distinct spectra [22,23]. Recently, coupled dynamics of a movable mirror and atoms trapped in the standing-wave light field of a cavity were studied [24]. It was shown that the dipole potential in which the atoms move is modified due to the back action of the atoms and that the position of the atoms can become bistable. New possibilities for cavity optomechanics by combining the tools of cavity electrodynamics with those of ultracold gases is the motivation of the present work. Here we show that different quantum states of ultracold gases in optical lattice, confined in a cavity can be distinguished by the steady-state displacement spectrum of the movable mirror (cantilever). The atomic and cantilever back action shifts the cavity resonance. The laser pump is shown to coherently control the dynamics of the mirror. Changing the pump intensity, one can switch between stable and bistable regimes. Due to coupling between the condensate wave function and the cantilever, mediated by the cavity photons, the cantilever displacement spectrum is continuously modified as the condensate moves across the Brillouin zone. We also show that in the presence of atom-atom interactions, the coupling of the mechanical oscillator, the cavity field fluctuations and the condensate fluctuations (Bogoliubov mode) leads to the splitting of the normal mode into three modes (normal-mode splitting).

II. CANTILEVER DISPLACEMENT SPECTRA AS A PROBE OF QUANTUM PHASES OF ULTRACOLD ATOMS

We consider an elongated cigar-shaped Bose-Einstein condensate (BEC) of N two-level ^{87}Rb atoms in the $|F=1\rangle$ state with mass m and frequency ω_a of the $|F=1\rangle \rightarrow |F'=2\rangle$ transition of the D_2 line of ^{87}Rb, strongly interacting with a quantized single standing-wave cavity mode of frequency ω_c (Fig. 1). The standing wave that forms in the cavity results in a one-dimensional optical lattice potential. The cavity field is also coupled to external fields incident from one side mirror. It is well known that high-Q optical cavities can significantly isolate the system from its environment, thus strongly reducing decoherence and ensuring that the light field remains quantum mechanical for the duration of the experiment. We also assume that the induced resonance frequency shift of the cavity is much smaller than the longitudinal-mode spacing, so that we restrict the model to a single longitudinal mode. In order to create an elongated BEC, the frequency of the har-

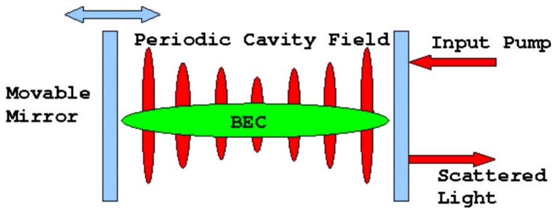

FIG. 1. (Color online) Optomechanical realization of parametric coupling of a mechanical oscillator to a cavity optical mode and Bose-Einstein condensate trapped in an optical lattice inside the cavity.

monic trap along the transverse direction should be much larger than one in the axial (along the direction of the optical lattice) direction. The system is also coherently driven by a laser field with frequency ω_p through the cavity mirror with amplitude η. The cavity mode is coupled to a mechanical oscillator (movable mirror) with frequency Ω_m via the dimensionless parameter $\epsilon = (x_0/\Omega_m)\frac{d\omega_p}{dx}|_{x=0}$. Here $x_0 = \sqrt{\hbar/2m_{eff}\Omega_m}$ is the zero-point motion of the mechanical mode, m_{eff} is its effective mass. It is well known that high-Q optical cavities can significantly isolate the system from its environment, thus strongly reducing decoherence and ensuring that the light field remains quantum mechanical for the duration of the experiment. The harmonic confinement along the directions perpendicular to the optical lattice is taken to be large so that the system effectively reduces to one dimension. This system is modeled by the optomechanical Hamiltonian (H_{om}) in a rotating wave and dipole approximation.

$$H_{om} = \frac{p^2}{2m} - \hbar\Delta_a\sigma^+\sigma^- - \hbar\Delta_c\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{a}_m^\dagger\hat{a}_m - i\hbar g(x)[\sigma^+\hat{a} - \sigma^-\hat{a}^\dagger] - i\eta(\hat{a} - \hat{a}^\dagger) + \hbar\epsilon\Omega_m\hat{a}^\dagger\hat{a}(\hat{a}_m + \hat{a}_m^\dagger), \tag{1}$$

where $\Delta_a = \omega_p - \omega_a$ and $\Delta_c = \omega_p - \omega_c$ are the large atom-pump and cavity-pump detuning, respectively. Here σ^+, σ^- are the Pauli matrices. The atom-field coupling is written as $g(x) = g_0\cos(kx)$. Here $\hat{a}$ and $\hat{a}_m$ are the annihilation operators for a cavity photon and the mechanical mode respectively. The input laser field populates the intracavity mode which couples to the cantilever through the radiation pressure and the atoms through the dipole interaction. The field in turn is modified by the back action of the atoms and cantilever. It is important to notice the nonlinearity in Eq. (1) arising from the coupling between the intracavity intensity and the position quadrature of the cantilever. The system we are considering is intrinsically open as the cavity field is damped by the photon leakage through the massive coupling mirror and the cantilever is connected to a bath at finite temperature. In the absence of the radiation-pressure coupling, the cantilever would undergo a pure Brownian motion driven by its contact with the thermal environment. Here we also assume that there is no direct coupling between the atoms and the cantilever, though this coupling could give rise to some interesting physics. The effects of the direct atom-mirror interaction can be neglected by assuming that a few lattice sites near the center of the cavity are appreciably populated. Since the detuning Δ_a is large, spontaneous emission is negligible and we can adiabatically eliminate the excited state using the Heisenberg equation of motion $\sigma^- = \frac{i}{\hbar}[H_{om}, \sigma^-]$. This yields the single-particle Hamiltonian

