Spectral conditions for entanglement witnesses versus bound entanglement

Dariusz Chruściński, Andrzej Kossakowski, and Gniewomir Sarbicki
Institute of Physics, Nicolaus Copernicus University, Grudziądzka 5/7, 87-100 Toruń, Poland
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It is shown that entanglement witnesses constructed via the family of spectral conditions are decomposable, i.e., cannot be used to detect bound entanglement. It supports several observations that bound entanglement reveals highly nonspectral features.

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I. INTRODUCTION

One of the most important problems of quantum information theory [1,2] is the characterization of mixed states of composed quantum systems. In particular, it is of primary importance to test whether a given quantum state exhibits quantum correlation, i.e., whether it is separable or entangled. For low-dimensional systems, there exists a simple necessary and sufficient condition for separability. The celebrated Peres-Horodecki criterium [3] states that a state of a bipartite system living in $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$ is separable if and only if its partial transpose is positive. Unfortunately, for higher-dimensional systems, there is no single *universal* separability condition. Apart from positive partial transpose (PPT) criterion, there are several separability criteria available in the literature (see [2,4] for the review). However, each of them defines only a necessary condition.

The power and simplicity of Peres-Horodecki criterion comes from the fact that it is based on the spectral property: to check for PPT, one simply checks the spectrum of $\rho^\Gamma = (\mathbb{1} \otimes T)\rho$. Another simple spectral separability test is known as the reduction criterion [5]

$$\mathbb{I}_A \otimes \rho_B \geq \rho \quad \text{and} \quad \rho_A \otimes \mathbb{I}_B \geq \rho, \tag{1}$$

where $\rho_A = \mathrm{Tr}_B\, \rho (\rho_B = \mathrm{Tr}_A\, \rho)$ is the reduced density operator. However, reduction criterion is weaker that Peres-Horodecki one, i.e., any PPT state does satisfy Eq. (1) as well.

Actually, there exist other criteria which are based on spectral properties [6]. For example, it turns out that separable states satisfy so-called entropic inequalities

$$S(\rho) - S(\rho_A) \geq 0 \quad \text{and} \quad S(\rho) - S(\rho_B) \geq 0, \tag{2}$$

where S denotes the von Neumann entropy. This means that in the case of separable states, the whole system is more disordered than its subsystems. Actually, these inequalities may be generalized [7–9] for Rényi entropy (or equivalently Tsallis entropy). Another spectral tool was proposed by Nielsen and Kempe [10] and it is based on the majorization criterion

$$\lambda(\rho_A) \succ \lambda(\rho) \quad \text{and} \quad \lambda(\rho_B) \succ \lambda(\rho), \tag{3}$$

where $\lambda(\rho)$ and $\lambda(\rho_{A(B)})$ denote vectors consisting of eigenvalues of ρ and $\rho_{A(B)}$, respectively. Recall, that if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two stochastic vectors, then $x \prec y$ if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}, \quad k = 1, \ldots, n-1, \tag{4}$$

where $x^{\downarrow} (1 \leq i \leq n)$ are components of vector x rearranged in decreasing order $(x_1^{\downarrow} \geq \cdots \geq x_n^{\downarrow})$ and similarly for $y_i^{\downarrow}$. Actually, majorization can be shown [11] to be a more stringent notion of disorder than entropy in the sense that if $x \prec y$, then it follows that $H(x) \geq H(y)$, where $H(x)$ stands for the Shanon entropy of the stochastic vector x.

Interestingly, both criteria, i.e., entropic inequalities (2) and majorization relations (3) follow from the reduction criterion (1) [9,12]. It means that they cannot be used to detect bound entanglement. In particular, since PPT criterion $\rho^\Gamma \geq 0$ implies Eq. (1), the above spectral tests are useless in searching for PPT entangled states.

