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Complete positivity conditions for quantum qutrit channels

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We present an analysis of complete positivity constraints on qutrit-qubit and qutrit-qutrit quantum channels that have a form of affine transformations of a generalized Bloch vector. We show that, in general, the complete positivity constraints for qutrit channels' parameters reveal nonlinearities and cannot be reduced to piecewise linear conditions defining a convex simplex. We discuss qutrit-qubit entanglement breaking channels and show Kraus representation for qutrit-qutrit damping channels.

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I. INTRODUCTION

The characterization of quantum channels aka trace preserving completely positive maps (CPMs) is one of the central questions of the quantum information theory. The concept of quantum channels appears naturally in quantum communications theory and is a direct generalization of the channels known in classical information theory [1]. The typical question concerning quantum channels concerns various types of capacities (cf. [2]), as well as entanglement transmission and/or breaking properties [3–5]. The very same concept of trace preserving CPMs is more frequently used to describe processes that physically may take place and lead to evolution of quantum states [6–8].

Before even starting to investigate the properties of quantum channels, it is already of great importance to be able to parametrize them in an efficient and useful way. So far, this has been achieved for the case of maps carrying states of one qubit into states of one qubit. The full analysis of quantum qubit-qubit channels has been presented in Ref. [9]. This paper gives a connection between the dynamics of a two-level system, represented in terms of Bloch equations, and the quantum channel formalism [9]. One derives here strict mathematical conditions and bounds that the parameters that appear in Bloch formalism must obey. These mathematical conditions and bounds are the direct consequence of the fact that each physical process corresponds to a completely positive map.

These results are of great importance and have various nontrivial applications. For example, in Ref. [10] the authors have used the results of [9] to completely characterize renormalization group transformations of a certain class of quantum states (the simplest instance of the so-called matrix product states). Such analysis is very much desired for other classes of states, but it requires a better understanding of the properties of CPMs acting in higher dimensional spaces. Similarly, determination whether a given map is entanglement breaking (EB), or not, is equivalent to checking whether a certain bipartite state is separable or not. Knowing that the channel is EB allows to realize the channel experimentally via the set of local measurements [positive-operator-valued measure (POVM)] [3,5]. The property of partial EB implies that the channel (when acting on bipartite states) can produce entangled states with limited Schmidt number (for definitions see [11]).

In this paper, we aim at the analysis of the simplest CPMs for qutrit channels. Qutrit states are the states belonging to three-dimensional Hilbert space and, in analogy to qubit case, one can use a generalized Bloch formalism to describe their evolution [12,13]. We consider here two cases as follows.

(i) Qutrit-qubit CPMs: these maps may be considered as quantum communication channels with ancillas or with assistance. Alternatively, one can look at them as describing the reduced evolution of a system plus ancilla (resumed by performing a measurement POVM).

(ii) Qutrit-qutrit maps: these maps have standard interpretations as quantum channels or dynamical transformations.

In our approach we tend to use the dynamical interpretation; i.e., we investigate the evolution of a generalized qutrit Bloch vector that evolves within a Bloch ball. As a natural choice, we investigate qutrit channels of the general form of linear transformations of the qutrit Bloch vector. In particular we focus on affine transformations acting on qutrit Bloch vectors.

Our main results can be summarized as follows:

(i) We derive the complete positivity constraints for qutrit-qubit channels (Secs. V A and V B).

(ii) We characterize qutrit-qubit entanglement breaking channels (Sec. V C).

(iii) We derive the complete positivity (CP) constraints for damping qutrit-qutrit channels previously obtained by Dixit and Sudarshan [14] using a different method (Sec. VI C).

(iv) We derive the CP constraints for qutrit-qutrit channels with damping and shifting (Secs. VI D and VI E) and present also the Kraus representation for qutrit damping channels (Sec. VI C 2).

(v) We present an interesting class of "false" qutrit-qubit channels (Sec. VI F).

(vi) We discuss the PPT qutrit-qutrit channels (Sec. VI H) and two-qutrit states defined by the qutrit-qutrit channels (Sec. VI G).

Our results show that contrary to the qubit-qubit case and quite generally, CP constraints on channel parameters lead in the qutrit-qubit or qutrit-qutrit cases to nonlinear structures

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in the parameter space, that do not correspond to simple simplexes. We point out that there exist qutrit-qubit channels that are entanglement breaking and discuss their properties. We find the class of qutrit channels that corresponds to two qutrit states that cannot be described only by the magic simplex states [15]. Interestingly, the qubit tetrahedronlike simplex structures reappear in a specific class of qutrit channels (Sec. VI F).

The paper is organized according to the obtained results. It does contain, however, other sections, where we introduce the necessary quantum state description (Sec. II), complete positivity issues (Sec. III), and Choi-Jamiolkowski-Sudarshan isomorphism (Sec. III). We also recall the physical background of the Bloch vector qubit and qutrit formalism.

II. STATE DESCRIPTION

Let us first recall the idea behind the Bloch formalism. This, in qubit case, corresponds to the choice of representation of the qubit state density operator: the basis of Pauli matrices σ_i [6,16],

$$\rho_{qb} = \frac{1}{2}(\mathbb{I} + \vec{b}\cdot\vec{\sigma}), \tag{1}$$

where $\vec{b}$ is a three-dimensional, real Bloch vector, describing the qubit state and satisfying $\vec{b}^2 \le 1$ (equality for pure states). Qubit states occupy entirely the Bloch ball. In a similar way we can represent a qutrit state, a state belonging to three-dimensional Hilbert space. In qutrit case the choice of representation is set to be the basis of Gell-Mann matrices λ_i, the generators of SU(3) group [17]

$$\rho_{qt} = \frac{1}{3}(\mathbb{I} + \sqrt{3}\vec{n}\cdot\vec{\lambda}). \tag{2}$$

Here $\vec{n}$ is a generalized Bloch vector, real, and eight dimensional. Qutrit states can be characterized by condition $\vec{n}^2 \le 1$. However, qutrit case is more sophisticated: pure states are states for which two conditions are satisfied [12,17]:

$$\vec{n}^2 = 1, \quad \vec{n} * \vec{n} = \vec{n}, \tag{3}$$

where $*$ product is defined as $(\vec{A}*\vec{B})_i = d_{ijk}A_jB_k$, with d_{ijk} being a totally symmetric tensor [12]. Qutrit states belong to a generalized Bloch ball; qutrit pure states belong to the unit sphere $\mathcal{S}^7 = \{\vec{n} \in \mathrm{R}^8 : \vec{n}^2 = 1\}$. However, physical qutrit states do not occupy entirely the generalized Bloch ball. The pure qutrit states [states satisfying conditions (3)] form a subset of the unit sphere. They can be parametrized with four parameters [17] as follows:

$$|\Psi\rangle = e^{\imath\chi_1}\sin\theta\cos\phi|0\rangle + e^{\imath\chi_2}\sin\theta\sin\phi|1\rangle + \cos\theta|2\rangle, \tag{4}$$

where $0 \le \theta, \phi < \frac{\pi}{2}, 0 \le \chi_1, \chi_2 < 2\pi$ and overall phase was omitted. Hence, the set of pure qutrit states is a four-dimensional subset of a seven-dimensional sphere [13].