$$H_0 = \frac{p^2}{2m} - \hbar\Delta_c\hat{a}^\dagger\hat{a} + \cos^2(kx)[V_{\rm cl}(\mathbf{r}) + \hbar U_0\hat{a}^\dagger\hat{a}] - i\eta(\hat{a} - \hat{a}^\dagger) + \hbar\Omega_m\hat{a}_m^\dagger\hat{a}_m + \hbar\epsilon\Omega_m\hat{a}^\dagger\hat{a}(\hat{a}_m + \hat{a}_m^\dagger). \tag{2}$$

The parameter $U_0 = \frac{g_0^2}{\Delta_a}$ is the optical lattice barrier height per photon and represents the atomic backaction on the field [25]. $V_{\rm cl}(\mathbf{r})$ is the external classical potential. Here we will always take $U_0 > 0$. In this case the condensate is attracted to the nodes of the light field, and hence the lowest bound state is localized at these positions which leads to a reduced coupling of the condensate to the cavity compared to that for $U_0 < 0$. Along x, the cavity field forms an optical lattice potential of period $\lambda/2$ and depth $(\hbar U_0\langle\hat{a}^\dagger\hat{a}\rangle + V_{\rm cl})$. We now write the Hamiltonian in a second quantized form including the two-body interaction term,

$$H = \int d^3x\Psi^\dagger(\vec{r})H_0\Psi(\vec{r}) + \frac{1}{2}\frac{4\pi a_s\hbar^2}{m}\int d^3x\Psi^\dagger(\vec{r})\Psi^\dagger(\vec{r})\Psi(\vec{r})\Psi(\vec{r}), \tag{3}$$

where $\Psi(\vec{r})$ is the field operator for the atoms. Here a_s is the two-body s-wave scattering length. The corresponding optomechanical-Bose-Hubbard (OMBH) Hamiltonian can be derived by writing $\Psi(\vec{r}) = \Sigma_j\hat{b}_j w(\vec{r} - \vec{r}_j)$, where $w(\vec{r} - \vec{r}_j)$ is the Wannier function and $\hat{b}_j$ is the corresponding annihilation operator for the bosonic atom at the j^{th} site. Retaining only the lowest band with nearest-neighbor interaction, we have

$$H = E_0\sum_j \hat{b}_j^\dagger\hat{b}_j + E\sum_j(\hat{b}_{j+1}^\dagger\hat{b}_j + \hat{b}_{j+1}\hat{b}_j^\dagger) + (\hbar U_0\hat{a}^\dagger\hat{a} + V_{\rm cl}) \times\left\{J_0\sum_j\hat{b}_j^\dagger\hat{b}_j + J\sum_j(\hat{b}_{j+1}^\dagger\hat{b}_j + \hat{b}_{j+1}\hat{b}_j^\dagger)\right\} + \frac{U}{2}\sum_j\hat{b}_j^\dagger\hat{b}_j^\dagger\hat{b}_j\hat{b}_j - \hbar\Delta_c\hat{a}^\dagger\hat{a} - i\hbar\eta(\hat{a} - \hat{a}^\dagger) + \hbar\Omega_m\hat{a}_m^\dagger\hat{a}_m + \hbar\epsilon\Omega_m\hat{a}^\dagger\hat{a}(\hat{a}_m + \hat{a}_m^\dagger) \tag{4}$$

where

$$U = \frac{4\pi a_s\hbar^2}{m}\int d^3x|w(\vec{r})|^4,$$

$$E_0 = \int d^3xw(\vec{r} - \vec{r}_j)\left\{\left(-\frac{\hbar^2\nabla^2}{2m}\right)\right\}w(\vec{r} - \vec{r}_j),$$

$$E = \int d^3xw(\vec{r} - \vec{r}_j)\left\{\left(-\frac{\hbar^2\nabla^2}{2m}\right)\right\}w(\vec{r} - \vec{r}_{j\pm1}),$$

$$J_0 = \int d^3xw(\vec{r} - \vec{r}_j)\cos^2(kx)w(\vec{r} - \vec{r}_j),$$

$$J = \int d^3xw(\vec{r} - \vec{r}_j)\cos^2(kx)w(\vec{r} - \vec{r}_{j\pm1}). \tag{5}$$

The OMBH Hamiltonian derived above is valid only for weak atom-field nonlinearity [26]. It has been shown [27] that the intracavity field intensity is bistable, and leads to a bistable optical lattice potential. The position of the individual lattice wells is bistable as well since a mirror displacement l_m displaces each optical lattice well by l_m in the same direction. However, we consider a regime where $l_m/(\pi/k) \ll 1$, and thus we ignore this effect on the Wannier function used above. The nearest-neighbor nonlinear interaction terms are usually very small compared to the onsite interaction and are neglected as usual. We now write down the Heisenberg-Langevin equation of motion for the bosonic field operator $\hat{b}_j$, the internal cavity mode $\hat{a}$ and the mechanical mode $\hat{a}_m$ as

