Approximate phase-coherent states and their generation

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We show that phase-coherent states of a single-mode quantized electromagnetic field are well approximated by certain superpositions of the squeezed vacuum and one-photon states. We further show that the squeezed vacuum and one-photon states can be understood, approximately, as superpositions of phase-coherent states. Our approximate phase-coherent states may be generated by a degenerate parametric down-converter acting on a prepared input state of the form $|0\rangle+\alpha|1\rangle$. The input superposition state itself can be prepared from a coherent state $|\alpha\rangle$ by a quantum scissors device.

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I. INTRODUCTION

The phase-coherent states (PCSs), here denoted $|z\rangle$, of a single-mode quantized field (a harmonic oscillator) described by the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, respectively, are eigenstates of the Susskind-Glogower (SG) [1] phase operator $\hat{e}=(\hat{a}\hat{a}^\dagger)^{-1/2}\hat{a}$ according to $\hat{e}|z\rangle=z|z\rangle$ [2], where z is a complex number within the unit circle, $0\leq|z|<1$. In terms of the photon number states, the PSCs have the form

$$|z\rangle=(1-|z|^2)^{1/2}\sum_{n=0}^{\infty}z^n|n\rangle. \tag{1}$$

These states are closely related to the so-called phase states given by

$$|\theta\rangle=\sum_{n=0}^{\infty}e^{in\theta}|n\rangle, \tag{2}$$

which are also eigenstates of the SG phase operator, i.e., $\hat{e}|\theta\rangle=e^{i\theta}|\theta\rangle$. These are identical to the unnormalized form of the states of Eq. (1), the states $|z\rangle\rangle\equiv\sum_{n=0}^{\infty}z^n|n\rangle$, if we set $z=\rho e^{i\theta}$ and take the limit that $\rho\to 1$ from below, such that

$$|\theta\rangle=\lim_{\rho\to 1-}|\rho e^{i\theta}\rangle\rangle. \tag{3}$$

The phase states are not normalizable and thus are not physical states. But the PCSs of Eq. (1) are normalizable and are thus physical states that share some of the important properties of the unphysical phase states.

As far as we are aware, the PCSs were first introduced by Lerner *et al.* [3], who noticed that the photon number distribution for the state resembles a thermal distribution in that

$$P_n^{(\mathrm{PSC})}=|\langle n|z\rangle|^2=(1-|z|^2)|z|^{2n}=\frac{\bar{n}^n}{(1+\bar{n})^{n+1}}, \tag{4}$$

where $\bar{n}=|z|^2/(1-|z|^2)$ is the average photon number. If we set $z=e^{-\beta\hbar\omega/2}$, we obtain exactly a thermal distribution, though here we have a pure state. The distribution is pictured in Fig. 1 for $z=0.7$, which gives an average photon number $\bar{n}=0.96$. The phase-coherent states, because they are eigenstates of the SG phase operator, are "coherent" in the sense that [2]

$$\langle z|\hat{e}^{\dagger N}\hat{e}^M|z\rangle=z^{*N}z^M, \tag{5}$$

just as the ordinary coherent states

$$|\alpha\rangle=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{6}$$

are coherent states in the sense that

$$\langle\alpha|\hat{a}^{\dagger N}\hat{a}^M|\alpha\rangle=\alpha^{*N}\alpha^M. \tag{7}$$

Also, like the ordinary coherent states, the PCSs are not orthogonal

$$\langle z'|z\rangle=\frac{[(1-|z'|^2)(1-|z|^2)]^{1/2}}{1-z'^*z}. \tag{8}$$

The phase-coherent states have been studied by many authors who have noted their connection to certain representations of the Lie algebra su(1,1) [3,4]. They were reintroduced by Shapiro *et al.* [2] as optimum phase states for both the Süssman and the reciprocal peak likelihood [2,5,6] measures of the phase uncertainty. In a different context, they could serve as seed states [7–9] in sampling canonical phase distributions through unconventional heterodyne detection as described by D'Ariano and Sacchi [10]. Further, D'Ariano and Macchiavello [11] showed that the PCS maintain phase coherence under amplification and thus are privileged states for phase-based communication channels [12].

D'Ariano [13] and D'Ariano *et al.* [14] previously presented schemes for the generation of PCS. They noted that the two-mode squeezed vacuum state [15]

$$|z\rangle_{ab}=(1-|z|^2)^{1/2}\sum_{n=0}^{\infty}z^n|n\rangle_a|n\rangle_b \tag{9}$$

has exactly the same set of expansion coefficients as do the PCS. The state $|z\rangle_{ab}$ is an eigenstate of the product of SG phase operators of the two modes a and b [16], i.e.,

$$(\hat{a}\hat{a}^\dagger)^{-1/2}(\hat{b}\hat{b}^\dagger)^{-1/2}\hat{a}\hat{b}|z\rangle_{a,b}=z|z\rangle_{a,b}. \tag{10}$$