III. COMPLETELY POSITIVE TRACE PRESERVING MAPS

It was shown in [18,19] that physical transformations must not only be positivity preserving but there exist more subtle conditions to satisfy; these are called CP conditions [7,18]. The classification of qubit channels according to complete positivity is well known [9]. We want to present a similar analysis for qutrit channels.

We represent a physical system with Hilbert space $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded operators on $\mathcal{H}$; a linear map $\Phi : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is completely positive if for every positive integer m the map,

$$\Phi^{(m)} = \Phi \otimes \mathbb{I}^{(m)} : \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}^{(m)} \mapsto \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}^{(m)}, \tag{5}$$

is positive (where $\mathbb{I}^{(m)}$ is the identity operator on the algebra $\mathcal{M}^{(m)}$ of $m \times m$ complex matrices) [18]. This can be interpreted by saying that Φ acts on a subsystem A of a larger Hilbert space and there is a reservoir (or subsystem B) on which we act with unit operator $\mathbb{I}^{(m)}$. Here, we do not know the dimension of the reservoir and therefore $\Phi^{(m)}$ must be positive for any m. It was shown that every CPM has an operator-sum or Kraus representation [20],

$$\Phi^{CPM}(\rho) = \sum_i \mathcal{K}_i \rho \mathcal{K}_i^\dagger, \tag{6}$$

with $\mathcal{K}_i$ being a set of Kraus operators satisfying $\Sigma_i \mathcal{K}_i^\dagger \mathcal{K}_i = \mathbb{I}$.

To evaluate whether a given transformation Φ (a linear map) is completely positive we need to construct the so-called dynamical (or Choi-Sudarshan) matrix of the order $N^2 \times N^2$ (N is the dimension of the system of interest). We will denote the dynamical matrix with D_Φ [7,18]. Dynamical matrix represents uniquely channel action. We denote with E_{jk} $N \times N$ matrix with 1 at position (j,k) and zeros elsewhere. The map Φ is CPM if and only if:

$$D_\Phi = \sum_{i,j=1}^{N} \Phi(E_{ij}) \otimes E_{ij}, \tag{7}$$

is positive semidefinite ($D_\Phi \ge 0$).

Channel Φ must preserve hermiticity of density matrix and therefore its dynamical matrix must be Hermitian: $D_\Phi = D_\Phi^\dagger$. Trace preserving of the density operators means that the partial trace of D_Φ with respect to the first subsystem (A) gives the unit operator for the second subsystem: $\mathrm{Tr}_A\, D_\Phi = \mathbb{I}$. To evaluate the entries of dynamical matrix, we need to compute the action of the channel Φ on E_{jk}. The non-negativity of the dynamical matrix is directly given by the non-negativity of its eigenvalues. However, if finding the eigenvalues is not successful one can analyze the characteristic polynomial of D_Φ. If we denote the latter with $P_\Phi(x)$ then, by the Descartes' theorem [21], the number of negative roots of $P_\Phi(x)$ is equal to the number of sign changes in the list of coefficients of $P_\Phi(-x)$ [dynamical matrix D_Φ is Hermitian, therefore all its eigenvalues are real and the characteristic polynomial $P_\Phi(x)$ has real coefficients].

On the other hand, $N^2 \times N^2$, positive and Hermitian dynamical matrix D_Φ must correspond to a density operator acting on an N^2-dimensional Hilbert space. This relation is up to normalization factor, since $\mathrm{Tr}\, D_\Phi = N$. Hence ρ_Φ

$=\frac{1}{N}D_\Phi$ is a proper density matrix that we can write as

$$\rho_\Phi = \frac{1}{N}D_\Phi = \frac{1}{N}\sum_{i,j=1}^{N}\Phi(|i\rangle\langle j|)\otimes(|i\rangle\langle j|) = \frac{1}{N}D_\Phi = \frac{1}{N}\sum_{i,j=1}^{N}\Phi(E_{ij}) \otimes E_{ij}. \quad (8)$$

The set of density operators defined by dynamical matrices is only a subset of density matrices in N^2-dimensional Hilbert space since dynamical matrices must satisfy $\mathrm{Tr}_A\, D_\Phi = \mathbb{I}$. The fact that completely positive maps Φ^{CPM}, which are represented uniquely by the dynamical matrices, correspond to states is known as the Choi-Jamiołkowski-Sudarshan isomorphism [22].

Therefore, when analyzing quantum channels we can reinterpret it as an analysis of bipartite quantum states. In case of qutrit-qubit channels the density operator of the bipartite state is six dimensional, and for qutrit channels—nine dimensional.

IV. QUANTUM QUBIT CHANNELS

A. Bloch equations

To recall the qubit case analysis, we can start with a two-level quantum system and its evolution. The latter can be written by means of Bloch equations that are equations for components of Bloch vector $\vec{b}=(u,v,w)$. If we take, for instance, decoherence of a two-level atom, these equations read

$$\dot{u} = -\frac{1}{T_u}u - \Delta v,$$

$$\dot{v} = -\frac{1}{T_v}v + \Delta u + \Omega w,$$

$$\dot{w} = -\frac{1}{T_w}(w - w_{eq}) - \Omega w, \quad (9)$$

and represent the evolution of the system. Here Ω and Δ are Rabi frequency and detuning, respectively. $\frac{1}{T_i}$ stands for decay rates for the atomic dipole $(i=u,v)$ and decay rate of the atomic inversion $(i=w)$. These equations, when put together, give rise to an affine transformation of the qubit Bloch vector that is governed by the parameters listed above. Any physical process amounts to a transformation of the qubit state that is already a completely positive map. However, *not* every affine transformation of the (qubit) Bloch vector will be a completely positive map.

