Propagation of partially coherent solitons in saturable logarithmic media: A comparative analysis

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Dynamic self-similar solutions of the nonlinear Schrödinger equation are derived describing the propagation of partially coherent solitons in media with saturable logarithmic nonlinearity. The analysis is based on both the Wigner and the coherent density formalisms and it is shown that although the approaches involve different analysis, the solutions for the evolution of the physically relevant intensity distributions are equivalent.

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I. INTRODUCTION

During recent years, a strong research effort has followed the observation that soliton formation is not exclusively associated with coherent waves, but can occur also for partially coherent waves [1,2]. In situations where the response time of the medium is long compared to the characteristic time of the statistical wave intensity fluctuations of the field, the medium experiences only the statistical average of the wave intensity and concomitantly the nonlinear change of the refractive index also depends only on the statistical average of the wave intensity rather than on the instantaneous intensity as for the case of coherent waves.

Several theories have been developed for describing propagation of partially incoherent waves in nonlinear optical media: the mutual coherence function approach [3], the self-consistent multimode theory [4], the coherent density method [5], and the Wigner approach [6]. These methods can be viewed as nonlinear generalizations of previous classical methods for analyzing linear propagation of partially coherent light and have been shown to be equivalent [7,8], although the choice of the most suitable approach may depend on the nature of the physical problem to be investigated.

Most interest has been focused on soliton behavior in nonlinear Kerr media. However, few analytical solutions have been found of the corresponding nonlinear Schrödinger (NLS) equation where the nonlinearity is determined by the statistical average of the wave intensity, $\langle|\psi|^2\rangle$, with ψ denoting the slowly varying wave envelope function. A more benign type of nonlinearity from the point of view of analysis is the logarithmic nonlinearity where the change in refractive index is proportional to $\ln\langle|\psi|^2\rangle$. In this case, explicit soliton solutions of Gaussian form have been found for the coherent case [9] and later for partially coherent waves using the coherent density approach [10] and the coupled-mode theory [11] (although in the latter cases, only for the stationary case), as well as by means of the mutual coherence approach [12]. The purpose of the present work is to reconsider the problem of propagation of partially coherent light solitons in media with logarithmic nonlinearity with special emphasis on the dynamic propagation of nonstationary solitons. A comparative analysis is made by using two alternative approaches based on both the Wigner function and the coherent density function where the stationary solution obtained in [10] is generalized to the dynamic case. Although the different approaches involve different analyses, the solutions for the evolution of the physically relevant intensity distributions of the solitons are shown to be the same. However, for the present application, the Wigner approach seems to provide a more direct solution procedure, similar to that of the mutual coherence approach, and gives a clearer picture of the influence of the partial coherence on the soliton dynamics.

II. WIGNER APPROACH

Propagation of partially coherent light in a logarithmic saturable nonlinear medium is determined (in normalized variables) by the following nonlinear Schrödinger equation [10–13]:

$$i\frac{\partial \Psi}{\partial z} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2} + \ln\langle|\Psi|^2\rangle\Psi = 0. \tag{1}$$

In the Wigner approach for describing propagation of partially coherent waves, Eq. (1) is transformed to phase space by means of the Wigner transform

$$\rho(x,p,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\Psi^*(x+\xi/2,z)\Psi(x-\xi/2,z)\rangle e^{ip\xi}d\xi. \tag{2}$$

The Wigner distribution function $\rho(x,p,z)$ is a quasiprobability distribution function that in the present problem can be shown to satisfy the following Wigner-Moyal evolution equation:

$$\frac{\partial \rho}{\partial z} + p\frac{\partial \rho}{\partial x} + 2\ln\langle|\Psi|^2\rangle\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}\right)\rho = 0, \tag{3}$$

where the arrows in the sine operator indicate the direction of the respective operations. This equation is to be solved together with the following relation between the averaged wave field intensity and the Wigner function:

$$\langle|\Psi|^2\rangle = \int_{-\infty}^{\infty}\rho(x,p,z)dp. \tag{4}$$

The analytically simplifying property of the logarithmic nonlinearity is that it allows Gaussian-shaped solutions (cf. Refs. [9–13]). Thus, in order to solve Eq. (3), we look for a self-similar Gaussian solution of the form