Of course, the two-mode squeezed vacuum state is generated by the interaction Hamiltonian

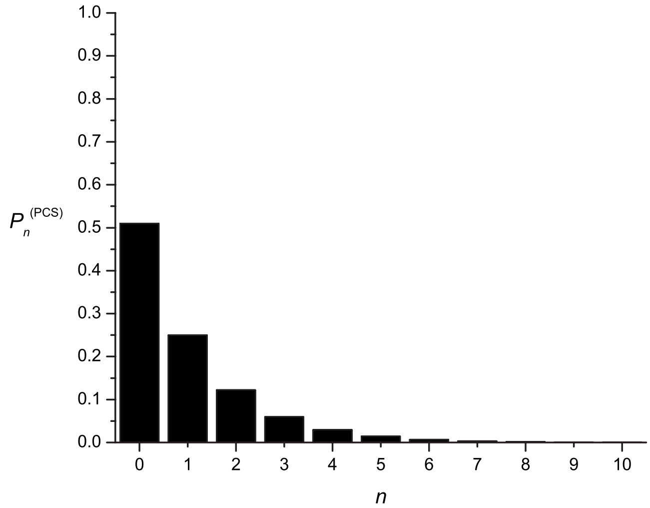

FIG. 1. The photon number distribution for the PSC plotted against photon number for $z=0.7$.

$$\hat{H}_{\mathrm{I}}=i\hbar\chi(\hat{a}\hat{b}-\hat{a}^{\dagger}\hat{b}^{\dagger}), \tag{11}$$

where χ is proportional to a second-order nonlinear susceptibility $\chi^{(2)}$. The scheme in Refs. [13,14] requires a device that performs the photon number recombination

$$|n\rangle_a|n\rangle_b|0\rangle_c \rightarrow |0\rangle_a|0\rangle_b|n\rangle_c. \tag{12}$$

Up-conversion is required, but, in order not to alter the coefficients, an interaction with intensity-dependent couplings of a form not realizable by any known optical device is necessary. However, D'Ariano *et al.* [14] showed that it is possible to replace the intensity-dependent factors by their averages such that one could use the standard trilinear interaction of the form

$$\hat{H}_{\mathrm{I}} \sim \chi^{(2)}[\hat{a}\hat{b}\hat{c}^{\dagger}+\hat{a}^{\dagger}b^{\dagger}\hat{c}], \tag{13}$$

where the frequencies for the modes satisfy the relation $\omega_c=\omega_a+\omega_b$. Acting on the initial state $|z\rangle_{ab}|0\rangle_c$, this interaction *approximately* produces the state $|0\rangle_{ab}|z\rangle_c$, that is, that mode c is approximately in a PCS. The approach just described requires two $\chi^{(2)}$ interactions where both signal and idler modes (the down-converted modes) of the first interaction are taken as inputs of the second interaction in such a way as to perform twin- beam upconversion.

II. SUPERPOSITIONS OF SQUEEZED VACUUM AND ONE-PHOTON STATES

In the present work, we take a rather different approach to generating approximate PCS. A single-mode squeezed vacuum state has a thermal-like photon number distribution as does a squeezed one-photon state, though only with every other photon number state populated; only the even for the former and only the odd for the latter. We therefore consider a superposition of the squeezed vacuum and one-photon states as possible approximate realizations of the PCS. We choose parameters of the states and for the superposition based on the maximization of its fidelity with a desired PCS.