B. Affine transformations on qubit Bloch vectors

The analysis of completely positive trace preserving maps on $\mathcal{M}_2$ (complex two-dimensional matrices) has been studied extensively [9,16] and gives the answer to the problem. Without loss of generality one can analyze qubit channels that transform qubit Bloch vector according to

$$\Phi^{qb}:\vec{b} \mapsto \vec{b}' = \Lambda^{qb}\vec{b} + \vec{t}^{qb}, \quad (10)$$

where matrix $\Lambda^{qb}=\mathrm{diag}\{\Lambda_1^{qb},\Lambda_2^{qb},\Lambda_3^{qb}\}$ consists of damping eigenvalues Λ_i^{qb} and $\vec{t}^{qb}=(t_1^{qb},t_2^{qb},t_3^{qb})$ is a translation. The image of Bloch sphere of pure states $(\vec{b}^2=1)$ under such transformation is the ellipsoid

$$\left(\frac{u'-t_1^{qb}}{\Lambda_1^{qb}}\right)^2 + \left(\frac{v'-t_2^{qb}}{\Lambda_2^{qb}}\right)^2 + \left(\frac{w'-t_3^{qb}}{\Lambda_3^{qb}}\right)^2 = 1, \quad (11)$$

with its center defined by $\vec{t}^{qb}$ and its axes by Λ_1^{qb}. The set of conditions on both Λ_i^{qb} and t_i^{qb} can be found in [9,16]. When we limit ourselves just to damping (or diagonal) qubit channels (meaning $\vec{t}^{qb}=0$), then the set of allowed Λ_i^{qb} form a tetrahedron structure [7,9]. This structure reappears also in the space of two qubit states.

V. QUTRIT-QUBIT CHANNELS

In this section we will analyze channels that transform quantum qutrit states to quantum qubit states. The action of such a channel can be shown as

$$\rho_{qutrit} = \frac{\vec{n}'\cdot\vec{\lambda}'}{\sqrt{3}} \to \rho_{qubit} = \frac{\vec{b}'\cdot\vec{\sigma}'}{2}, \quad (12)$$

where we used the notation: $\vec{\lambda}'=(\sqrt{\frac{2}{3}}\mathbb{I}_3,\vec{\lambda})$, $\vec{\sigma}'=(\mathbb{I}_2,\vec{\sigma})$, $\vec{n}'=(\frac{1}{\sqrt{2}},\vec{n})$, $\vec{b}'=(1,\vec{b})$, and $\vec{n},\vec{b}$ are the usual qutrit and qubit Bloch vectors, respectively. Since the states are characterized by their Bloch vectors $(\vec{n},\vec{b})$, one can rewrite the channel action as

$$\Phi^{3-2}:\vec{n}' \to \vec{b}' = \Lambda^{3-2}\vec{n}', \quad (13)$$

where Λ^{3-2} is a 4×9 matrix containing both damping and translation parameters and has the following form:

$$\Lambda^{3-2} = \begin{pmatrix} \sqrt{2} & 0\ldots 0 \\ \sqrt{2}\vec{l}_{(3)} & L_{(3\times 8)} \end{pmatrix} \quad (14)$$

(the numbers in the brackets indicate object size). In general, all the entries of L matrix can be nonzero—hence the information encoded in eight-dimensional qutrit Bloch vector is used, together with translation $\vec{l}$, to construct a three-dimensional qubit Bloch vector. The question is what are the complete positivity limits imposed on this transformation. For simplicity, we divide this problem into two steps: the first one—investigate a transformation which does not include a translation (only damping), and the second one—add translations to the damping.

A. Damping qutrit-qubit channels

To analyze complete positivity constraints on the qutrit-qubit channels which act on qutrit Bloch vector only by means of damping matrix $L=(\Lambda_{ij})_{i=1,2,3;j=1,\ldots,8}$, we construct corresponding dynamical matrix D_L,

$$D_L \equiv \sum_{i,j=1}^{3} \Phi^L(|i\rangle\langle j|) \otimes |i\rangle\langle j|. \tag{15}$$

In general, we keep all the parameters Λ_{ij} nonzero and assume that $|\Lambda_{ij}| \leq 1$. This channel transforms the Bloch vectors according to

$$\Phi^L:\vec{n} \to \vec{b} = \sum_{j=1}^{8} (\Lambda_{1j}n_j, \Lambda_{2j}n_j, \Lambda_{3j}n_j). \tag{16}$$

The question is what the CP-allowed values of $\Lambda_{ij}$are. If we apply Descartes' theorem to the characteristic polynomial $P_L(x)$ of dynamical matrix D_L, we get five inequalities which must be satisfied to guarantee that D_L does not have any negative eigenvalues. These inequalities are, in principle, nonlinear in Λ_{ij} parameters since they correspond to coefficients p_k of characteristic polynomial $P_L(x)=\Sigma_{s=0}^{6} p_s x^s$ with $0 \leq k \leq 4$. The simplest CP condition in this case, given by

$$\sum_{i=1,2,3;j=1}^{8} \Lambda_{ij}^2 < 10, \tag{17}$$

corresponds to a ball in the space of all parameters Λ_{ij} (quadratic condition). The rest of the CP conditions obtained in this manner are inequalities of higher order in channel parameters (up to sixth order)—there is no linear condition given by this method; a second inequality (third order in channel parameters) is shown in Appendix A. For some channels the list of nonlinear CP conditions might be equivalent to some linear CP conditions (defining hyperplanes).

Below we show two examples of damping qutrit-qubit channels—one of them reveals a polyhedronlike structure (a) in the space of CP-allowed channel parameter values; the other one shows nonlinear structure (b).