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$$\rho(x,p,z) = A(z)\exp[-a(z)x^2 - b(z)p^2 + c(z)xp], \quad (5)$$

which has the corresponding intensity distribution

$$\langle|\Psi|^2\rangle = \sqrt{\frac{\pi}{b}}A\,\exp\left[-\frac{4ab-c^2}{4b}x^2\right]. \quad (6)$$

Inserting this ansatz into Eq. (3), we note that the sine operator truncates after its first term and after identifying coefficients for x and p, we find the following evolution equations for the parameter functions:

$$A'(z) = 0, \quad (7)$$

$$a'(z) = -\frac{c}{2b}(4ab - c^2), \quad (8)$$

$$b'(z) = c, \quad (9)$$

$$c'(z) = 2a - (4ab - c^2). \quad (10)$$

The first evolution equation immediately gives $A(z)=A_0=$const, i.e., the amplitude of the Wigner distribution is a constant. The remaining nonlinear equations for the parameters a,b, and c are easily rewritten in the form of two invariants and a single equation for the parameter b,

$$4ab - c^2 = C_1 = \text{const}; \quad a + \frac{C_1}{2}\ln b = C_2 = \text{const}, \quad (11)$$

$$[b'(z)]^2 = 4C_2 b - 2C_1 b\ln b - C_1. \quad (12)$$

Equation (12) describes the evolution of the dynamic intensity width parameter $b(z)$ (cf. Eq. (6) and note that $4ab-c^2=C_1=$const) and is equivalent to that obtained in [12] using the mutual coherence function approach.

The constants of motion found above are closely related to the properties of the moments of the Wigner function, which generally can be defined as, cf. [15],

$$\overline{x^i p^j} \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^i p^j \rho(x,p,z)\,dx\,dp. \quad (13)$$

For the zeroth-order moment, one finds

$$\int_{-\infty}^{\infty}\langle|\Psi|^2\rangle dx \equiv N = \text{const} \quad (14)$$

and the second-order moments give

$$\frac{d}{dz}\overline{x^2} = 2\overline{xp}, \quad (15)$$

$$\frac{d}{dz}\overline{xp} = \overline{p^2} - N, \quad (16)$$

$$\overline{p^2} - 2\int_{-\infty}^{\infty}\langle|\Psi|^2\rangle\ln\langle|\Psi|^2\rangle dx \equiv H = \text{const}. \quad (17)$$

The constants N and H correspond to conservations of the energy and of the Hamiltonian of the system, respectively. Using these moments, it is straightforward to show that the combination $4ab-c^2=C_1$ is a constant directly related to the energy whereas the other combination, $a+(C_1/2)\ln b=C_2$, is associated with the Hamiltonian.

The argument of the exponential in Eq. (5) can be written in a form which gives some further insight to the interpretation of the parameters

$$ax^2 + bp^2 - cxp = \frac{4ab-c^2}{4b}x^2 + b\left(p - \frac{c}{2b}x\right)^2. \quad (18)$$

The factor $\frac{c}{2b}x$ in the combination $p-\frac{c}{2b}x$ may be interpreted as a "chirp" function since it implies a shift of the variable p in the Wigner function such that one can write (cf. [14])

$$\rho_C(x,p,z) = \rho_{UC}\left(x, p - \frac{c}{2b}x, z\right). \quad (19)$$

where the subscripts C and UC denote chirped and unchirped Wigner distributions, respectively. A complementary result is obtained if the Wigner function is subjected to a translation transformation

$$\rho_T(x,p,z) = \rho_{UT}(x - \alpha(z), p - \beta(z), z), \quad (20)$$

where indices T and UT denote translated and untranslated Wigner functions, respectively. If $\alpha(z)=\beta_0 z+\alpha_0$ and $\beta(z)=\beta_0=$const, Eq. (5) remains a solution and is given by the form

$$\rho(x,p,z) = A(z)\exp\{-a(z)[x-\alpha(z)]^2 - b(z)[p-\beta(z)]^2 + c(z)[x-\alpha(z)][p-\beta(z)]\}, \quad (21)$$

with the corresponding intensity distribution

$$\langle|\Psi|^2\rangle = \sqrt{\frac{\pi}{b}}A\,\exp\left[-\frac{4ab-c^2}{4b}(x-\alpha)^2\right]. \quad (22)$$