The most general approach to characterize quantum entanglement uses a notion of an entanglement witness (EW) [13,14]. A Hermitian operator W defined on a tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is called an EW if and only if (1) $\mathrm{Tr}(W\sigma_{\mathrm{sep}}) \geq 0$ for all separable states σ_{sep} and (2) there exists an entangled state ρ such that $\mathrm{Tr}(W\rho) < 0$ (one says that ρ is detected by W) [15]. It turns out that a state is entangled if and only if it is detected by some EW [13]. There was a considerable effort in constructing and analyzing the structure of EWs [16–21] (see also [2] for the review). However, the general construction of these objects is not known.

In the recent paper [22], we proposed a class of entanglement witnesses. Their construction is based on the family of spectral conditions. Therefore, they do belong to the family of spectral separability tests. This class recovers many well-known examples of entanglement witnesses. In the present paper, we show that similarly to other spectral tests, our class of witnesses cannot be used to detect PPT entangled states. It means that these witnesses are decomposable.

The paper is organized as follows. In the next section, we recall the construction of entanglement witnesses from [22]. Section III presents several examples from the literature which do fit our class. Section IV contains our main result—proof of decomposability. Final conclusions are collected in the last section.

II. CONSTRUCTION OF THE SPECTRAL CLASS

Any entanglement witness W can be represented as a difference $W = W_+ - W_-$, where both W_+ and W_- are semipositive operators in $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. However, there is no general method to recognize that W defined by $W_+ - W_-$ is indeed an EW. One particular method based on spectral properties of W was presented in [22]. Let $\psi_\alpha(\alpha=1,\ldots,D=d_A d_B)$ be an orthonormal basis in $\mathcal{H}_A\otimes\mathcal{H}_B$ and denote by P_α the corresponding projector $P_\alpha=|\psi_\alpha\rangle\langle\psi_\alpha|$. It leads therefore to the following spectral resolution of identity:

$$\mathbb{I}_A\otimes\mathbb{I}_B=\sum_{\alpha=1}^{D}P_\alpha. \tag{5}$$

Now, take D semipositive numbers $\lambda_\alpha\geq 0$ such that λ_α is strictly positive for $\alpha>L$ and define

$$W_-=\sum_{\alpha=1}^{L}\lambda_\alpha P_\alpha, W_+=\sum_{\alpha=L+1}^{D}\lambda_\alpha P_\alpha, \tag{6}$$

where L is an arbitrary integer $0<L<D$. This construction guarantees that W_+ is strictly positive and all zero modes and strictly negative eigenvalues of W are encoded into W_-. Consider normalized vector $\psi\in\mathcal{H}_A\otimes\mathcal{H}_B$ and let

$$s_1(\psi)\geq\cdots\geq s_d(\psi)$$

denote its Schmidt coefficients ($d=\min\{d_A,d_B\}$). For any $1\leq k\leq d$, one defines k norm of ψ by the following formula [23]:

$$\|\psi\|_k^2=\sum_{j=1}^{k}s_j^2(\psi). \tag{7}$$

It is clear that

$$\|\psi\|_1\leq\|\psi\|_2\leq\cdots\leq\|\psi\|_d. \tag{8}$$

Note that $\|\psi\|_1$ gives the maximal Schmidt coefficient of ψ, whereas due to the normalization, $\|\psi\|_d^2=\langle\psi|\psi\rangle=1$. In particular, if ψ is maximally entangled then

$$\|\psi\|_k^2=\frac{k}{d}. \tag{9}$$

Equivalently, one may define k norm of ψ by

$$\|\psi\|_k^2=\max_{\phi}|\langle\psi|\phi\rangle|^2, \tag{10}$$

where the maximum runs over all normalized vectors ϕ such that $\mathcal{R}_{\rm Sch}(\psi)\leq k$ (such ϕ is usually called k separable). Recall that a Schmidt rank of $\psi[\mathcal{R}_{\rm Sch}(\psi)]$ is the number of nonvanishing Schmidt coefficients of ψ. One calls entanglement witness W a Schmidt rank k EW if $\langle\psi|W|\psi\rangle\geq 0$ for all ψ such that $\mathcal{R}_{\rm Sch}(\psi)\leq k$. The main result of [22] consists in the following