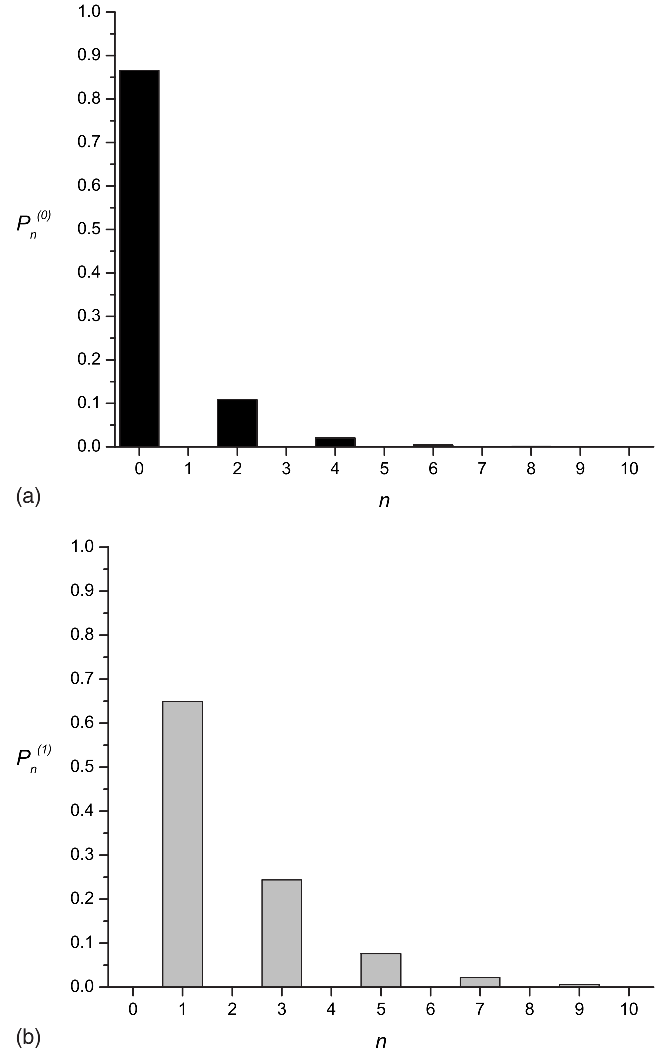

FIG. 2. Photon number distribution versus photon number for (a) the squeezed vacuum state and (b) the squeezed one-photon state, both for the choice $\xi=0.5$.

Generation of the superposition requires just one degenerate down-conversion, a $\chi^{(2)}$ interaction acting on an initial state consisting of a superposition of the vacuum, and one-photon states. We suggest that the initial superposition state can be generated by a quantum scissors process, which does require a single photon, which could be produced via a nondegenerate $\chi^{(2)}$ process, though quantum scissoring itself can be performed with linear optics. There would actually be an advantage to this method in that one photon from the nondegenerate down-conversion could be detected and thus herald the production of its twin, which is used for the scissoring process.

We begin by considering the squeezed vacuum and squeezed one-photon states given, respectively, by

$$|\xi,0\rangle=(1-|\xi|^2)^{1/4}\sum_{m=0}^{\infty}\left[\frac{\Gamma(m+1/2)}{m\,!\,\Gamma(1/2)}\right]^{1/2}\xi^m|2m\rangle, \tag{14}$$

and

$$|\xi,1\rangle = (1-|\xi|^2)^{3/4}\sum_{m=0}^{\infty}\left[\frac{\Gamma(m+3/2)}{m\,!\,\Gamma(3/2)}\right]^{1/2}\xi^m|2m+1\rangle. \tag{15}$$

Note that $\langle\xi,1|\xi,0\rangle=0$ as the states $|\xi,0\rangle$ and $|\xi,1\rangle$ contain only even and odd number states, respectively. The states are generated by the action of the squeeze operator $\hat{S}(\beta)=\exp[\frac{1}{2}(\beta^*\hat{a}^2-\beta\hat{a}^{\dagger 2})]$, where $\beta=re^{i\theta}$, r being the squeeze parameter, on the vacuum and one-photon states, respectively. The parameter $\xi=-e^{i\theta}\tanh r$ and clearly $|\xi|<1$. In the squeezed vacuum state, only the even number states are populated while in the squeezed one-photon state only the odd appears. The corresponding photon number distributions are given by

$$P_n^{(0)} = |\langle n|\xi,0\rangle|^2 = (1-|\xi|^2)^{1/2}\left[\frac{\Gamma(m+1/2)}{(m)\,!\,\Gamma(1/2)}\right]|\xi|^{2m}\delta_{n,2m}, \tag{16}$$

and

$$P_n^{(1)} = |\langle n|\xi,1\rangle|^2 = (1-|\xi|^2)^{3/2}\left[\frac{\Gamma(m+3/2)}{(m)\,!\,\Gamma(3/2)}\right]|\xi|^{2m+1}\delta_{n,2m+1}. \tag{17}$$

In Fig. 2, we plot these distributions against n for the same value of ξ. They are both thermal-like (they are generated a spontaneous process); but every other photon number state, the odd in the former, the even in the latter, is missing. This suggests that a superposition of the squeezed vacuum and squeezed one-photon states might be close to the to the phase-coherent states. But note that when juxtaposed on the same graph, as in Fig. 3, the distributions do not smoothly match the distribution for the phase-coherent state of Fig. 1, there being an oscillation in the distribution not present for the phase-coherent states. Thus, a balanced superposition of the form $(|\xi,0\rangle+e^{i\delta}|\xi,1\rangle)/\sqrt{2}$ cannot represent or approximate a phase-coherent state.