Since the combination $4ab-c^2$ is a constant of motion, the parameter b characterizes the width of the intensity distribution. A solution of the equation for the evolution of this parameter in the nonstationary case, Eq. (12), has not been found in explicit analytical form. However, the stationary case is easily analyzed since the stationary widths of the Wigner distribution in x and p are determined directly from Eqs. (8)–(10) using the conditions $a'(z)=b'(z)=c'(z)=0$, which directly imply that $c=0$ and $b=1/2$. Assuming that the stochastic variation of the partially coherent wave has a Gaussian spectral distribution, $J(\theta)=(1/\sqrt{2\pi\theta_0^2})\exp[-\theta^2/(2\theta_0^2)]$, where the parameter θ_0 characterizes the width of the spectrum, i.e., the degree of partial incoherence, the parameter b can easily be expressed in terms of a and θ_0, *viz.* $b=(a+2\theta_0^2)^{-1}$ (cf. [15]), and the squared inverse width of the stationary intensity distribution is found to be determined by $a=2(1-\theta_0^2)$ in agreement with the corresponding results of [10,12].

III. COHERENT DENSITY APPROACH

It is instructive to carry out the corresponding analysis in terms of the coherent density formalism. Such an analysis was reported in [10], but only for the stationary case. In the present work, we will generalize this by considering the dynamic situation. The evolution equation for the coherent density function $F(x,\theta,z)$ corresponding to the NLS equation, Eq. (1), is

$$i\left(\frac{\partial F}{\partial z}+\theta\frac{\partial F}{\partial x}\right)+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}+\ln[I_N(x,z)]F=0, \tag{23}$$

together with the (separable) initial condition $F(x,\theta,0)=\sqrt{J(\theta)}\phi(x)$, where $J(\theta)$ will be taken as the normalized Gaussian spectrum used above and $\phi(x)$ is the initial spatial profile (also to be assumed Gaussian).

The intensity is coupled to the coherent density function through the relation

$$I_N(x,z)=\int_{-\infty}^{\infty}|F(x,\theta,z)|^2 d\theta. \tag{24}$$

By means of a simple transformation $F(x,\theta,z)=f(x,\theta,z)\exp(-i\theta x+i\theta^2 z/2)$, Eq. (23) may be put in the same form as the NLS equation, Eq. (1),

$$i\frac{\partial f}{\partial z}+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}+\ln[I(x,z)]f=0, \tag{25}$$

with

$$I(x,z)=\int_{-\infty}^{\infty}|f(x,\theta,z)|^2 d\theta. \tag{26}$$

Equation (23) has previously been solved under the assumption of a nonmoving solution with a stationary intensity profile [10]. Generalizing this solution, a nonstationary and moving solution of Eq. (25) would need an ansatz of the following form:

$$f(x,\theta,z)=A(z)\sqrt{J(\theta)}\exp\left\{-\frac{[x-\alpha(z)-\theta\xi(z)]^2}{2a(z)^2}+i[\delta(z)+\theta\rho(z)+\beta(z)x+\theta\mu(z)x+\theta^2\sigma(z)+\zeta(z)x^2]\right\}. \tag{27}$$

The corresponding intensity distribution is

$$I(x,z)=\frac{aA^2}{\sqrt{a^2+2\theta_0^2\xi^2}}\exp\left[-\frac{(x-\alpha)^2}{a^2+2\theta_0^2\xi^2}\right]. \tag{28}$$

The parameters in this ansatz can be given the following interpretation. $A(z)$ is an amplitude function and $a(z)$ is the width in the coherent limit. The parameter $\alpha(z)$ may be considered as a coherent translation while $\xi(z)$ is a partially coherent translation that is proportional to θ. Correspondingly, $\beta(z)$ is a coherent and $\mu(z)$ a partially coherent frequency and translation velocity respectively. Finally, $\zeta(z)$ determines the chirp variation and $\delta(z)$ is a θ independent phase while $\rho(z)$ and $\sigma(z)$ are phase parameters involving the θ dependence.