$$\dot{\hat{b}}_j = -i\left(U_0\hat{a}^\dagger\hat{a} + \frac{V_{\rm cl}}{\hbar}\right)\{J_0\hat{b}_j + J(\hat{b}_{j+1} + \hat{b}_{j-1})\} - \frac{iE}{\hbar}\{\hat{b}_{j+1} + \hat{b}_{j-1}\} - \frac{iU}{\hbar}\hat{b}_j^\dagger\hat{b}_j\hat{b}_j - \frac{iE_0}{\hbar}\hat{b}_j, \quad (6)$$

$$\dot{\hat{a}} = -iU_0\left\{J_0\sum_j \hat{b}_j^\dagger\hat{b}_j + J\sum_j (\hat{b}_{j+1}^\dagger\hat{b}_j + \hat{b}_{j+1}\hat{b}_j^\dagger)\right\}\hat{a} + \eta + i\{\Delta_c - \epsilon\Omega_m(\hat{a}_m + \hat{a}_m^\dagger)\}\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\xi_p(t), \quad (7)$$

$$\dot{\hat{a}}_m = \left(-i\Omega_m - \frac{\Gamma_m}{2}\right)\hat{a}_m - i\epsilon\Omega_m\hat{a}^\dagger\hat{a} + \sqrt{\Gamma_m}\xi_m(t). \quad (8)$$

Here κ and Γ_m characterizes the dissipation of the optical and mechanical degree of freedom respectively. Here, we follow a semiclassical theory by considering *noncommuting* noise operators for the input field, i.e., $\langle\xi_p(t)\rangle=0$, $\langle\xi_p^\dagger(t')\xi_p(t)\rangle=n_p\delta(t'-t)$, $\langle\xi_p(t')\xi_p^\dagger(t)\rangle=(n_p+1)\delta(t'-t)$, and a *classical* thermal noise input for the mechanical oscillator, i.e., $\langle\xi_m(t)\rangle=0$, $\langle\xi_m^\dagger(t')\xi_m(t)\rangle=\langle\xi_m(t')\xi_m^\dagger(t)\rangle=n_m\delta(t'-t)$, in Eqs. (7) and (8). The quantities n_m and n_p are the equilibrium occupation numbers for the mechanical and optical oscillators, respectively. We consider a deep lattice formed by a strong classical potential $V_{\rm cl}(\mathbf{r})$, so that the overlap between Wannier functions is small. Thus, we can neglect the contribution of tunneling by putting $E=0$ and $J=0$. Under this approximation, the matter-wave dynamics is not essential for light scattering. In experiments, such a situation can be realized because the time scale of light measurements can be much faster than the time scale of atomic tunneling. One of the well-known advantages of the optical lattices is their extremely high tunability. Thus, tuning the lattice potential, tunneling can be made very slow [28].

The steady-state value of the position quadrature $x_{m,s}=\hat{a}_{m,s}+\hat{a}_{m,s}^\dagger$ (the subscript s denotes the steady-state value) is found as

$$x_{m,s} = \frac{-8\epsilon\Omega_m^2\hat{a}_s^\dagger\hat{a}_s}{4\Omega_m^2 + \Gamma_m^2}, \quad (9)$$

where,

$$\hat{a}_s^\dagger\hat{a}_s = \frac{\eta^2}{(\Delta_c - U_0J_0\hat{N} - \epsilon\Omega_m x_{m,s})^2 + \kappa^2/4}. \quad (10)$$

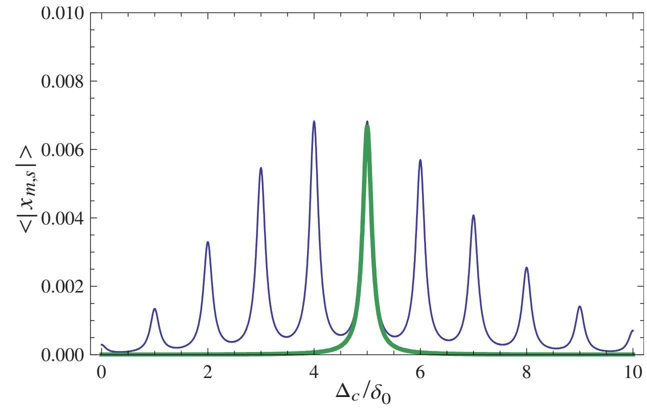

FIG. 2. (Color online) Steady-state displacement spectra of the cantilever. The single Lorentzian (thick line) for the MI reflects the nonfluctuating atom number. Many Lorentzians for SF (thin line) reflect atom number fluctuations. Here we have used $\Omega_m/\delta_0=30$, $\Gamma_m/\delta_0=0.001$, $\kappa/\delta_0=0.1$, $\epsilon=0.0067$, and the pump intensity is $\eta/\delta_0=0.2$.

Here $\hat{N}=\hat{b}_j^\dagger\hat{b}_j$. The equation for $\hat{a}_s^\dagger\hat{a}_s$ is very important for the understanding the physics behind this problem. From it we clearly see how the cantilever and atom dynamics affects the steady state of the intracavity field. The coupling to the mirror and the atoms shifts the cavity resonance frequency and changes the field inside the cavity in a way to induce a new stationary intensity. The change occurs after a transient time depending on the response of the cavity and the strength of the coupling to the cantilever and the ultracold atoms. Following [22] Eqs. (9) and (10) allows to express $x_{m,s}$ as a function $f(\hat{n}_1,\ldots,\hat{n}_M)$ of atomic occupation number operators and calculate their expectation values for prescribed atomic states $|\Psi\rangle$.