We therefore consider a more general superposition of the form

$$|\psi_{\rm SS}\rangle = \frac{1}{\sqrt{1+|\alpha|^2}}(|\xi,0\rangle + \alpha|\xi,1\rangle), \tag{18}$$

which in terms of the number states can be expanded as $|\psi\rangle=\Sigma_{n=0}^{\infty}C_n|n\rangle$, where

$$C_n = \begin{cases} \dfrac{1}{\sqrt{1+|\alpha|^2}}(1-|\xi|^2)^{1/4}\left[\dfrac{\Gamma\left(\dfrac{n+1}{2}\right)}{\left(\dfrac{n}{2}\right)!\,\Gamma\left(\dfrac{1}{2}\right)}\right]^{1/2}\xi^{n/2}, & n \text{ even} \\[2ex] \dfrac{\alpha}{\sqrt{1+|\alpha|^2}}(1-|\xi|^2)^{3/4}\left[\dfrac{\Gamma\left(\dfrac{n+2}{2}\right)}{\left(\dfrac{n-1}{2}\right)!\,\Gamma\left(\dfrac{3}{2}\right)}\right]^{1/2}\xi^{(n-1)/2}, & n \text{ odd}. \end{cases} \tag{19}$$

We then search for parameters ξ and α such that $|\psi\rangle$ is a close as possible to a target phase-coherent state $|z\rangle$. To this end, we calculate the fidelity

$$F = |\langle z|\psi_{\rm SS}\rangle|^2 = (1-|z|^2)\left|\sum_{n=0}^{\infty} z^{*n}C_n\right|^2 \tag{20}$$

and numerically maximize it with respect to ξ and α for a given value of z. We assume all these parameters to be real.

In Table I, we list for a series of values of z the corresponding values of ξ and α obtained through maximizing F, along with the corresponding value of F itself. We see that the fidelity remains high, $F>0.9$, over the entire range considered. In Fig. 4, we plot together the photon number distributions of the corresponding phase-coherent states $P_n^{(\rm PSC)}$ and that of the superposition state $P_n^{(\rm SS)}=|\langle n|\psi_{\rm SS}\rangle|^2=|C_n|^2$ for select values of the parameters from Table I. For low values of z (0.1, 0.3, and 0.5), we see that the photon number distributions are very closely matched. But as z increases to 0.7 and 0.9, the matching is not so good even though we still have $F>0.9$. Thus, there appears to be smearing in the fidelity for these higher values of z. As a further comparison, we plot the quasiprobability distribution known at the function Q function, given, for as an arbitrary pure state $|\psi\rangle=\Sigma_{n=0}^{\infty}d_n|n\rangle$ are

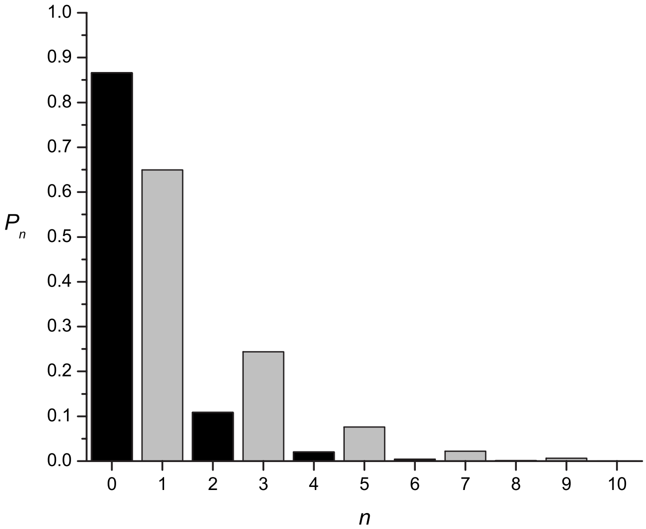

FIG. 3. A composite of Figs. 2(a) and 2(b) for the purpose of comparison.

$$Q(x,y) = |\langle\alpha|\psi\rangle|^2 = e^{-|\alpha|^2}\left|\sum_{n=0}^{\infty}\frac{(\alpha^*)^n d_n}{n!}\right|^2, \tag{21}$$

where we have set $\alpha = x + iy$. In Figs. 5(a) and 5(b), we plot to corresponding Q functions for the PCS and the superposition state, respectively, for $z=0.3$, and in Figs. 6(a) and 6(b) we repeat for $z=0.9$. For the former value of z, the Q functions are indistinguishable. For the latter, there is only a slight difference. Both are elongated along the y direction and compressed along the x direction, indicating the presence of quadrature squeezing. The similarities of the Q functions of the exact and approximate phase states is an independent measure of the closeness of the two states. The squeezing properties of the PCS were discussed by Kuang and Chen [17]. The Wigner function, determined from

$$W(x,y) = \frac{2}{\pi}\langle\hat{D}(\alpha)(-1)^{\hat{a}^\dagger\hat{a}}\hat{D}(-\alpha)\rangle, \tag{22}$$

we also calculate for our states and are shown in Figs. 7 and 8. Again, we find close resemblances of the Wigner functions between the exact and approximate phase states. Note also regions of phase space where the Wigner function is negative, an indication of the nonclassicality of the phase-coherent states.