Theorem 1. Let $\sum_{\alpha=1}^{L}\|\psi_\alpha\|_k^2<1$. If the following spectral conditions are satisfied:

$$\lambda_\alpha\geq\mu_k,\quad \alpha=L+1,\ldots,D, \tag{11}$$

where

$$\mu_\ell:=\frac{\sum_{\alpha=1}^{L}\lambda_\alpha\|\psi_\alpha\|_\ell^2}{1-\sum_{\alpha=1}^{L}\|\psi_\alpha\|_\ell^2}, \tag{12}$$

then W is a k EW. If moreover $\sum_{\alpha=1}^{L}\|\psi_\alpha\|_{k+1}^2<1$ and

$$\mu_{k+1}>\lambda_\alpha,\quad \alpha=L+1,\ldots,D, \tag{13}$$

then W being a k EW is not a $(k+1)$ EW.

III. EXAMPLES

Surprisingly this simple construction recovers many well-known examples of EWs.

Example 1. Flip operator in $d_A=d_B=2$,

$$W=\begin{pmatrix}1&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&1\end{pmatrix}, \tag{14}$$

where dots represent zeros. Its spectral decomposition has the following form: $W_-=\lambda_1 P_1$,

$$\lambda_1=\lambda_2=\lambda_3=\lambda_4=1,$$

and

$$\psi_1=\frac{1}{\sqrt{2}}(|12\rangle-|21\rangle),$$

$$\psi_2=\frac{1}{\sqrt{2}}(|12\rangle+|21\rangle),$$

$$\psi_3=|11\rangle,\quad \psi_4=|22\rangle.$$

One finds $\mu_1=1$ and hence condition (11) is trivially satisfied $\lambda_\alpha\geq\mu_1$ for $\alpha=2,3,4$. We stress that our construction does not recover flip operator in $d>2$. It has $d(d-1)/2$ negative eigenvalues. Our construction leads to at most $d-1$ negative eigenvalues.

Example 2. Entanglement witness corresponding to the reduction map

$$\lambda_1=d-1,\quad \lambda_2=\cdots=\lambda_D=1$$

and

$$W_-=P_d^+,\quad W_+=\mathbb{I}_d\otimes\mathbb{I}_d-P_d^+, \tag{15}$$

where P_d^+ denotes maximally entangled state in $\mathbb{C}^d\otimes\mathbb{C}^d$. Again, one finds $\mu_1=1$ and hence condition (11) is trivially satisfied $\lambda_\alpha\geq\mu_1$ for $\alpha=2,\ldots,D=d^2$. Now, since ψ_1 corresponds to the maximally entangled state, one has $1-\|\psi_1\|_2^2=(d-2)/d<1$. Hence, condition (13)

$$\mu_2=2\frac{d-2}{d-1}>\lambda_\alpha,\quad \alpha=2,\ldots,D \tag{16}$$

implies that W is not a Schmidt rank 2 EW.

Example 3. A family of k EW in $\mathbb{C}^d \otimes \mathbb{C}^d$ defined by [24]

$$\lambda_1 = pd - 1, \quad \lambda_2 = \ldots = \lambda_D = 1,$$

with $p \geq 1$ and

$$W_- = P_d^+, \quad W_+ = \mathbb{I}_d \otimes \mathbb{I}_d - P_d^+. \tag{17}$$

Clearly, for $p=1$, it reproduces the reduction EW. Now, conditions (11) and (13) imply that if

$$\frac{1}{k+1} < p \leq \frac{1}{k}, \tag{18}$$

then W is k but not $(k+1)$ EW.

Example 4. A family of EWs in $\mathbb{C}^3 \otimes \mathbb{C}^3$ defined by [25]

$$W[a,b,c] = \begin{pmatrix} a & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a \end{pmatrix}, \tag{19}$$

with $a,b,c \geq 0$. Necessary and sufficient conditions for $W[a,b,c]$ to be an EW are

(1) $0 \leq a < 2$,

(2) $a+b+c \geq 2$,

(3) if $a \leq 1$, then $bc \geq (1-a)^2$.