For the Mott state $\langle\hat{n}_j\rangle_{\rm MI}=q_j$ atoms are well localized at the jth site with no number fluctuations. It is represented by a product of Fock states, i.e., $|\Psi\rangle_{\rm MI}=\Pi_{j=1}^M|q_j\rangle_j\equiv|q_1,\ldots,q_M\rangle$, with expectation values

$$\langle f(\hat{n}_1,\ldots,\hat{n}_M)\rangle_{\rm MI} = f(q_1,\ldots,q_M), \quad (11)$$

For simplicity we consider equal average densities $\langle\hat{n}_j\rangle_{\rm MI}=N/M\equiv n_0$.

In SF state, each atom is delocalized over all sites leading to local number fluctuations. It is represented by superposition of Fock states corresponding to all possible distributions of N atoms at M sites: $|\Psi\rangle_{\rm SF}=\Sigma_{q_1,\ldots,q_M}\sqrt{N!/M^N}/\sqrt{q_1!\ldots q_M!}|q_1,\ldots,q_M\rangle$. Expectation values of light operators can be calculated from

$$\langle f(\hat{n}_1,\ldots,\hat{n}_M)\rangle_{\rm SF} = \frac{1}{M^N}\sum_{q_1,\ldots,q_M}\frac{N!}{q_1!\ldots q_M!}f(q_1,\ldots,q_M), \quad (12)$$

representing a sum of all possible "classical" terms. Thus, all these distributions contribute to scattering from a SF, which is obviously different from $\langle f(\hat{n}_1,\ldots,\hat{n}_M)\rangle_{\rm MI}$ with only a single contributing term.

In Fig. 2, we represent $\langle|x_{m,s}|\rangle$ as a function of Δ_c/δ_0,

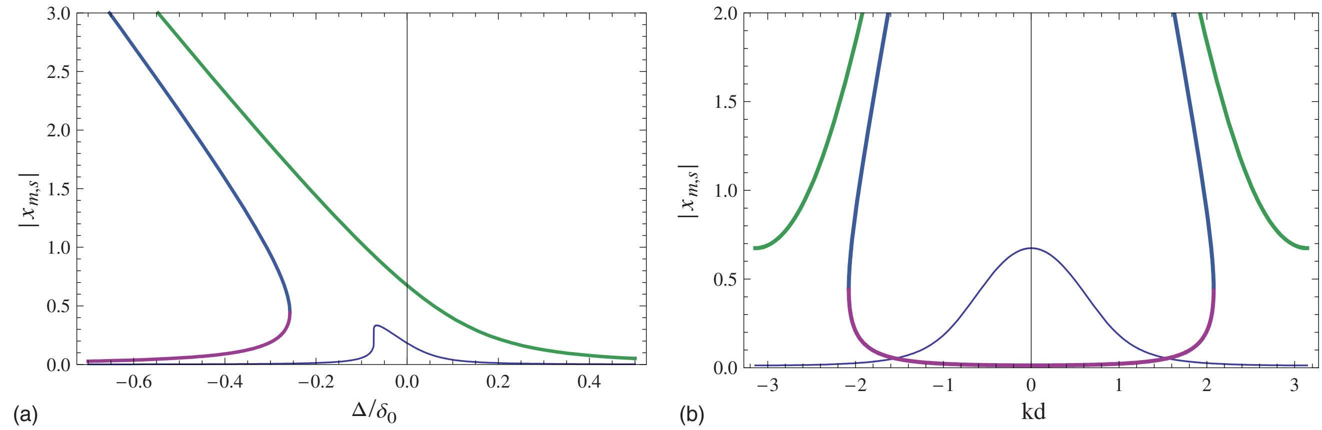

FIG. 3. (Color online) (a) Steady-state displacement spectrum as a function of Δ/δ_0 for $k=0$. The thin curve corresponds to pump intensity $\eta/\delta_0=0.2$ and the thick line corresponds to $\eta/\delta_0=1.0$. For the larger pump intensity, we find three steady-state solutions, with two of them being stable. The system prepared below the resonance will follow the steady-state branch until reaching the lower turning point, where a nonsteady-state dynamics is excited. The parameters taken are, $\epsilon=0.0067$, $\Omega_m/\delta_0=30$, $\Gamma_m/\delta_0=0.001$ and $\kappa/\delta_0=0.08$. (b): The displacement spectrum modified by the atomic backaction through the cavity photons as a function of the quasimomentum (kd) for two different values of Δ_c/δ_0. For $\Delta_c/\delta_0=-0.5$, a bistable behavior is seen (thick line) when the condensate is at the edge of the Brillouin zone ($kd=\pm\pi$). This bistable behavior is absent when $\Delta_c/\delta_0=0.5$ (thin line).

where $\delta_0=U_0J_0$. Clearly for the SF case, the displacement spectra are a sum of Lorentzians with different dispersion shifts. A comblike structure is seen if each Lorentzian is resolved. In the Mott state however, a single Lorentzian (thick line) is noticed. These were the typical structures predicted for the transmission spectra [22]. Since the coupling between the cantilever and the condensate wave function is mediated by the cavity field, we find that the displacement spectrum of the cantilever maps the distribution function of the ultracold atoms. An important condition to resolve the Lorentzians is $\kappa<\delta_0$ and this condition is easily met in present experiments [29]. Detecting the mirror's motion is straightforward, since the optical phase shift is directly proportional to the mirror's displacement. Typically, the lorentizian frequency spectrum of the mirror's position is obtained in this way. The peak width yields the total damping rate, including the effective optomechanical damping. The area under the spectrum reveals the variance of the mirror displacement, which is a measure of the effective temperature.

