Multiple-measurement Leggett-Garg inequalities

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(Received 4 August 2009; published 18 September 2009)

In the investigation of the quantum to classical transition, the Leggett-Garg inequality represents what Bell's inequality is for the study of entanglement and nonlocality: a definite quantitative test showing the inadequacy of familiar lines of thought in interpreting quantum phenomena. Here we discuss the generalization of the Leggett-Garg inequality to an arbitrary number of measurements in the perspective of its use as a tool for characterizing nondestructive measurement devices.

DOI: 10.1103/PhysRevA.80.034102 PACS number(s): 03.65.Ta

The transition from quantum to classical world is considered one the most intriguing questions of contemporary physics. Despite its countless successes in correctly describing nature at a microscopic level, quantum mechanics has no clear indication of why quantum interferences and superpositions are not commonly experienced in the classical word. The most famous illustrating example dates back to Schrödinger's cat paper [1]: it shows that an immediate mapping of quantum properties to macroscopic objects seems to fight against our common sense. Remarkably, systems which should be considered at least mesoscopic—thus intuitively well under the domain of classical mechanics—have been prepared and measured in a genuinely quantum state, although under strictly controlled conditions [2–6]. Several solutions have been suggested to explain the transition to ordinary-life observations: either decoherence via coupling of the system with the environment [7], a spontaneous collapse of the wave function [8], or, more recently, coarse-grained measurement [9].

Leggett and Garg pointed out that such investigation should proceed by formalizing the implicit assumptions we expect to be fulfilled in classical word [10]. This would allow us to derive a quantitative test, following the example of Bell's celebrated inequality [11]. They claimed that for an object to make us comfortable with our intuition, its observations should behave according to the two postulates of *macroscopic realism* and *noninvasiveness of the measurement*. Specifically, these require the object to sit in only one of the macroscopically distinct states and that such state can be identified without affecting its future evolution. As expected, quantum mechanics does not obey the Leggett-Garg inequality (LGI): measurements on the same system at different times show correlations which cannot be explained invoking the macrorealism postulates. Very recently, experiments have confirmed the predicted violation [12,13].

Very recently, some studies have been made concerning the generalization of the LGI: Kofler and Brukner have extended LGI to systems or arbitrarily dimensions [9], while Jordan and co-workers included the possibility of performing weak measurements [14,15]. Here we present an approach to generalize the LGI in the form of Jordan *et al.* to an arbitrary number of measurements. Our work goes in the direction of developing the LGI as a tool for characterizing nondestructive measurement devices, in the same way that Bell's inequality is widely adopted these days to test the quality of entanglement.

In our model, we describe a two-level system—adequately represented by a vector on the Bloch sphere—which evolves through a series of unitary rotations and nondemolition measurements. Without loosing generality, we can assume that the initial pure state of the system is on the positive direction of the $\mathbf{Z}$ axis: $\rho_0=|0\rangle\langle 0|$. We also consider rotations $\mathbf{R}_y(\theta)$ by angle θ around the $\mathbf{Y}$ axis and nondestructive measurements $\mathcal{M}$ along the $\mathbf{Z}$ axis, following Ref. [14]. A representation of this elementary sequence is shown in Fig. 1(a). The apparatus in Fig. 1(a) is replicated several times each time collecting a measurement result which will be used later to reconstruct correlations between two successive measurements [Fig. 1(b)]. In the following, we work in the limit of zero disturbance measurements: those can give the maximal violation of the LGI, as they entirely preserve the correlations between measurement outcomes.