Example 1. As a first example we take a channel that transforms qutrit Bloch vector according to

$$\Phi_1:\vec{n} \to \vec{b}_1 = (0,0,\Lambda_{33}n_3 + \Lambda_{38}n_8). \tag{18}$$

The analysis of its dynamical matrix characteristic polynomial gives the following (nontrivial) CP inequalities

$$\frac{5}{3} + (\Lambda_{33}^2 + \Lambda_{38}^2)^2 - 4(\Lambda_{33}^2 + \Lambda_{38}^2) > 0,$$

$$\frac{1}{3} + (\Lambda_{33}^2 + \Lambda_{38}^2)^2 - \frac{4}{3}(\Lambda_{33}^2 + \Lambda_{38}^2) > 0,$$

$$(4\Lambda_{38}^2 - 1)[1 + (3\Lambda_{33}^2 - \Lambda_{38}^2)^2 - 2(\Lambda_{33}^2 + \Lambda_{38}^2)] < 0, \tag{19}$$

where the last one of these is in fact Det $D_{\Phi_1} > 0$. However, it turns out that in this case the dynamical matrix describing Φ_1 channel has a diagonal form. Therefore it is straightforward to write down CP conditions for its eigenvalues,

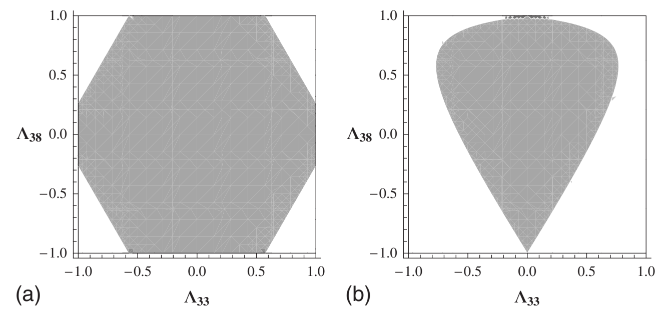

FIG. 1. The figures show CP-allowed region for damping qutrit-qubit channels that have the form (a) given by Eq. (18) and (b) given by Eq. (21). $\Lambda_{33}, \Lambda_{38}$ are dimensionless channel parameters.

$$1 \pm \sqrt{3}\Lambda_{33} + \Lambda_{38} \geq 0,$$

$$1 \pm \sqrt{3}\Lambda_{33} - \Lambda_{38} \geq 0,$$

$$1 \pm 2\Lambda_{38} \geq 0. \tag{20}$$

Both sets of CP inequalities (19) and (20) give the same CP-allowed region of parameters $\Lambda_{33}, \Lambda_{38}$ shown on Fig. 1(a).

Example 2. On the other hand, we can change a channel given in example 1, to have a channel that transforms the Bloch vector according to

$$\Phi_2:\vec{n} \to \vec{b}_2 = (\Lambda_{33}n_1, \Lambda_{38}n_2, \Lambda_{33}n_3 + \Lambda_{38}n_8). \tag{21}$$

The channel is simple enough to find the eigenvalues of the corresponding dynamical matrix explicitly what gives the following CP constraints,

$$1 \pm \Lambda_{38} > 0,$$

$$2 - \sqrt{3}\Lambda_{33} \pm \sqrt{3\Lambda_{33}^2 - 6\Lambda_{33}\Lambda_{38} + 4\Lambda_{38}^2} > 0,$$

$$2 + \sqrt{3}\Lambda_{33} \pm \sqrt{3\Lambda_{33}^2 + 6\Lambda_{33}\Lambda_{38} + 4\Lambda_{38}^2} > 0. \tag{22}$$

The corresponding CP region of allowed channel parameters is presented in Fig. 1(b).

B. General qutrit-qubit channels—damping and shifting

The next step is to investigate qutrit-qubit channels that also include translations in Λ^{3-2} matrix. Therefore the transformation on qutrit Bloch vector is the most general one [producing a qubit state, Eq. (13)]. Since we do allow the general form of Λ^{3-2}, the analytical treatment of the corresponding dynamical matrix D_{3-2},

$$D_{3-2} \equiv \sum_{i,j=1}^{3} \Phi^{3-2}(|i\rangle\langle j|) \otimes |i\rangle\langle j|, \tag{23}$$

becomes sophisticated. Again we analyze the characteristic polynomial of dynamical matrix D_{3-2} and check when it has no negative roots (eigenvalues). As before, there are five CP inequalities which must be satisfied to have a physical transformation. They are all nonlinear in parameters Λ_{ij}, t_i. The simplest one is given by

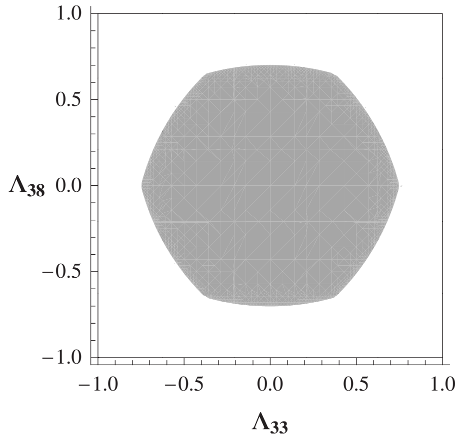

FIG. 2. The figure shows an example of a qutrit-qubit channel [Eq. (25)] that involves damping and shifting of Bloch vector; $\Lambda_{33}, \Lambda_{38}$ are dimensionless channel parameters.

$$\sum_{i=1,2,3;j=1}^{8} \Lambda_{ij}^2 + 2\sum_{i=1}^{3} l_i^2 < 10, \tag{24}$$

and in Appendix A we show explicitly one more (cubic in Λ_{ij}, l_i).

For some channels the nonlinear CP conditions might simplify to linear constraints on channel parameters.

Example. As an example we take the modified channel Φ_1 given as an example of damping qutrit-qubit channel. Here, we add a translation that introduces nonzero $b_{i=1,2}$,

$$\Phi_3:\vec{n} \to \vec{b}_3 = (\Lambda_{33}, \Lambda_{38}, \Lambda_{33}n_3 + \Lambda_{38}n_8). \tag{25}$$

The CP inequalities can be given directly by the eigenvalues of the corresponding dynamical matrix, since the channel has a fairly simple form,

$$1 \pm \sqrt{\Lambda_{33}^2 + \Lambda_{38}^2} \geq 0,$$

$$2 \pm \sqrt{7\Lambda_{33}^2 - 2\sqrt{3}\Lambda_{33}\Lambda_{38} + 5\Lambda_{38}^2} \geq 0,$$

$$2 \pm \sqrt{7\Lambda_{33}^2 + 2\sqrt{3}\Lambda_{33}\Lambda_{38} + 5\Lambda_{38}^2} \geq 0. \tag{26}$$

The CP region for this channel, shown on Fig. 2, is a modified region that we have seen on Fig. 1(a). Adding translation caused some shrinking and smoothening—the structure is no longer linear.