Inserting this ansatz into Eqs. (25) and (26), separating real and imaginary parts, and collecting powers of x and θ yield a complicated system of ten coupled nonlinear parameter equations. By a careful inspection of these equations, it is found that this system can be significantly simplified by using a more convenient ansatz for the phase. Thus, instead of the ansatz given by Eq. (27), we write

$$f(x,\theta,z)=A(z)\sqrt{J(\theta)}\exp\left\{-\frac{[x-\alpha(z)-\theta b(z)/\sqrt{2\theta_0^2}]^2}{2a^2(z)}+i\left\{\delta(z)+\theta\rho(z)+\beta(z)[x-\alpha(z)/2]+\theta\frac{\mu(z)}{\sqrt{2\theta_0^2}}[x-\alpha(z)-\theta b(z)/(2\sqrt{2\theta_0^2})]+\theta^2\sigma(z)+\frac{\zeta(z)}{2a(z)}[x-\alpha(z)-\theta b(z)/\sqrt{2\theta_0^2}]^2\right\}\right\}. \tag{29}$$

The intensity distribution remains the same as before except for the rescaling of $\xi(\xi\to b/\sqrt{2\theta_0^2})$,

$$I(x,z)=\frac{aA^2}{\sqrt{a^2+b^2}}\exp\left[-\frac{(x-\alpha)^2}{a^2+b^2}\right]. \tag{30}$$

Inserting the new ansatz into Eqs. (25) and (26), separating real and imaginary parts, and matching powers of x and θ, the following system of ten coupled nonlinear equations for the parameters is obtained:

$$A'(z)=-\frac{A\zeta}{2a}, \tag{31}$$

$$\delta'(z)=-\frac{1}{2a^2}+\ln\left(\frac{aA^2}{\sqrt{a^2+b^2}}\right), \tag{32}$$

$$\alpha'(z)=\beta, \tag{33}$$

$$\beta'(z)=0, \tag{34}$$

$$\rho'(z)=0, \tag{35}$$

$$\sigma'(z)=0, \tag{36}$$

$$a'(z)=\zeta, \tag{37}$$

$$b'(z)=\mu, \tag{38}$$

$$\zeta'(z)=\frac{1}{a^3}-\frac{2a}{a^2+b^2}, \tag{39}$$

$$\mu'(z)=-\frac{2b}{a^2+b^2}. \tag{40}$$

It is immediately seen that $\beta(z)=\beta_0$, $\alpha(z)=\beta_0 z+\alpha_0$, $\rho(z)=\rho_0$, and $\sigma(z)=\sigma_0$. Furthermore, Eq. (32) for the phase contribution $\delta(z)$ is unimportant in the present context since it does not affect the dynamics of the intensity distribution, Eq. (30). However, the remaining system of coupled parameter equations contains important information about the dynamics of the solution. From Eqs. (31) and (37), we directly infer that $aA^2=\text{const}$. Thus, the main features of the soliton dynamics is contained in the last four equations which can be written as the following second-order system of coupled differential equations for the two parameters a and b that determine the dynamics of the intensity distribution:

$$a''(z) = \frac{1}{a^3} - \frac{2a}{a^2+b^2}, \tag{41}$$

$$b''(z) = -\frac{2b}{a^2+b^2}. \tag{42}$$

The Hamiltonian of this system is easily identified as

$$H \equiv \frac{1}{2}(a')^2 + \frac{1}{2}(b')^2 + \frac{1}{2a^2} + \ln(a^2+b^2) = \text{const}. \tag{43}$$

This relation could alternatively have been derived by noting that the constants of motion, Eqs. (14) and (17) of the Wigner formalism, generalize to

$$\int_{-\infty}^{\infty} I(x,z)dx \equiv \tilde{N} = \text{const} \tag{44}$$

and

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left|\frac{\partial f(x,\theta,z)}{\partial x}\right|^2 d\theta - 2I(x,z)\ln I(x,z)\right)dx \equiv \tilde{H} = \text{const} \tag{45}$$

in terms of the coherent density function.

The physically important quantity characterizing the width of the intensity profile is $W(z)\equiv a^2(z)+b^2(z)$ [cf. Eq. (30)]. Using Eqs. (37)–(43), it is possible to derive the following evolution equation for the parameter W:

$$[W'(z)]^2 = 8HW - 8W\ln W - C, \tag{46}$$

where C is a constant of integration. Comparing to the corresponding result in the Wigner case [Eq. (12)], it is clear that the dynamic behaviors of the intensity widths in the two representations are identical.