A family $W[a,b,c]$ generalizes celebrated Choi indecomposable witness corresponding to $a=b=1$ and $c=0$. Now, spectral properties of $W[a,b,c] = W_+ - W_-$ read as follows: $W_- = \lambda_1 P_3^+$ and

$$\lambda_1 = 2 - a, \quad \lambda_2 = \lambda_3 = a + 1,$$

$$\lambda_4 = \lambda_5 = \lambda_6 = b, \quad \lambda_7 = \lambda_8 = \lambda_9 = c.$$

One finds $\mu_1 = (2-a)/2$ and hence condition (11) implies

$$a \geq 0, \quad b,c \geq \frac{2-a}{2}. \tag{20}$$

It gives therefore

$$a + b + c \geq 2 \tag{21}$$

and one easily shows that the conditions 3 is also satisfied. Summarizing, $W[a,b,c]$ belongs to our spectral class if and only if

(1) $0 \leq a < 2$,

(2) $b,c \geq (2-a)/2$.

Note that the Choi witness $W[1,1,0]$ does not belong to this class. It was shown [25] that $W[a,b,c]$ is decomposable if and only if $a \geq 0$ and

$$bc \geq \frac{(2-a)^2}{4}. \tag{22}$$

Hence, $W[a,b,c]$ from our spectral class is always decomposable. In particular, $W[0,1,1]$ reproduces the EW corresponding to the reduction map in $d=3$. Note that there are entanglement witnesses $W[a,b,c]$ which are decomposable, i.e., satisfy Eq. (22), but do not belong to or spectral class. Similarly one can check when $W[a,b,c]$ defines rank 2 EW. One finds $\mu_2 = 2(2-a)$ and hence condition (11) implies

(1) $1 \leq a < 2$,

(2) $b,c \geq 2(2-a)$.

Clearly, any rank 2 EW from our class is necessarily decomposable. It was shown [25] that all rank 2 EWs from the class $W[a,b,c]$ are decomposable.

Interestingly, all examples show one characteristic feature—entanglement witnesses satisfying spectral conditions (11) are decomposable. In the next section, we show that it is not an accident.

IV. DECOMPOSABILITY OF THE SPECTRAL CLASS

Indeed, we show that if entanglement witness W does satisfy Eq. (11) with $k=1$, then it is necessarily decomposable. It means that if ρ is PPT, then it cannot be detected by W,

$$\rho^\Gamma \geq 0 \Rightarrow \mathrm{Tr}(\rho W) \geq 0. \tag{23}$$

To prove it, note that

$$W = A + B, \tag{24}$$

where

$$A = \sum_{\alpha=L+1}^{D} (\lambda_\alpha - \mu_1) P_\alpha \tag{25}$$

and

$$B = \mu_1 \mathbb{I}_A \otimes \mathbb{I}_B - \sum_{\alpha=1}^{L} (\lambda_\alpha + \mu_1) P_\alpha. \tag{26}$$

Now, since $\lambda_\alpha \geq \mu_1$, for $\alpha = L+1, \ldots, D$, it is clear that $A \geq 0$. The partial transposition of B reads as follows:

$$B^\Gamma = \mu_1 \mathbb{I}_A \otimes \mathbb{I}_B - \sum_{\alpha=1}^{L} (\lambda_\alpha + \mu_1) P_\alpha^\Gamma. \tag{27}$$