We notice from Table I that the values of α that we obtain numerically are very close to the corresponding assigned values of z. This suggests that the PCS can be approximated by

$$|z\rangle \approx \frac{1}{\sqrt{1+|z|^2}}[|\xi,0\rangle + z|\xi,1\rangle]. \tag{23}$$

Evidently, we can then write

$$|-z\rangle \approx \frac{1}{\sqrt{1+|z|^2}}[|\xi,0\rangle - z|\xi,1\rangle] \tag{24}$$

and thus it follows that

TABLE I. We list the values of ξ, α, and the fidelity F of Eq. (20) obtained by numerically maximizing F for the given values of z.

z	ξ	α	F
0.1	0.0139671	0.0999974	0.999999
0.2	0.0539696	0.199931	0.99997
0.3	0.115547	0.299606	0.999716
0.4	0.19417	0.398845	0.998709
0.5	0.286957	0.497774	0.996107
0.6	0.393056	0.597011	0.990875
0.7	0.513417	0.697778	0.981808
0.8	0.650852	0.802258	0.9672
0.9	0.810526	0.914413	0.943402

$$|\xi,0\rangle \approx \frac{1}{2}(1+|z|^2)^{1/2}[|z\rangle + |-z\rangle], \tag{25}$$

exactly the result on the right-hand side obtained by Gerry *et al.* [18] for normalized eigenstates of the square of the SG phase operator, i.e.,

$$\hat{e}^2(|z\rangle + |-z\rangle) = z^2(|z\rangle + |-z\rangle), \tag{26}$$

where

$$\hat{e}^2 = \left(\frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}}\hat{a}\right)^2. \tag{27}$$

On the other hand, as shown by Agarwal [16], the *exact* squeezed vacuum state satisfies the eigenvalue problem

$$\frac{1}{\hat{a}\hat{a}^\dagger}\hat{a}^2|\xi,0\rangle = \xi|\xi,0\rangle, \tag{28}$$

where the operator on the left-hand side is the *approximate* square of $\hat{e}$ obtained by ignoring the commutation relation $[\hat{a},\hat{a}^\dagger]=\hat{I}$ in Eq. (27). Comparing Eqs. (26) and (28), we conclude that we should have $\xi \approx z^2$, a relation that appears to be born out in our results in Table I. Obviously, we can write the approximate squeezed one-photon state as

$$|\xi,1\rangle \approx \frac{1}{2z}(1+|z|^2)^{1/2}[|z\rangle - |-z\rangle], \tag{29}$$

the right-hand side of which is also an eigenstate of $\hat{e}^2$, whereas the left-hand side is an eigenstate of the approximate form of this operator as given on the left-hand side of Eq. (28) again leading to $\xi \approx z^2$. Thus, we can write our approximate phase state as

$$|z\rangle \approx \frac{1}{\sqrt{1+|z|^2}}[|z^2,0\rangle + z|z^2,1\rangle], \tag{30}$$