The result in the stationary case can be obtained as follows. The ansatz function must be matched to the (nonseparable) initial condition

$$f(x,\theta,0) = A\sqrt{J(\theta)}\exp\left(-\frac{x^2}{2a_0^2} + i\theta x\right). \tag{47}$$

This requires

$$\alpha(0)=\beta(0)=\delta(0)=\rho(0)=\sigma(0)=\zeta(0),$$

$$b(0)=0, \quad a(0)=a_0, \quad \mu(0)/\sqrt{2\theta_0^2}=1. \tag{48}$$

The condition $W=a^2+b^2=\text{const}$ implies $W=a_0^2$ and requires $W''=H-\ln W-1=0$. The value of the Hamiltonian is obtained by noting that $a'(0)=\zeta(0)=0$ and $b'(0)=\mu(0)=\sqrt{2\theta_0^2}$, which implies

$$1 = H - \ln W = \frac{1}{2}[b'(0)]^2 + \frac{1}{2a_0^2} = \theta_0^2 + \frac{1}{2a_0^2} = \theta_0^2 + \frac{1}{2W} \tag{49}$$

and the stationary width (squared) is given by $W=1/[2(1-\theta_0^2)]$, in agreement with the result obtained by the Wigner analysis.

In order to analyze the nonstationary case, the integration constant C in Eq. (46) can be replaced by a new more convenient invariant, $\tilde{C}$, defined as

$$\tilde{C}^2 \equiv \frac{C}{4} - 1 = \frac{b^2}{a^2} + (a\mu - b\zeta)^2. \tag{50}$$

The invariance of this quantity can also be proved by using the Poisson bracket $\{H,\tilde{C}\}$ corresponding to the Hamiltonian given by Eq. (43).

Equation (50) can be viewed as expressing a trigonometric identity by making the following substitutions:

$$\frac{b}{a} = \tilde{C}\sin\left(\int_0^z \frac{1}{a^2}dz' + D\right), \tag{51}$$

$$a\mu - b\zeta = \tilde{C}\cos\left(\int_0^z \frac{1}{a^2}dz' + D\right), \tag{52}$$

where D is a constant determined by initial conditions. Equation (51) directly expresses b as a function of a and can be used to decouple the system (41) and (42) to obtain a single (integrodifferential) equation for the parameter $a(z)$,

$$a''(z) = \frac{1}{a^3} - \frac{2}{a\left[1+\tilde{C}^2\sin^2\left(\int_0^z \frac{1}{a^2}dz' + D\right)\right]}. \tag{53}$$

This equation implies a qualitatively new feature for the partially coherent dynamics in the sense that the dynamics depends, not only on the instantaneous state of the system in terms of the coherent width a and the displacement b associated with the partial coherence, but also on the history of the dynamics through the integral of $1/a^2$. Equation (53) may be further rewritten as an ordinary differential equation of third order by introducing a new variable $q(z) \equiv \int_0^z \frac{1}{a^2}dz' + D$, which implies

$$a = \frac{1}{\sqrt{q'}}, \quad b = \frac{\tilde{C}}{\sqrt{q'}}\sin q. \tag{54}$$

Substituting these relations into the Hamiltonian given by Eq. (43), a complicated second-order equation is obtained for $q(z)$,

$$H = \frac{1}{8}\frac{q''^2}{q'^3} + \frac{\tilde{C}^2}{2}\left[q'^{1/2}\cos q - \frac{q''}{2q'^{3/2}}\sin q\right]^2 + \frac{q'}{2} + \ln\left[\frac{1}{q'}(1+\tilde{C}^2\sin^2 q)\right]. \tag{55}$$