The simplest LGI can be written for three measurements,

$$\langle B_3\rangle=\langle\mathcal{M}^{(1)}\mathcal{M}^{(2)}\rangle+\langle\mathcal{M}^{(2)}\mathcal{M}^{(3)}\rangle-\langle\mathcal{M}^{(1)}\mathcal{M}^{(3)}\rangle\leq 1. \tag{1}$$

Notice that each measurement $\mathcal{M}^{(i)}$ is preceded by a rotation $\mathbf{R}_y^{(i-1)}(\theta^{(i-1)})$, except for initial preparation. For the sake of simplicity in the notation, we leave this as implicit. The dem-

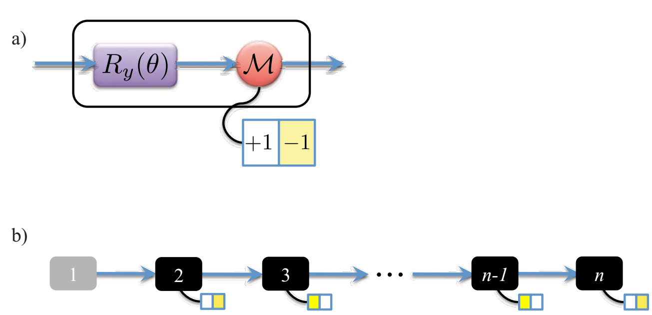

FIG. 1. (Color online) Scheme of the multimeasurement LGI setup. (a) The procedure consists of a series of $n-1$ replicas of the apparatus illustrated. It performs a rotation around the $\mathbf{Y}$ axis, followed by a nondestructive measurement on the $\mathbf{Z}$ axis. As an outcome, the ± 1 result of the weak measurement is provided to the experimenter. (b) A chain of the replicas—represented by the black boxes $2,3,\ldots,n-1,n$—is formed, starting with a strong measurement apparatus 1, which acts as the state preparation.

onstration of such bound is immediate: if, following prescriptions of Leggett and Garg, one assumes that the measurement outcomes are predetermined and unaffected by previous measurements, one should assign the values ± 1 to each of them, similarly to the Greenberger-Horne-Zeiliner proof of Bell's theorem [16], or in proofs of noncontextuality [17]. Jordan *et al.* [14] showed that $\mathcal{M}_1$ can actually consist of a standard measurement and coincide with the state preparation. Direct inspection of quantum mechanical predictions,

$$\langle B_3 \rangle = \cos(\theta^{(1)}) + \cos(\theta^{(2)}) - \cos(\theta^{(1)} + \theta^{(2)}), \tag{2}$$

reveals that the optimal violation $\langle B_3 \rangle = \frac{3}{2}$ is achieved for $\theta^{(1)} = \theta^{(2)} = \frac{\pi}{3}$.

Extension to four measurements was suggested by Leggett and Garg [10] and considered by Kofler and Brukner [9],

$$\langle B_4 \rangle = \langle \mathcal{M}^{(1)}\mathcal{M}^{(2)} \rangle + \langle \mathcal{M}^{(2)}\mathcal{M}^{(3)} \rangle + \langle \mathcal{M}^{(3)}\mathcal{M}^{(4)} \rangle - \langle \mathcal{M}^{(1)}\mathcal{M}^{(4)} \rangle \leq 2. \tag{3}$$

It bears close resemblance with the Clauser-Horne-Shimony-Holt form of Bell's inequality [18], where the measurement times take the place of polarizer settings [9,10]. Maximal violation $\langle B_4 \rangle = 2\sqrt{2}$ is attained for $\theta^{(i)} = \frac{\pi}{4}$. It is legitimate to ask whether this is actually the best way to obtain a four-measurement LGI [19]. Besides the parallel with Bell's inequality, this form of the LGI has a clear underlying physical motivation: the system is monitored as it evolves by observing correlations between consecutive measurements, and eventually the preparation and the last outcome are compared. A violation of such LGI tells us that the information we collected during such evolution is not consistent with the final observation assuming macrorealism. To identify such approach, we will refer to it as final consistency analysis (FCA), as it calls the system to a final account on how it behaved in between measurements.