C. Qutrit-qubit entanglement breaking channels

By means of Choi-Jamiołkowski-Sudarshan isomorphism we know that dynamical matrices of qutrit-qubit channels correspond to qubit-qutrit states. Density operator given by

$$\rho_{3-2} = \frac{1}{3}D_{3-2} = \frac{1}{3}\sum_{r,s=1}^{3} \Phi^{3-2}(|r\rangle\langle s|) \otimes |r\rangle\langle s| \tag{27}$$

represents a 2×3 qubit-qutrit system. However, states given by Eq. (27) only form a subset in the space of all qubit-qutrit states, since the condition $\mathrm{Tr}_2\, D_{3-2}=\mathbb{I}_3$ must be satisfied. One can ask about the entanglement of the state represented by ρ_{3-2}. We know that for qubit-qutrit states there exist necessary and sufficient condition for entanglement detection—Peres-Horodeccy criterion [23,24]—PPT. To check whether a state ρ_{3-2} is separable, we need to find out if

$$\xi = \frac{1}{3}\sum_{r,s=1}^{3} [\Phi^{3-2}(|r\rangle\langle s|)]^T \otimes |r\rangle\langle s| \tag{28}$$

is a proper density operator (positive semidefinite).

Here are two examples of states and channels—separable and entangled one, both satisfying the channel partial trace condition. The first one is given by a density matrix

$$\rho_{ex1} = \frac{1}{2}[P(|0,0\rangle + |1,1\rangle) + P(|0,1\rangle + |1,2\rangle) + P(|0,2\rangle + |1,0\rangle)], \tag{29}$$

where $P(|\Psi\rangle)$ denotes a projector onto the subspace spanned by $|\Psi\rangle$. The state ρ_{ex1} is separable and corresponds to a channel given by

$$\Phi_{ex1}:\vec{n} \to \vec{b}_{ex1} = \frac{1}{\sqrt{3}}\begin{pmatrix} n_1 + n_4 + n_6 \\ n_2 - n_5 + n_7 \\ 0 \end{pmatrix}, \tag{30}$$

which is a damping type channel.

The second example is given by a state

$$\rho_{ex2} = \frac{1}{6}[P(|0,0\rangle + |1,1\rangle) + P(|0,1\rangle + |1,2\rangle) + P(|0,0\rangle + |1,2\rangle)], \tag{31}$$

which is entangled (according to PPT criterion) and corresponds to a channel

$$\Phi_{ex2}:\vec{n} \to \vec{b}_{ex2} = \frac{1}{\sqrt{3}}\begin{pmatrix} n_1 + n_4 + n_6 \\ n_2 + n_5 + n_7 \\ n_3 + \sqrt{3}n_8 \end{pmatrix}. \tag{32}$$

Therefore Φ_{ex2} is not an entanglement breaking channel. Furthermore, we can analyze a class of similar channels that act on the Bloch vector in the following way:

$$\Phi_{ex3}:\vec{n} \to \vec{b}_{ex3} = \begin{pmatrix} \Lambda_{xy}(n_1 + n_4 + n_6) + \Lambda_{xy}\Lambda_z, \\ \Lambda_{xy}(n_2 + n_5 + n_7) + \Lambda_{xy}\Lambda_z, \\ \Lambda_z(n_3 + n_8) + \Lambda_{xy}\Lambda_z \end{pmatrix}. \tag{33}$$

In Fig. 3 we show region of CP-allowed values of Λ_{xy}, Λ_z together with a region for which the channel Φ_{ex3} is entanglement breaking.

VI. QUANTUM QUTRIT CHANNELS

A. Qutrit Bloch equations

As before, we can start the analysis of transformations on qutrit quantum states with the analysis of a three-level atom for which we can write down Bloch equations. The three-level atom is not the only possible physical realization

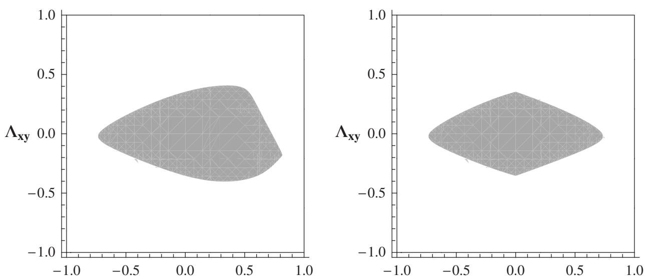

FIG. 3. (a) CP-allowed region for channel Φ_{ex3}, (b) the region of entanglement breaking channels of Φ_{ex3} type; Λ_{xy} and Λ_z are dimensionless channel parameters.

[25,26] but it is very illustrative. The analog of the Bloch vector for the case of a three-level atom was in the beginning introduced as a (eight-dimensional, real) coherent vector $\vec{S}$ [27], in which components (denoted as u,v,w) were defined as

$$u_{jk} = \rho_{jk} + \rho_{kj},$$

$$v_{jk} = \imath(\rho_{jk} - \rho_{kj}),$$

$$w_{jk} = -\sqrt{\frac{2}{l(l+1)}}(\rho_{11} + \rho_{22} + \cdots + \rho_{ll} - l\rho_{l+1,l+1}), \quad (34)$$

with $1 \leq j < k \leq 3$ and $1 \leq l \leq 2$. Now, as an example of the physical system we can take a three-level atom for which nonzero dipole moments are between levels 1 and 2, and 2 and 3. The atom interacts with the electric field (two electromagnetic waves incident on the atom) and we assume that detunings are the same ($\Delta_{12} = -\Delta_{23} = \Delta$). The corresponding Bloch equations for coherent vector $\vec{S}$ are

$$\dot{u}_{12} = \Delta v_{12} + \beta v_{13},$$

$$\dot{u}_{23} = -\Delta v_{23} - \alpha v_{13},$$

$$\dot{u}_{13} = \beta v_{12} - \alpha v_{23},$$

$$\dot{v}_{12} = -\Delta u_{12} - \beta u_{13} + 2\alpha w_1,$$

$$\dot{v}_{23} = \Delta u_{23} + \alpha u_{13} - \beta w_1 + \sqrt{3}\beta w_2,$$

$$\dot{v}_{13} = -\beta u_{12} + \alpha u_{23},$$

$$\dot{w}_1 = -2\alpha v_{12} + \beta v_{23},$$

$$\dot{w}_2 = -\sqrt{3}\beta v_{23}, \quad (35)$$

where α, β are related to two Rabi frequencies [27]. These equations are a generalization of the equations we have seen in the qubit case.

In this work we use a slightly different notation for the qutrit vector—we already have introduced qutrit Bloch vector $\vec{n}$ related to the choice of Gell-Mann matrices basis (in some works generalized Bloch vectors are also called coherent vectors [12]). These two vectors ($\vec{S}$ and $\vec{n}$) are of course equivalent.