where the squeezed vacuum and one-photon states are approximately

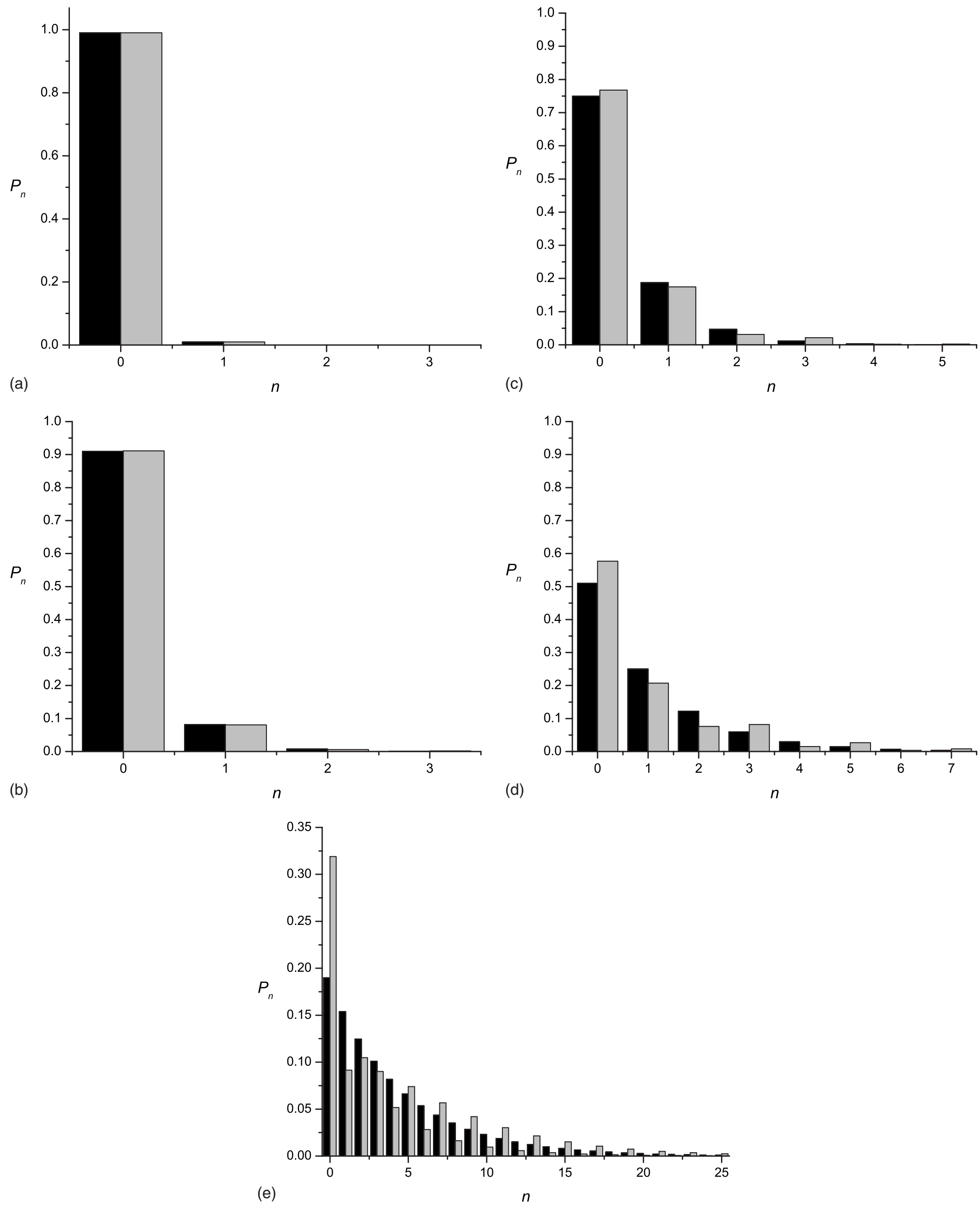

FIG. 4. Plots of wherein the bars on the left (black) are from the photon number distributions of the PCS and those on the right (grey) are from the superposition state whose parameters are determined by the numerical maximization of the fidelity as given in Table I, for (a) z=0.3, (b) z=0.4, (c) z=0.5, (d) z=0.7, and (e) z=0.9.

$$|z^2,0\rangle \approx \frac{1}{2}(1+|z|^2)^{1/2}[|z\rangle + |-z\rangle], \tag{31}$$

and

$$|z^2,1\rangle \approx \frac{1}{2z}(1+|z|^2)^{1/2}[|z\rangle - |-z\rangle], \tag{32}$$

respectively. The point of these last two equations, with $\xi \approx z^2$, is that to a good approximation the squeezed vacuum and one-photon states form Schrödinger cat states, the even and odd phase cats, respectively, analogous to the forms of cat states given in terms of the ordinary coherent states of Eq. (6), i.e., the even and odd cat states $|\alpha\rangle \pm |-\alpha\rangle$ [19,20]. Interestingly, a one-photon squeezed state well approximates the odd cat state $|\alpha\rangle - |-\alpha\rangle$ for low $|\alpha|$. The even and odd cat states are being considered as a basis for some optical quantum computer schemes [21].

III. GENERATION OF THE SUPERPOSITION STATES

We now discuss a method for producing our states. They may be generated by the action of a degenerate parametric down-converter on a prepared superposition state of the form

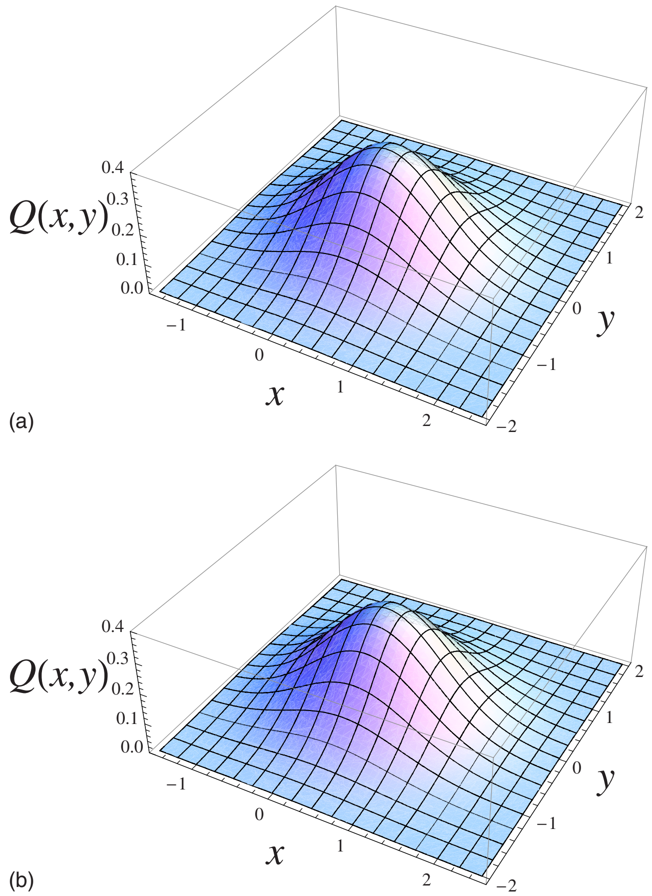

FIG. 5. (Color online) The Q function plotted as a function of x and y for $z=0.3$ for (a) the PCS state and (b) the superposition state.