There is an alternative choice for constructing generalized LGIs, which we will call subsequent consistency analysis (SCA). The basic idea is that we do not wait until the end to check, but we rather cut the evolution into smaller steps and test the self-consistency of each of those steps. By this procedure, we obtained a new inequality in a chained form,

$$\langle \widetilde{B}_4 \rangle = \langle \mathcal{M}^{(1)}\mathcal{M}^{(2)} \rangle + 2\langle \mathcal{M}^{(2)}\mathcal{M}^{(3)} \rangle + \langle \mathcal{M}^{(3)}\mathcal{M}^{(4)} \rangle - \langle \mathcal{M}^{(1)}\mathcal{M}^{(3)} \rangle - \langle \mathcal{M}^{(2)}\mathcal{M}^{(4)} \rangle \leq 2. \tag{4}$$

This consists of two separate three-measurement LGIs. The first inequality coincides with the one we considered above in a three-measurement experiment, while the second one involves a second triplet, starting from the second measurement. It is possible to show that the correlation function [Eq. (2)] is obtained also when preparing the initial state with a weak measurement, therefore $\langle \widetilde{B}_4 \rangle = 2\langle B_3 \rangle$. Maximal violation is $\langle \widetilde{B}_4 \rangle = 3$, attained for $\theta^{(i)} = \frac{\pi}{3}$. A frequent monitoring of our system is then able to reveal a failure of a macrorealistic model better than with a single comparison. Incidentally, we observe that combining the two approaches by writing the inequality $\langle \widetilde{B}_4 \rangle - \langle \mathcal{M}^{(1)}\mathcal{M}^{(2)} \rangle$ presents no benefits. The difference between the macrorealistic bound $\langle \widetilde{B}_4 \rangle - \langle \mathcal{M}^{(1)}\mathcal{M}^{(2)} \rangle \leq 3$ and the maximal quantum value $\langle \widetilde{B}_4 \rangle - \langle \mathcal{M}^{(1)}\mathcal{M}^{(2)} \rangle = 4$ remains the same as the one obtained by SCA.

The clear advantage of SCA as compared to the FCA becomes evident when one considers the scaling with the number of measurements n; the macrorealistic limit is the same in both cases,

$$\langle B_n \rangle \leq n - 2, \tag{5}$$

$$\langle \widetilde{B}_n \rangle \leq n - 2. \tag{6}$$

In the first case, the correlations give

$$\langle B_n \rangle = \sum_{i=1}^{n-1} \cos \theta^{(i)} - \cos \sum_{i=1}^{n-1} \theta^{(i)}, \tag{7}$$

which is maximized in the symmetric case $\theta^{(i)} = \frac{\pi}{n}$, reaching the value

$$\langle B_n \rangle = n \cos \frac{\pi}{n}. \tag{8}$$

The scaling does not compare favorably to the one achieved by chaining $(n-2)$ consecutive three-measurement LGIs,

$$\langle \widetilde{B}_n \rangle = \tfrac{3}{2}(n-2). \tag{9}$$

More generally, in an experiment involving n different measurements—including the preparation—one can consider all possible LGIs obtained by selecting k measurements; we will indicate such inequalities as k m LGIs. If one considers, for instance, all 3 and 4 m inequalities, the corresponding LGIs for equally spaced rotations are written as [20]

$$\langle B_{n,3} \rangle = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \sum_{l=j+1}^{n-1} \cos(j-i)\theta + \cos(l-j)\theta - \cos(l-i)\theta \leq \binom{n}{3}, \tag{10}$$

$$\langle B_{n,4} \rangle = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \sum_{p=j+1}^{n-1} \sum_{q=p+1}^{n-1} \cos(j-i)\theta + \cos(p-j)\theta + \cos(q-p)\theta - \cos(q-i)\theta \leq 2\binom{n}{4}. \tag{11}$$