Parameters that appear in the qutrit Bloch equations have physical background; therefore, the resulting affine transformation is a completely positive map. However, our question is the opposite: given an arbitrary affine transformation on qutrit Bloch vector what are the conditions on its parameters which guarantee complete positivity?

B. Affine transformations of qutrit Bloch vectors

Having in mind the question stated above, we will look at transformations of qutrit Bloch vector that have a form

$$\Phi: \vec{n} \mapsto \vec{n}' = \Lambda\vec{n} + \vec{t}, \quad (36)$$

where $\Lambda = \mathrm{diag}\{\Lambda_1, \ldots, \Lambda_8\}$ consists of eight damping coefficients and $\vec{t}$ is an eight-dimensional translation. The image of the set of pure states under this transformation is

$$\sum_{i=1}^{8} \left(\frac{n_i' - t_i}{\Lambda_i}\right)^2 = 1, \quad (37)$$

together with the condition for $*$ product $\vec{n} * \vec{n} = \vec{n}$,

$$\frac{n_i' - t_i}{\Lambda_i} = d_{ijk}\frac{n_j' - t_j}{\Lambda_j}\frac{n_k' - t_k}{\Lambda_k}. \quad (38)$$

On the other hand, parameters Λ_i, t_i must satisfy

$$\sum_i (\Lambda_i n_i + t_i)^2 \leq 1, \quad (39)$$

according to the requirement $\vec{n}'^2 \leq 1$. However, complete positivity is a much stronger condition than the condition saying that we cannot exceed value 1 for the length of Bloch vector. The latter, in qubit case, amounts only to the statement that the density operator must be a positive definite operator. In qutrit case, however, it is even less than that since not every point within the $\mathcal{S}^7$ sphere corresponds to density operator.

To construct dynamical matrix D_Φ, we apply the channel action to $E_{jk} \mapsto \Phi(E_{jk})$, representing it in the basis of Gell-Mann matrices: $E_{jk} = \frac{1}{\sqrt{3}} n_\alpha^{jk} \lambda_\alpha$ (where $\alpha \in \{0, \ldots, 8\}$, $\lambda_0 = \sqrt{\frac{2}{3}}\mathbb{I}$, and n_α^{jk} can be interpreted as an analog of Bloch vector).

We will first look at channels that consist only of a damping matrix and do not have a translation. These channels are in fact unital since they leave the maximally mixed state unchanged (they are called bistochastic maps [9]). Later on, we will look at channels that include also translations of Bloch vector.

C. CPM conditions for damping qutrit channels

In qubit case, the action of the damping channel can be written as

$$\vec{b} \to \vec{b}' = \Lambda^{qb}\vec{b}, \quad \Lambda^{qb} = \mathrm{diag}\{\Lambda_1^{qb}, \Lambda_2^{qb}, \Lambda_3^{qb}\}, \quad (40)$$

whereas for qutrits we have

$$\vec{n} \to \vec{n}' = \Lambda\vec{n}, \quad \Lambda = \mathrm{diag}\{\Lambda_1, \ldots, \Lambda_8\}. \quad (41)$$

In both cases, we assume that the nature of Λ_i parameters is quasidamping, hence $|\Lambda_i| \leq 1$. This comes from the fact that

$\vec{n}^2 \leq 1$ at all times, therefore, the change in any initial Bloch vector will lead to a vector within the (generalized) Bloch ball. Dynamical matrix D_Φ for a qutrit channel of form (36) must be positive semidefinite in order to correspond to CPM. There are nine eigenvalues d_i that must be non-negative to satisfy positivity of D_Φ. The first six eigenvalues give rise to conditions that can be written as

$$
\begin{aligned}
1-\Lambda_8+\frac{3}{2}(\Lambda_4-\Lambda_5) &\geq 0,\\
1-\Lambda_8-\frac{3}{2}(\Lambda_4-\Lambda_5) &\geq 0,\\
1-\Lambda_8+\frac{3}{2}(\Lambda_6-\Lambda_7) &\geq 0,\\
1-\Lambda_8-\frac{3}{2}(\Lambda_6-\Lambda_7) &\geq 0,\\
1-\Lambda_8+\frac{3}{2}(\Lambda_1-\Lambda_2)+\frac{3}{2}(\Lambda_8-\Lambda_3) &\geq 0,\\
1-\Lambda_8-\frac{3}{2}(\Lambda_1-\Lambda_2)+\frac{3}{2}(\Lambda_8-\Lambda_3) &\geq 0. \qquad (42)
\end{aligned}
$$

These conditions alone lead to the set of allowed Λ_i that has a polyhedronlike structure. In qubit case we have a similar set of equations for Λ^{qb} that define the tetrahedron structure. However, in qutrit case there are three remaining inequalities (given by eigenvalues $d_7, d_8, d_9 \geq 0$) which reveal coupling between all the parameters,

$$
\begin{aligned}
d_7(\Lambda_1,\ldots,\Lambda_8) &\geq 0,\\
d_8(\Lambda_1,\ldots,\Lambda_8) &\geq 0,\\
d_9(\Lambda_1,\ldots,\Lambda_8) &\geq 0. \qquad (43)
\end{aligned}
$$

Because of their numerical complexity they are discussed in Appendix B. Matrix D_Φ is Hermitian, therefore eigenvalues $d_{7,8,9}$ must be real. For some cases, three conditions [Eq. (43)] reduce to just two (see Appendix B). All the inequalities characterize the set of allowed $\{\Lambda^{CPM}\}$, in other words, channel parameters Λ_i for which Φ is a CPM. Similar analysis was obtained in [14]—the set of six eigenvalues defining hyperplanes. However, the remaining part of the non-negativity problem was solved not by finding eigenvalues ($d_{7,8,9}$) directly but by analyzing coefficients of a polynomial (characteristic polynomial of a submatrix). In that way it was possible to obtain one more linear inequality defining a seventh hyperplane.

In principle, parameters Λ_i can be time dependent; still, conditions (42) and (43) must be satisfied for any time t to have a CPM. If we assume, for example, that

$$\frac{dn_i(t)}{dt} = \gamma_i n_i(t), \qquad (44)$$

then time evolution of the Bloch vector $\vec{n}(t)$ is given by

$$n_i(t) = e^{\gamma_i t} n_i(0). \qquad (45)$$

We can then identify $\Lambda_i = e^{\gamma_i t}$ and conditions on Λ_i will impose conditions on γ_i. For this type of evolution, one can write the Lindblad equation for qutrit density operator $\rho(t)$ corresponding to the channel action. Some more details on relation between complete positivity and master equation and Lindblad operators can be found in [8,16,28].