III. BISTABLE BEHAVIOR

We now consider the case of large number of atoms and hence treat the BEC within the mean-field framework and assume the tight-binding approximation where we replace $\hat{b}_j$ by ϕ_j and look for solutions in the form of Bloch waves $\phi_j=u_k\exp(ikjd)\exp(-i\mu t/\hbar)$ [30]. Here μ is the chemical potential, d is the periodicity of the lattice and $\frac{1}{M}\Sigma_j\hat{b}_j^\dagger\hat{b}_j=n_0$ (atomic number density), M is the total number of lattice sites. From Eqs. (9) and (10), we obtain a cubic equation in $x_{m,s}$.

$$x_{m,s}^3-\frac{2\Delta}{\epsilon\Omega_m}x_{m,s}^2+\frac{(\Delta^2+\kappa^2/4)}{\epsilon^2\Omega_m^2}+\frac{8\eta^2}{\epsilon(4\Omega_m^2+\Gamma_m^2)}=0. \quad (13)$$

Here, $\Delta=\Delta_c-U_0N[J_0+2J\cos(kd)]$. Note that now we do not ignore the tunneling term J. A plot of $|x_{m,s}|$ versus Δ/δ_0 for two different values of the pump parameter η is shown in Fig. 3. Clearly for higher pump intensity the system shows a bistable behavior. For pump rates higher than a critical value, we find three steady-state solutions for the mirror displacement, with two of them being stable. The system prepared below resonance will follow the steady-state branch until reaching the lower turning point, where a nonsteady-state dynamics is excited. This dynamics is governed by the time scale of the mechanical motion of the mirror because the cavity damping is almost two orders of magnitude faster ($\Gamma_m\ll\kappa$). Figure 3(b) shows the steady-state displacement spectra $|x_{m,s}|$ as a function of the quasimomentum for two different values of the cavity detuning $\Delta_c/\delta_0=0.5$ (thin line) and $\Delta_c/\delta_0=-0.5$ (thick line). The cantilever phonons develop a quasimomentum dependence due to strong coupling with the condensate mediated by the cavity photons. The atomic back action modifies both the cavity field and the cantilever displacement. As the condensate moves across the Brillouin zone, the atom-field interaction changes and as a result the cantilever displacement spectra is continuously modified. For $\Delta_c/\delta_0=-0.5$, a bistable behavior is seen when the condensate is at the edge of the Brillouin zone (here $\Delta=0$). This bistable behavior is absent when $\Delta_c/\delta_0=0.5$. The position of the condensate in the Brillouin zone is easily manipulated by accelerating the condensate.

IV. DYNAMICS OF SMALL FLUCTUATIONS: NORMAL-MODE SPLITTING

Here we show that the coupling of the mechanical oscillator, the cavity field fluctuations and the condensate fluctuations (Bogoliubov mode) leads to the splitting of the normal mode into three modes [normal-mode splitting (NMS)]. The optomechanical NMS however involves driving three parametrically coupled nondegenerate modes out of equilibrium. The NMS does not appear in the steady-state spectra but

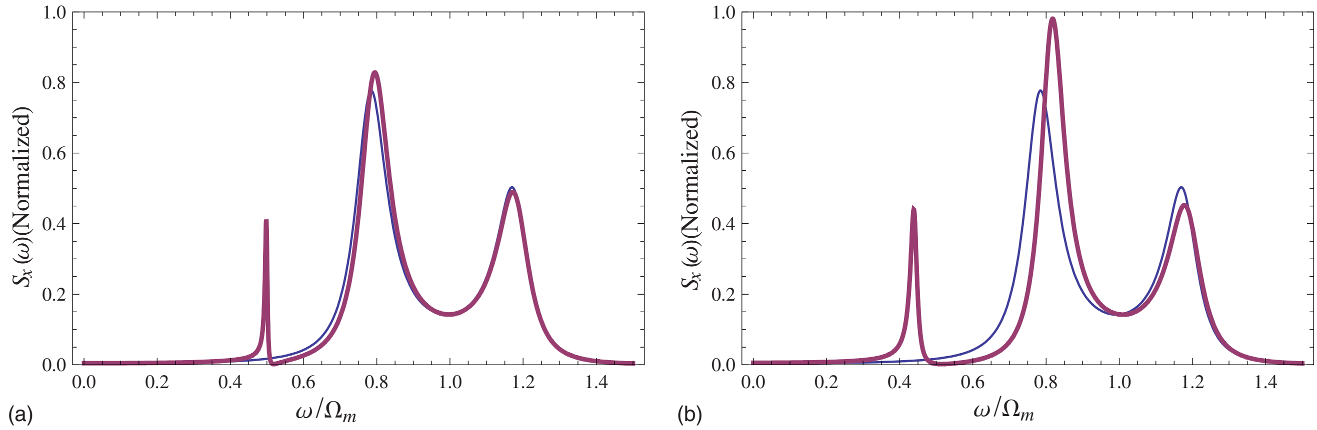

FIG. 4. (Color online) (a) Normalized plot of the displacement spectrum $S_x(\omega)$ for two values of the atomic two-body interaction, $U_{eff}/\Omega_m=0$ (thin line) and $U_{eff}/\Omega_m=0.3$ (thick line). Parameters chosen are: $g_m/\Omega_m=0.4$, $\kappa/\Omega_m=0.1$, $\Gamma_m/\Omega_m=0.01$, $g_c/\Omega_m=0.1$, $\Delta_d/\Omega_m=-1$, and $\nu/\Omega_m=0.01$. A clear NMS is observed in the presence of atom-atom interaction. (b) Normalized plot of the displacement spectrum $S_x(\omega)$ for the same parameters as (a) but with a stronger atom-photon coupling $g_c/\Omega_m=0.2$. The NMS is now much more prominent compared to that in (a).