Let us recall that the spectrum of the partial transposition of rank-1 projector $|\psi\rangle\langle\psi|$ is well known: the nonvanishing eigenvalues of $|\psi\rangle\langle\psi|^\Gamma$ are given by $s_\alpha^2(\psi)$ and $\pm s_\alpha(\psi) s_\beta(\psi)$, where $s_1(\psi) \geq \cdots \geq s_d(\psi)$ are Schmidt coefficients of ψ. Therefore, the smallest eigenvalue of B^Γ (call it $b_{\min}$) satisfies

$$b_{\min} \geq \mu_1 - \sum_{\alpha=1}^{L} (\lambda_\alpha + \mu_1) \|\psi_\alpha\|_1^2 \tag{28}$$

and using the definition of μ_1 [cf. Eq. (12)], one gets

$$b_{\min} \geq 0, \tag{29}$$

which implies $B^\Gamma \geq 0$. Hence, due to the formula (24), the entanglement witness W is decomposable.

Interestingly, saturating the bound (11), i.e., taking

$$\lambda_\alpha = \mu_1, \quad \alpha = L+1, \ldots, D, \tag{30}$$

one has $A=0$ and hence $W=Q^\Gamma$, with $Q=B^\Gamma \geq 0$, which shows that the corresponding positive map $\Lambda: \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ defined by

$$\Lambda(X) = \mathrm{Tr}_A(WX^T \otimes \mathbb{I}_B) \tag{31}$$

is completely copositive, i.e., $\Lambda \circ T$ is completely positive. Note that

$$\Lambda(X) = \mu_1 \mathbb{I}_B \,\mathrm{Tr}\, X - \sum_{\alpha=1}^{L} (\mu_1 + \lambda_\alpha) F_\alpha X F_\alpha^\dagger, \tag{32}$$

where F_α is a linear operator $F_\alpha: \mathcal{H}_A \to \mathcal{H}_B$ defined by

$$\psi_\alpha = \sum_{i=1}^{d_A} e_i \otimes F_\alpha e_i \tag{33}$$

and $\{e_1, \ldots, e_{d_A}\}$ denotes an orthonormal basis in $\mathcal{H}_A$. In particular, if $L=1$, i.e., there is only one negative eigenvalue, then formula (32) (up to trivial rescaling) gives

$$\Lambda(X) = \kappa \mathbb{I}_B \,\mathrm{Tr}\, X - F_1 X F_1^\dagger, \tag{34}$$

with

$$\kappa = \frac{\mu_1}{\mu_1 + \lambda_1} = \|\psi_1\|_1^2. \tag{35}$$

It reproduces a positive map (or equivalently an EW $W = \kappa \mathbb{I}_A \otimes \mathbb{I}_B - P_1$) which is known to be completely copositive [4,20,26]. If $d_A = d_B = d$ and ψ_1 is maximally entangled, that is, $F_1 = U/\sqrt{d}$ for some unitary $U \in U(d)$, then one finds for $\kappa = 1/d$ and the map (34) is unitary equivalent to the reduction map $\Lambda(X) = UR(X)U^\dagger$, where $R(X) = \mathbb{I}_d \,\mathrm{Tr}\, X - X$.

Finally, let us observe that EWs presented in examples 1–3 are not only decomposable but completely copositive, i.e., $W^\Gamma \geq 0$. Moreover, the flip operator (14) and the EW corresponding to the reduction map do satisfy (30). EW from example 4 fitting our spectral class is in general only decomposable but $W[a,b,c]^\Gamma \ngeq 0$. Its partial transposition becomes positive if in addition to $b,c \geq (2-a)/2$ it satisfies $bc \geq 1$. Note that condition (30) implies in this case

$$b = c = a + 1 = \frac{2-a}{2},$$

which leads to $a=0$ and $b=c=1$. This case, however, corresponds to the standard reduction map in $\mathbb{C}^3$.

V. CONCLUSIONS

We have shown that the spectral class of entanglement witnesses constructed recently in [22] contains only decomposable EWs, that is, it cannot be used to detect PPT entangled state. This observation supports other results such as entropic inequalities (2) and majorization relations (3) which are also defined via spectral conditions and turned out to be unable to detect bound entanglement. We conjecture that "spectral tools" are inappropriate in searching for bound entanglement which shows highly nonspectral features.

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