1. Structure of the set $\{\Lambda^{CPM}\}$

For qubit case, the allowed values of damping parameters $\{\Lambda_{qb,i}^{CPM}\}_{i=1,2,3}$ form a characteristic structure (tetrahedron, [9]). We are interested in the structure that appears in qutrit case. The main obstacle here is the size of parameter space. We have eight parameters on which we impose our CPM constraints. We can investigate the $\{\Lambda^{CPM}\}$ set projecting it onto subspaces. On Fig. 4 we show projections of $\{\Lambda^{CPM}\}$ onto the various subspaces in eight-dimensional space of parameters $\Lambda_1,\ldots,\Lambda_8$. The *dark regions* in these figures correspond to these values of Λ_i which are *satisfying CP conditions*. There are many projections that can be obtained from the conditions that we have derived. In principle, the structure of the set of $\{\Lambda^{CPM}\}$ is not simply a generalization of a tetrahedron. Since we have nonlinear conditions for Λ_i that couple all the parameters, the simple polyhedronlike structure [emerging from inequalities that are linear in Λ_i, Eq. (42)] is altered. In the figures, one can see combination of these linear and nonlinear structures.

2. Kraus representation of damping qutrit channels

Diagonalization of the dynamical matrix D_Φ leads not only to the answer about complete positivity of the channel, but also is the way to obtain operator-sum or Kraus representation of the channel action. Kraus operators are related to the eigenvectors of dynamical matrix D_Φ by reshaping operation [7]. The operator-sum representation for the damping qutrit channel is given by

$$
\begin{aligned}
\mathcal{K}_1^{damp} &= \sqrt{\frac{2+3\Lambda_4-3\Lambda_5-2\Lambda_8}{12}}\lambda_4,\\
\mathcal{K}_2^{damp} &= -\imath\sqrt{\frac{2-3\Lambda_4+3\Lambda_5-2\Lambda_8}{12}}\lambda_5,\\
\mathcal{K}_3^{damp} &= \sqrt{\frac{2+3\Lambda_6-3\Lambda_7-2\Lambda_8}{12}}\lambda_6,\\
\mathcal{K}_4^{damp} &= -\imath\sqrt{\frac{2-3\Lambda_6+3\Lambda_7-2\Lambda_8}{12}}\lambda_7,\\
\mathcal{K}_5^{damp} &= \sqrt{\frac{2+3\Lambda_1-3\Lambda_2-3\Lambda_3+\Lambda_8}{12}}\lambda_1,\\
\mathcal{K}_6^{damp} &= -\imath\sqrt{\frac{2-3\Lambda_1+3\Lambda_2-3\Lambda_3+\Lambda_8}{12}}\lambda_2, \qquad (46)
\end{aligned}
$$

and the three remaining Kraus operators $\mathcal{K}_{7,8,9}^{damp}$ are diagonal and have the following structure (d_j is the corresponding eigenvalue):

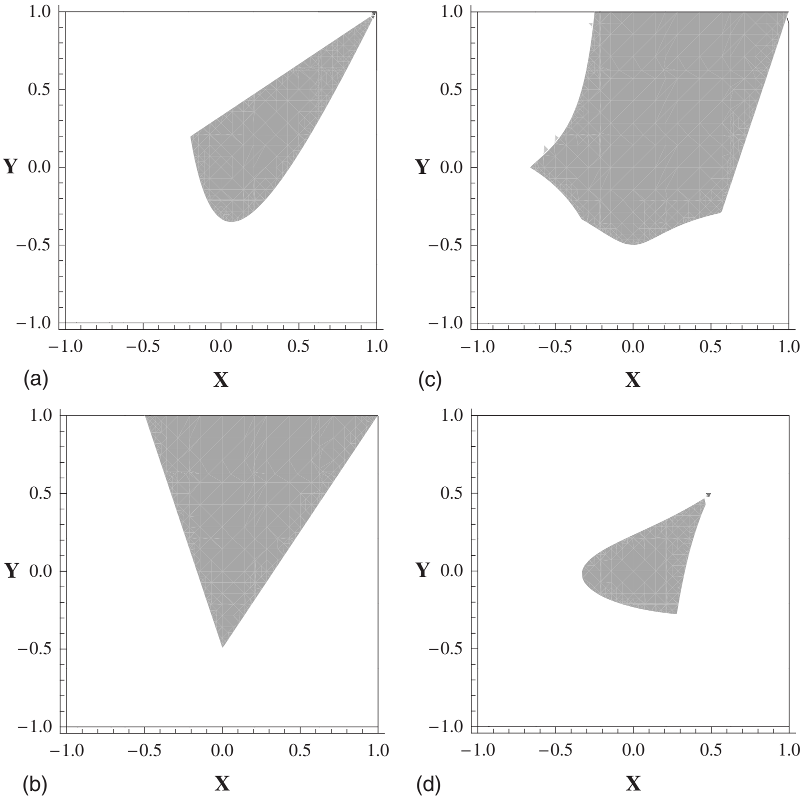

FIG. 4. The dark region in the figures shows values of damping (dimensionless) channel parameters that satisfy CP conditions. The following parametrization is assumed: (a) $\Lambda_1=\Lambda_2=Y$, $\Lambda_{i\neq 1,2}=X$, (b) $\Lambda_3=\Lambda_8=Y$, $\Lambda_{i\neq 3,8}=X$, (c) $\Lambda_3=X,\Lambda_8=Y$, $\Lambda_{i\neq 3,8}=XY$, and (d) $\Lambda_1=\Lambda_2=X$, $\Lambda_3=\Lambda_8=XY,\Lambda_{i\neq 1,2,3,8}=Y$.

$$\mathcal{K}_{j=7,8,9}^{damp} = \sqrt{\frac{d_j}{1+a_j^2+b_j^2}}\begin{pmatrix} a_j & 0 & 0 \\ 0 & b_j & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{47}$$

D. CPM conditions for qutrit channels based only on translations

In this section we will analyze shortly the constraints of complete positivity on the possible translations. The change in qutrit Bloch vector that we assume in this case will be of the form

$$\vec{n} \mapsto \vec{n}' = \vec{t}, \tag{48}$$

where $\vec{t}=(t_1,\ldots,t_8)$ is a translation introduced by a channel. In other words, Bloch vector $\vec{n}$ is damped to zero and moved to a specific vector given by $\vec{t}$. This type of a channel is nonunital. The CP constraints on parameters t_i of the introduced vector $\vec{t}$ are

$$\sum_{i=1}^{8} t_i^2 < \frac{1}{9},$$

$$-\frac{1}{27} + \sum_{i=1}^{8} t_i^2 + -3\sqrt{3}t_3(t_4^2+t_5^2-t_6^2-t_7^2) + -3t_8\left(2t_1^2+2t_2^2+2t_3^2 - t_4^2 - t_5^2 - t_6^2 - t_7^2 - \frac{2}{3}t_8^2\right) + 6\sqrt{3}(t_2t_4t_7 - t_2t_5t_6 - t_1t_4t_6 - t_1t_5t_7) < 0. \tag{49}$$

Below we show some examples of channels, for which we choose just two free parameters. Both of them reveal nonlinear structures in the channel parameter space.