rather manifests itself in the fluctuation spectra of the mirror displacement. To this end, we shift the canonical variables to their steady-state values (i.e., $\hat{a}\to\hat{a}_s+\hat{a}$, $\hat{a}_m\to\hat{a}_{m,s}+\hat{a}_m$, $\hat{b}_j\to\frac{1}{\sqrt{M}}(\sqrt{N}+\hat{b})$) and linearize to obtain the following Heisenberg-Langevin equations neglecting atomic losses due to heating:

$$\dot{\hat{b}}=-i\{\nu+2U_{eff}\}\hat{b}-iU_{eff}\hat{b}^\dagger-ig_c(\hat{a}+\hat{a}^\dagger), \quad (14)$$

$$\dot{\hat{a}}=\left(i\Delta_d-\frac{\kappa}{2}\right)\hat{a}-i\frac{g_m}{2}(\hat{a}_m+\hat{a}_m^\dagger)-ig_c(\hat{b}+\hat{b}^\dagger)+\sqrt{\kappa}\xi_p(t), \quad (15)$$

$$\dot{\hat{a}}_m=\left(-i\Omega_m-\frac{\Gamma_m}{2}\right)\hat{a}_m-i\frac{g_m}{2}(\hat{a}+\hat{a}^\dagger)+\sqrt{\Gamma_m}\xi_m(t). \quad (16)$$

Here, $U_{eff}=\frac{Un_0}{\hbar}$, $g_c=U_0J_0\sqrt{N}|\hat{a}_s|$, $\nu=U_0J_0|\hat{a}_s|^2+\frac{V_{cl}J_0}{\hbar}+\frac{E_0}{\hbar}$, and $\Delta_d=\Delta_c-U_0NJ_0-\epsilon\Omega_m x_{m,s}$ is the detuning with respect to the renormalized resonance and $\Delta_d<0$ leads to cooling. Note that $x_{m,s}$ is negative and hence makes Δ_d positive. The atomic and cantilever back action modifies the cavity detuning. The optomechanical coupling rate is given by $g_m=2\hat{a}_s\epsilon\Omega_m$ and $|\hat{a}_s|^2$ gives the mean resonator occupation number. In deriving the above equation, we have assumed that $\hat{a}_s$ to be real. As before, we assume negligible tunneling ($J=E=0$), and hence we drop the site index j from the atomic operators. We will always assume $\Gamma_m\ll\kappa$. Equations (14)–(16) and their Hermitian conjugates constitute a system of six first-order coupled operator equations, for which the Routh-Hurwitz criterion implies that the system is stable only for $g_m<\sqrt{(\Delta_d\Delta_d'+\kappa^2/4)\Omega_m/|\Delta_d|}$ and $\Delta_d\Delta_d'>0$, where $\Delta_d'=\Delta_d-2g_c/(\nu+3U_{eff})$. We transform to the quadratures: $X_m=\hat{a}_m+\hat{a}_m^\dagger$, $P_m=i(\hat{a}_m^\dagger-\hat{a}_m)$, $X_p=\hat{a}+\hat{a}^\dagger$, $P_p=i(\hat{a}^\dagger-\hat{a})$, $X_b=\hat{b}+\hat{b}^\dagger$, and $P_b=i(\hat{b}^\dagger-\hat{b})$. The displacement spectrum in Fourier space for $n_p=0$ is found as

$$S_x(\omega)=\frac{x_0^2}{2\pi}\Omega_m^2|\chi(\omega)|^2\left[\Gamma_m n_m-\frac{\Delta_d^2+\omega^2+\kappa^2/4}{2\Delta_d\Omega_m}\Gamma_s(\omega)\right], \quad (17)$$

where,

$$\chi^{-1}(\omega)=\Omega_m^2+2\Omega_m\Omega_s(\omega)-\omega^2-i\omega[\Gamma_m+\Gamma_s(\omega)], \quad (18)$$

$$\Omega_s(\omega)=\frac{\Delta_d g_m^2\Upsilon^4\Omega_\nu^2}{2\{\Upsilon^8+\kappa^2\omega^2\Omega_\nu^4\}}, \quad (19)$$

and

$$\Gamma_s(\omega)=-\frac{\kappa\Delta_d g_m^2\Omega_m\Omega_\nu^4}{\{\Upsilon^8+\kappa^2\omega^2\Omega_\nu^4\}}, \quad (20)$$

$$\Upsilon^4=\left\{\left(\frac{\kappa^2}{4}+\Delta_d^2-\omega^2\right)(\omega^2-(\nu+U_{eff})(\nu+3U_{eff}))-4\Delta_d g_c^2(\nu+U_{eff})\right\}, \quad (21)$$

$$\Omega_\nu^2=\{\omega^2-(\nu+U_{eff})(\nu+3U_{eff})\}. \quad (22)$$

This spectrum is characterized by a mechanical susceptibility $\chi(\omega)$ that is driven by thermal noise ($\propto n_m$) and by the quantum fluctuations of the radiation pressure (quantum back action) and quantum fluctuations of the condensate. Note that in the absence of the atom-photon coupling ($g_c=0$), the displacement spectrum [Eq. (17)] reduces to that found in [31].