Equation (55) can in general not be integrated further analytically. In fact, an explicit solution describing the dynamic variation of $q(z)$ is not even known for the coherent case when $\tilde{C}=0$ (cf. [9]). However, as discussed above, an exception is the stationary case when $W\equiv W_0=\text{const}$ and q is determined by the separable first-order equation

$$W_0 q'(z) = 1 + \tilde{C}^2 \sin^2 q(z), \tag{56}$$

which has the solution

$$q(z) = \arctan\left\{ \frac{1}{\sqrt{1+\tilde{C}^2}} \tan\left[\sqrt{1+\tilde{C}^2}\left(\frac{1}{W_0}z + q_0\right)\right]\right\}, \tag{57}$$

where q_0 is an integration constant. The variation of all parameters may then be obtained using this solution, in particular,

$$a^2(z) = \frac{W_0}{2(1+\tilde{C}^2)}\left\{2 + \tilde{C}^2 + \tilde{C}^2 \cos\left[2\sqrt{1+\tilde{C}^2}\left(\frac{1}{W_0}z + q_0\right)\right]\right\},$$

$$b^2(z) = \frac{W_0}{1+\tilde{C}^2}\tilde{C}^2 \sin^2\left[\sqrt{1+\tilde{C}^2}\left(\frac{1}{W_0}z + q_0\right)\right], \tag{58}$$

which agrees with the results obtained in [10]. The parameter $\tilde{C}$ characterizes the degree of coherence. In fact, $\tilde{C}=a_0\sqrt{2\theta_0^2}$ and W_0 may be determined by $\tilde{C}$ through Eq. (46)

$$4(1+\tilde{C}^2) = C = 8W_0(H - \ln W_0) = 8W_0. \tag{59}$$

Thus, the previous stationary result $W_0=a_0^2=1/[2(1-\theta_0^2)]$ is recovered.

IV. EQUIVALENCE OF SOLUTIONS

The equivalence of the solutions given by Eqs. (21) and (29) can be demonstrated explicitly by transforming the coherent density solution Eq. (29) into the Wigner domain. In a first step, the correlation function $K(x_1,x_2,z)$ can by obtained from Eq. (29) using the definition [16]

$$K(x_1,x_2,z) = \int_{-\infty}^{\infty} f^*(x_1,\theta,z) f(x_2,\theta,z)\, d\theta, \tag{60}$$

where the asterisk denotes complex conjugation. Changing to sum-difference coordinates $x=(x_1+x_2)/2$ and $\xi=x_1-x_2$, respectively, the mutual coherence function can be expressed as $K(x,\xi,z)$ (cf. [12]) from which a Fourier transform in the difference variable ξ gives the Wigner transform, according to Eq. (2). The result is

$$\rho(x,p,z) = \tilde{A}\exp[-\tilde{a}(x-\alpha)^2 - \tilde{b}(p-\beta)^2 + \tilde{c}(x-\alpha)(p-\beta)], \tag{61}$$

where

$$\tilde{A} = \frac{1}{\sqrt{1+\tilde{C}^2}} aA^2, \quad \tilde{a} = \frac{1}{1+\tilde{C}^2}\left(\zeta^2 + \mu^2 + \frac{1}{a^2}\right),$$

$$\tilde{b} = \frac{1}{1+\tilde{C}^2}(a^2 + b^2), \quad \tilde{c} = \frac{2}{1+\tilde{C}^2}(a\zeta + b\mu), \tag{62}$$

i.e., the solutions Eqs. (21) and (61) are identical.

V. CONCLUSIONS

The present analysis has reconsidered the problem of the dynamics of partially coherent solitons in media with logarithmic nonlinearity by using both the Wigner and coherent density function formalisms. Although the final results regarding the evolution of the intensity distribution of the solitons are equivalent, the Wigner approach (as well as the approach based on the mutual coherence function [12]) involves a more direct analysis than the coherent density approach in the sense that the Wigner approach is based on a smaller set of "natural" variables than the coherent density approach. In particular, the coherent density function approach requires a more comprehensive modeling of the amplitude and phase characteristics of the coherent density function, which plays the role of an auxiliary function for which not all parameters have a physical interpretation. An illustration of this is the fact that even in the case of a stationary intensity profile, the coherent density function involves significant parameter dynamics, although the physically relevant parameter combinations of amplitude and width of the intensity profile remain constant. On the other hand, it can be anticipated that the coherent density function approach should provide a more convenient frame work for studying other problems, e.g., the interaction between solitons in logarithmic nonlinear media (cf. [16]).

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