The violation of those two inequalities is inspected by numerical optimization; results are summarized by the histograms in Fig. 2 showing the difference between the maximal value achieved by quantum correlations $\langle B_{n,k} \rangle$ and the macrorealistic limit $L_{n,k}$ for the cases $k=3,4,5$. This clearly shows a competition between two different mechanisms. While a shorter subinequality can provide a larger violation by itself, it should also consider that the number of possible k m subinequalities grows with k, thus larger k provides more possible tests for macrorealism. Focusing, for instance, on the relative behavior of the 3 and 4 m chainings, it can see that the latter becomes more advantageous starting at $n \geq 6$; at this point the possibility of putting together more tests overcomes the one of having more efficient ones. In the same

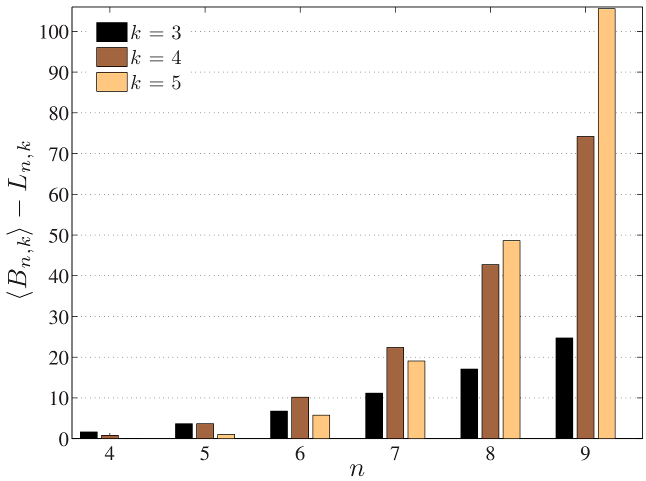

FIG. 2. (Color online) Violation of the Leggett-Garg inequality generalized by considering all groups of k out of n measurements. On the y axis we report the difference between the maximal value achievable with quantum correlations $\langle B_{n,k}\rangle$ and the macrorealistic limit $L_{n,k}$.

way, we observe that for $n \geq 8$ a 5 m chaining works better than a 4 m one. A reasonable conjecture is that there exists a value n_k for which the $(k+1)$ m inequality achieves better violation than the k m one.

While from the theoretical point of view zero-strength measurements are the most interesting limit, their exact implementation is impossible in a laboratory where they can only be performed with an arbitrarily small but finite strength. A recent test [12] has actually shown that a sufficiently weak measurement can yield violations of the LGI which are numerically close to the ideal case; nevertheless, it is likely that several small disturbances might seriously affect the maximal achievable violation. The SCA approach is quite robust against this problem, as it only tests one measurement at the time: the actual strength of the initial and final measurements at each stage does not enter in the correlation function of Eq. (1) [14,15].

In conclusion, we presented some possible generalizations of the Leggett-Garg inequality when considering an arbitrary number of measurements on the system. We have shown how it is possible to observe an enhancement in the violation of macrorealism by an appropriate selection of chained subinequalities. Each of these subinequalities involves a number k of measurements, which should be chosen depending on the number n of total measurements performed in the experiment.

We acknowledge stimulating discussion with A. Cabello and Ph. Grangier. This work was supported by project MCQM of the RTRA “Triangle de la Physique.”

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[19] Based on intuition, we build a hierarchical scale for LGIs based on the difference between the quantum expectation value and the classical bound. In different contexts, alternative definitions might provide a better insight: see A. Cabello, Phys. Rev. Lett. **97**, 140406 (2006); M. Barbieri, P. Mataloni, F. De Martini, G. Vallone, and A. Cabello,*ibid.* **97**, 140407 (2006). On a separate note, we observe that experimentalists might prefer to test the one including less terms, both for the sake of simplicity and because shorter chains might lead to smaller statistical uncertainties (see [12,13]).
[20] We have not been able to find an optimality proof of the symmetrical choice of angles, although optimality it is partially supported by some numerical calculations. Intuition suggests that this should be the case since biasing one of the angles to enhance the violation of one of the subinequalities in Eq. (10) or (11) might reduce the amount of violation in many others involving that same term.