In Fig. 4(a), we show a normalized plot of the displacement spectrum $S_x(\omega)$ for two values of the atomic two-body interaction, $U_{eff}/\Omega_m=0$ (thin line) and $U_{eff}/\Omega_m=0.3$ (thick line). Parameters chosen are: $g_m/\Omega_m=0.4$, $\kappa/\Omega_m=0.1$, $\Gamma_m/\Omega_m=0.01$, $g_c/\Omega_m=0.1$, $\Delta_d/\Omega_m=-1$, and $\nu/\Omega_m=0.01$. In the absence of the interactions, we observe the usual normal mode splitting into two modes and we find that due to a finite atom-atom interaction the normal mode splits up into three modes. The NMS is associated with the mixing between the mechanical mode and the fluctuation of the cavity field around the steady state and the fluctuations of the condensate (Bogoliubov mode) around the mean field. The origin of the fluctuations of the cavity field is the beat of the pump photons with the photons scattered from the condensate atoms. The frequency of the Bogoliubov mode in the low-momentum limit is $\approx\sqrt{U_{eff}}$. Hence in the absence of interactions, the Bogoliubov mode is absent and the system simply reduces to the case of two mode coupling (i.e coupling between the mechanical mode and the photon fluctuations). In the presence of finite atom-atom interaction, the mechanical mode, the photon mode, and the Bogoliubov mode forms a system of three coupled oscillators. It is important to note that splitting in the displacement spectrum occurs only for $g_m > \kappa/\sqrt{2}$ due to finite width of the peaks. Figure 4(b) illustrates the influence of increasing the atom-photon coupling $g_c=0.2$. The NMS is now much more prominent compared to the case with $g_c=0.1$. Similar three coupled oscillator experimental results where two coupled cavities, each containing three identical quantum wells [32] and one microcavity containing two quantum wells [33], have been reported. An important point to note is that in order to observe the NMS, the energy exchange between the three modes should take place on a time scale faster than the decoherence of each mode. Also the parameter regime in which NMS may appear implies cooling. On the positive detuning side, the observation of NMS is prevented by the onset of parametric instability. Experimentally, Normal-mode splitting of a system of large number of atoms coupled to the cavity field has been achieved recently [35]. It was observed that the NMS was observed only if the coupling between the atoms and the cavity was strong enough (strong cooperative coupling regime). This regime was achieved by increasing the atom numbers. One experimental limitation could be spontaneous emission which leads to momentum diffusion and hence heating of the atomic sample [17].

Strictly speaking, a three-dimensional trap geometry is essential to observe the superfluid to Mott insulator transition. Consequently instead of a linear array of atoms, a quasi-two-dimensional pancake-shaped condensates are formed. In a more complete theory, coupling between the inhomogeneous density in the radial plane and the density modulations along the axial direction should be taken into account. In actual experiments mode coupling are always present and this changes the Bogoliubov spectrum [36,37].

The biggest experimental challenge to observe the predicted dynamics is to have an optomechanical device with both a high optical finess (currently in the range from 10^3 to 10^5), and a high mechanical quality factor (10^3 to 10^5 for beams and cantilevers). An alternative approach [15,38] involving a membrane of thickness 50 nm inside a fixed optical cavity can circumvent this problem to some degree and has reached a finesse of 10^4 and a mechanical quality factor of 10^6.

To demonstrate that the dynamics investigated here are within experimental reach, we discuss the experimental parameters from [16,17,34]: a BEC of typically 10^5 ^{87}Rb atoms is coupled to the light field of an optical ultra high-finesse Fabry-Perot cavity. The atom-field coupling $g_0=2\pi \times 10.9$ MHz [16] ($2\pi\times 14.4$ [17]) is greater than the decay rate of the intracavity field $\kappa=2\pi\times 1.3$ MHz [16] ($2\pi \times 0.66$ MHz [17]). The temperature of the ultracold atomic gas $T \ll \hbar\kappa/k_B$ so that the coherent amplification or damping of the atomic motion is neglected. Typically atom-pump detuning is $2\pi\times 32$ GHz. From [34], the mechanical frequency $\Omega_m=2\pi\times 73.5$ MHz and $\Gamma_m=2\pi\times 1.3$ KHz. The coupling rate $g_m=2\pi\times 2.0$ MHz. The energy of the cavity mode decreases due to the photon loss through the cavity mirrors, which leads to a reduced atom-field coupling. Photon loss can be minimized by using high-Q cavities. Our proposed detection scheme relies crucially on the fact that coherent dynamics dominate over the losses. It is important that the characteristic time scales of coherent dynamics are significantly faster than those associated with losses (the decay rate of state-of-art optical cavities is typically 17 kHz [35]).