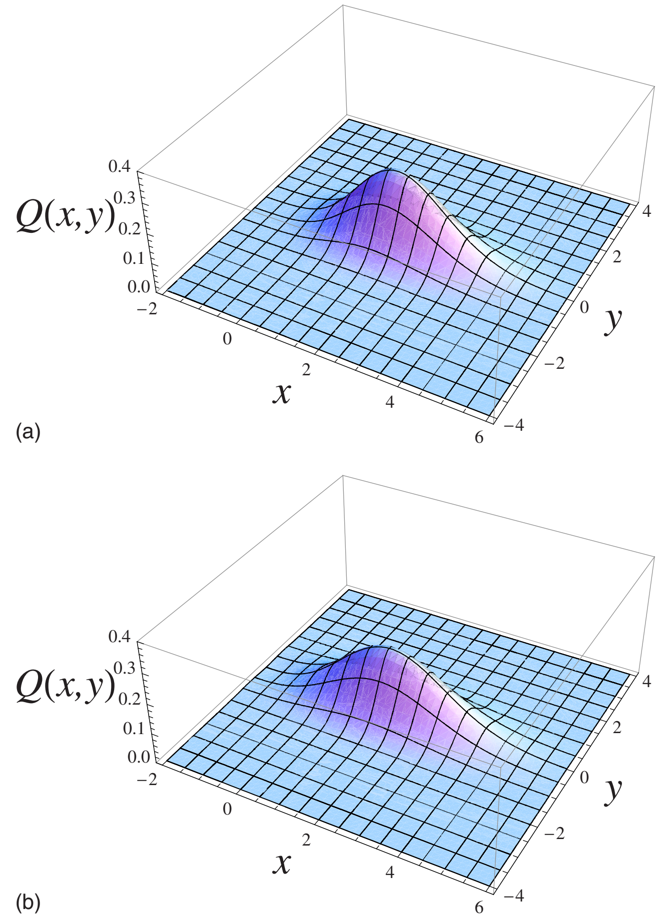

FIG. 6. (Color online) Same as Fig. 5 but for $z=0.9$.

$$|\psi_0\rangle = \frac{1}{\sqrt{1+|\alpha|^2}}(|0\rangle + \alpha|1\rangle). \tag{33}$$

The interaction Hamiltonian for a degenerate parametric down-converter can be written as

$$\hat{H}_{\mathrm{I}} = i\hbar\chi^{(2)}(\hat{a}^2\hat{b}^\dagger - \hat{a}^{\dagger 2}\hat{b}), \tag{34}$$

where the operators $\hat{b}$ and $\hat{b}^\dagger$ describe pump field and where $\chi^{(2)}$ is the second-order nonlinear susceptibility of the crystal. We shall the assume that the pump field is in a strong coherent state $|\gamma\rangle_b$ and we shall assume γ is real so that we may make the parametric approximation (PA) of replacing $\hat{b}$ and $\hat{b}^\dagger$ by γ to obtain

$$\hat{H}_{\mathrm{I}}^{\mathrm{PA}} = i\hbar\,\eta(\hat{a}^2 - \hat{a}^{\dagger 2}), \tag{35}$$

where we have set $\eta=\chi^{(2)}\gamma$. It then follows that $\exp[-i\hat{H}_{\mathrm{I}}t/\hbar]=\hat{S}(2\eta t)$ and thus that $|\psi_{\mathrm{SS}}\rangle=\hat{S}(2\eta t)|\psi_0\rangle$.

Lastly, we address the issue of the preparation of the initial state $|\psi_0\rangle$ in Eq. (33). Perhaps the best way of preparing the state is through the technique of optical state truncation, the device performing the truncation being known as a "quantum scissors," as developed by Pegg *et al.* [22]. The method, which involves a nonlocal effect, truncates any pure state of the form

$$|\Psi\rangle = \gamma_0|0\rangle + \gamma_1|1\rangle + \gamma_2|2\rangle + \ldots = \sum_{n=0}^{\infty}\gamma_n|n\rangle \tag{36}$$

to a superposition of the vacuum and one-photon states of the form

$$|\Psi'\rangle = \frac{1}{\sqrt{|\gamma_0|^2 + |\gamma_1|^2}}(\gamma_0|0\rangle + \gamma_1|1\rangle). \tag{37}$$

The quantum scissors device has been realized experimentally [23]. Of course, as we said earlier, one does need to use a single-photon source to perform the quantum scissor operation and that would typically come from a nondegenerate down-conversion process so that another $\chi^{(2)}$ nonlinear interaction is required, but that this could be of some advantage as the other twin photon can be used to herald the photon needed to perform that quantum scissoring process and, thus, signaling the generating of the initial state of the form of Eq. (37). There is also a dependence on outcomes of a measurement involving only linear optics. Obviously, truncating the coherent state of Eq. (6) in this fashion leads to the state of Eq. (33)