V. CONCLUSIONS

In summary we have analyzed cavity optomechanics with ultracold atoms. We showed that the steady-state displacement spectra of a cantilever coupled indirectly to a gas of ultracold atoms in an optical lattice through the cavity field are distinct for different quantum phases of equal densities. Further, we showed that in the mean field of the atoms, the cantilever displacement shows a bistable behavior for high pump intensities and exhibits a dependence on the position of the condensate in the Brillouin zone due to the back action of the condensate on the cantilever. The cantilever displacement spectrum shows a bistable behavior when the condensate is at the edge of the Brillouin zone. In the presence of atom-atom interactions, the coupling of the mechanical oscillator, the cavity field fluctuations and the condensate fluctuations (Bogoliubov mode) leads to the splitting of the normal mode into three modes (normal-mode splitting). The system described here shows a complex interplay between distinctly three systems namely, the nanomechanical cantilever, optical microcavity and the gas of ultracold atoms. The quantum state of the degenerate gas influences the cantilever displacement spectra via the cavity photons, while on the other hand the cantilever displacement modifies the cavity field which in turn modifies the properties of the condensate. The position of the condensate in the Brillouin zone and the atomic tunneling can influence the cantilever dynamics and at the same time the cantilever dynamics can be used to control the atomic dynamics.

[1] T. W. Hansch and A. L. Schawlow, Opt. Commun. **13**, 68 (1975).
[2] D. J. Wineland, R. E. Drullinger, and F. L. Walls, Phys. Rev. Lett. **40**, 1639 (1978).
[3] S. Chu, L. Hollberg, J. E. Bjorkholm, A. Cable, and A. Ashkin, Phys. Rev. Lett. **55**, 48 (1985).
[4] C. M. Caves Phys. Rev. Lett. **45**, 75 (1980); R. Loudon, *ibid.* **47**, 815 (1981).
[5] T. Corbitt and N. Mavalvala, J. Opt B: Quantum Semiclasssical Opt. **6**, S675 (2004).
[6] T. Corbitt, Y. Chen, E. Innerhofer, H. Muller-Ebhardt, D. Ottaway, H. Rehbein, D. Sigg, S. Whitcomb, C. Wipf, and N. Mavalvala, Phys. Rev. Lett. **98**, 150802 (2007).
[7] C. Höhberger-Metzger and K. Karrai, Nature (London) **432**, 1002 (2004).
[8] S. Gigan *et al.*, Nature (London) **444**, 67 (2006).
[9] O. Arcizet *et al.*, Nature (London) **444**, 71 (2006).
[10] D. Kleckner and D. Bouwmeester, Nature (London) **444**, 75 (2006).
[11] I. Favero *et al.*, Appl. Phys. Lett. **90**, 104101 (2007).
[12] C. Regal, J. Teufel, and K. Lehnert, Nature (London) **4**, 555 (2008).
[13] T. Carmon, H. Rokhsari, L. Yang, T. J. Kippenberg, and K. J. Vahala, Phys. Rev. Lett. **94**, 223902 (2005).
[14] A. Schliesser, P. DelHaye, N. Nooshi, K. J. Vahala, and T. J. Kippenberg, Phys. Rev. Lett. **97**, 243905 (2006).
[15] J. D. Thompson *et al.*, Nature (London) **452**, 72 (2008).
[16] F. Brennecke *et al.*, Science **322**, 235 (2008).
[17] K. W. Murch *et al.*, Nat. Phys. **4**, 561 (2008).
[18] C. Fabre, M. Pinard, S. Bourzeix, A. Heidmann, E. Giacobino, and S. Reynaud, Phys. Rev. A **49**, 1337 (1994); S. Mancini and P. Tombesi, *ibid.* **49**, 4055 (1994).
[19] F. Marquardt, J. G. E. Harris, and S. Girvin, Phys. Rev. Lett. **96**, 103901 (2006).
[20] V. B. Braginsky and F. Khalili, *Quantum Measurements* (Cambridge University Press, New York, 1992).
[21] M. Greiner, O. Mandel, T. Esslinger, T. Hansch, and I. Bloch, Nature (London) **415**, 39 (2002).
[22] I. B. Mekhov, C. Maschler, and H. Ritsch, Nat. Phys. **3**, 319 (2007).
[23] W. Chen, D. Meiser, and P. Meystre, Phys. Rev. A **75**, 023812 (2007).
[24] D. Meiser and P. Meystre, Phys. Rev. A **73**, 033417 (2006).
[25] C. Maschler and H. Ritsch, Phys. Rev. Lett. **95**, 260401 (2005).
[26] J. Larson *et al.*, Phys. Rev. Lett. **100**, 050401 (2008).
[27] W. Chen, K. Zhang, D. Goldbaum, M. Bhattacharya, and P. Meystre, e-print arXiv:0902.2427.
[28] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Phys. Rev. Lett. **81**, 3108 (1998).
[29] T. Bourdel, T. Donner, S. Ritter, A. Ottl, M. Kohl, and T. Esslinger, Phys. Rev. A **73**, 043602 (2006).
[30] A. Bhattacherjee, Opt. Commun. **281**, 3004 (2008).
[31] J. M. Dobrindt *et al.*, Phys. Rev. Lett. **101**, 263602 (2008).
[32] R. P. Stanley *et al.*, Appl. Phys. Lett. **65**, 2093 (1994).
[33] E. K. Lindmark *et al.*, Opt. Lett. **21**, 994 (1996).
[34] A. Schliesser *et al.*, Nat. Phys. **4**, 415 (2008).
[35] J. Klinner, M. Lindholdt, B. Nagorny, and A. Hemmerich, Phys. Rev. Lett. **96**, 023002 (2006).
[36] T. K. Ghosh and K. Machida, Phys. Rev. A **72**, 053623 (2005).
[37] A. Bhattacherjee, J. Phys. B **40**, 143 (2007).
[38] A. M. Jayich *et al.*, New J. Phys. **10**, 095008 (2008).