IV. CONCLUSIONS

In summary, we have shown that a superposition of squeezed vacuum and one-photon states can, to a good ap-

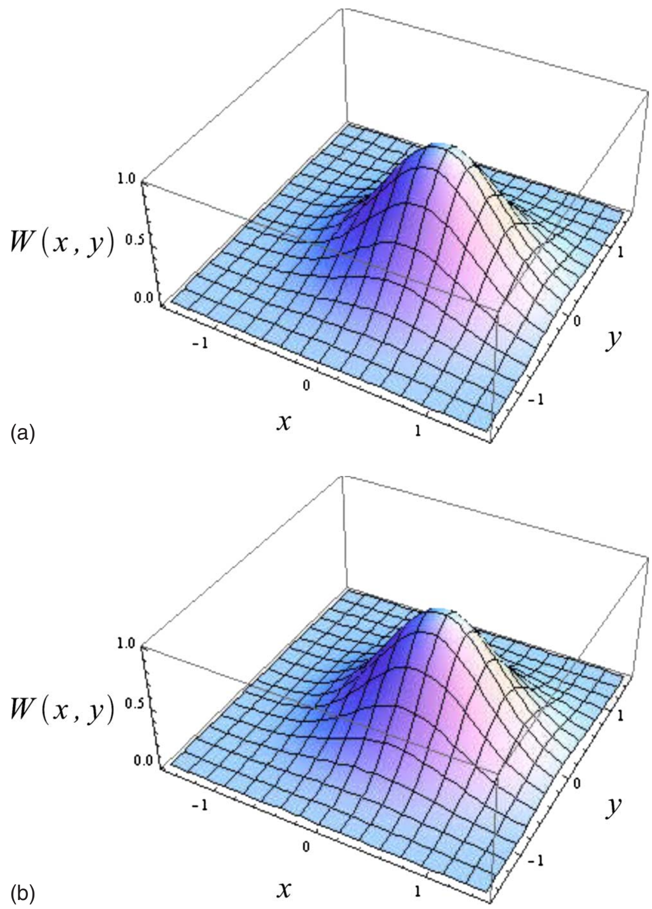

FIG. 7. (Color online) The Wigner function plotted as a function of x and y for $z=0.3$ for (a) the PCS state and (b) the superposition state.

proximation, represent phase-coherent states. We have further shown that the states can be generated by the action of a degenerate down-converter on a superposition of the vacuum and one one-photon states and that latter state prepared from a coherent state of the appropriate amplitude by the quantum scissors method of quantum state truncation. To verify the generation of our approximate phase states, it should be possible to perform quantum state tomography [24] to reconstruct the Wigner function and thus compare it with the Wigner function of the corresponding exact phase state.

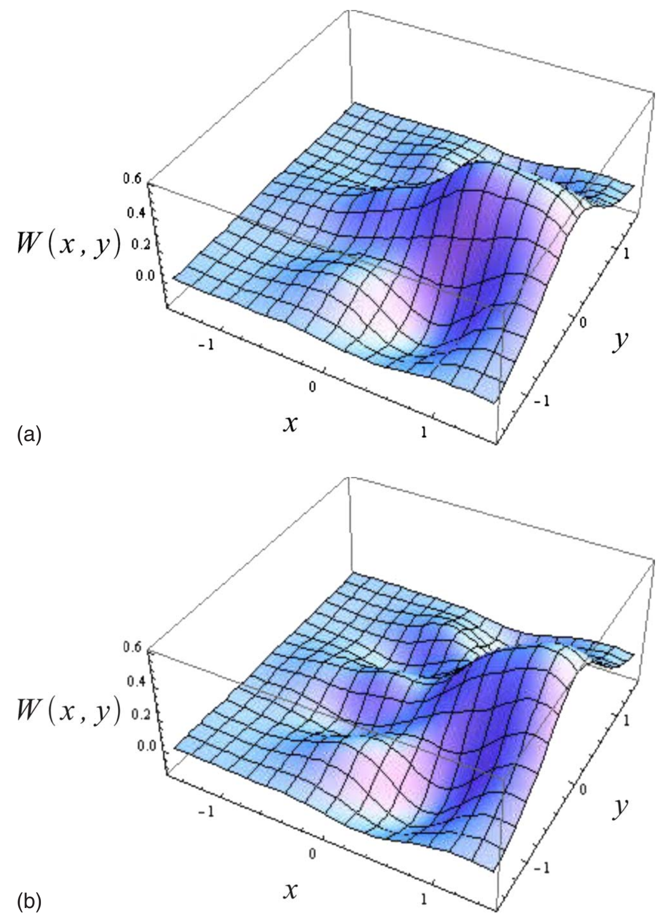

FIG. 8. (Color online) Same as Fig. 7 but for $z=0.9$.

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