Example 1. First, let us look at the translation of the form

$$T_1 = (t_1,t_1,t_3,0,0,0,0,0). \tag{50}$$

It turns out that effectively, parameters t_1,t_3 must satisfy

$$1 - 3\sqrt{3}\sqrt{2t_1^2+t_3^2} \geq 0, \tag{51}$$

what graphically is represented on Fig. 5(a)—the dark region corresponding to CP-allowed parameter values has an ellipsoid form.

Example 2. On the other hand, if we let the translation to have also the eight component different from zero and equal to the third component, therefore translation having a form

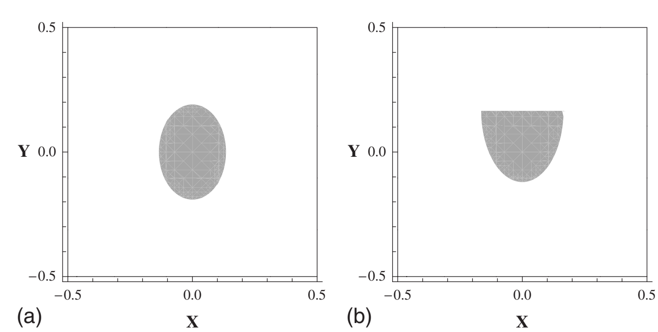

FIG. 5. The dark region in the figures shows these values of (dimensionless) parameters $t_1=X,t_3=Y$ that satisfy CP conditions, when translation acting on qutrit Bloch vector has the form (a) $T_1=(t_1,t_1,t_3,0,0,0,0,0)$; (b) $T_2=(t_1,t_1,t_3,0,0,0,0,t_3)$.

$$T_2 = (t_1, t_1, t_3, 0, 0, 0, 0, t_3), \tag{52}$$

then the allowed set of parameters t_1, t_3 is cut off with respect to t_3—this parameter must satisfy $t_3 \leq 1/6$. Also both parameters t_1, t_3 must satisfy

$$1 + 3t_3 - 3\sqrt{3}\sqrt{2t_1^2 + t_3^2} \geq 0. \tag{53}$$

This is shown on Fig. 5(b)—the ellipsoid shape from Fig. 5(a) is now cut off and shifted.

E. CPM conditions for qutrit channels—damping and shifting

We have seen what the CP conditions are for damping channels and investigated some examples of channels built only with translations. What occurs when these two effects combine? Let us take the channel that changes the Bloch vector according to

$$\Phi: \vec{n} \to \vec{n}' = \Lambda \vec{n} + \vec{t},$$

$$\Phi: \rho \to \rho' = \frac{1}{3}[\mathbb{I} + \sqrt{3}(\Lambda \vec{n} + \vec{t}) \cdot \vec{\lambda}]. \tag{54}$$

There are 16 parameters on which we impose CP constraints. Again, we construct a dynamical matrix that corresponds to Φ and analyze its characteristic polynomial with the method explained before. The eight CP constraints revealed in this way are, in principle, nonlinear, but for some examples they might show linear behavior. The simplest inequality that must be obeyed by Λ_i, t_i is given by

$$\sum_{i=1}^{8} (\Lambda_i^2 + 6t_i^2) + \frac{4}{\sqrt{3}} \sum_{j=1,4,6} \Lambda_j t_j < 8. \tag{55}$$

The rest is more sophisticated and gives rise to the set of allowed parameter values $\{\Lambda_i, t_i\}^{CPM}$. We project the latter on subspaces to present two examples of qutrit channels that contain both damping and shifting. The two channels are

$$\Phi_4: \vec{n} \to \vec{n}_4 = (Xn_1 + Y, \ldots, Xn_8 + Y), \tag{56}$$

$$\Phi_5: \vec{n} \to \vec{n}_5 = (Xn_1 + XY, Xn_2 + XY, Yn_3 + XY, \ldots, Yn_8 + XY), \tag{57}$$

where X, Y are channels' parameters. On Figs. 6(a) and 6(b) the dark regions correspond to the CP-allowed values of X, Y for Φ_4, Φ_5, respectively. We see that in case of Φ_4, when we take $X \to 1$, the second parameter $Y \to 0$ and the identity operator is reached (all $\Lambda_i = 1$). One cannot have a channel that leaves all Bloch vectors unchanged (meaning $\Lambda = \mathbb{I}$) and shifts the entire Bloch ball (by $\vec{t} \neq \vec{0}$)—complete positivity does not allow such a transformation. On the other hand, if we take in Φ_4 $X=0$ we see what the width is of possible uniform $(Y\vec{1})$ translation. Figure 6(b) illustrates CP-allowed region for Φ_5 that has more elaborate parametrization. In both cases, however, the structures are nonlinear.

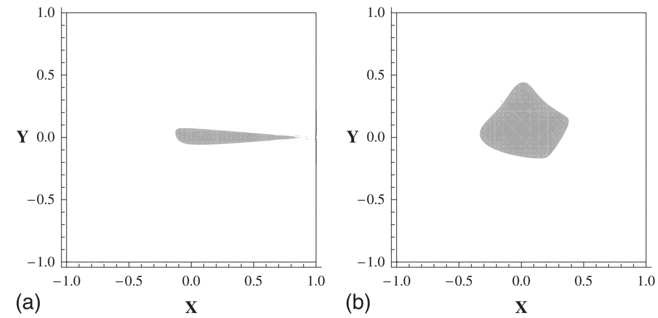

FIG. 6. The dark region in the figures shows the values of (dimensionless) channel parameters that satisfy CP conditions. The following parametrization is assumed: (a) $\Lambda_i = X, t_i = Y$ and (b) $\Lambda_{1,2} = X, \Lambda_{i \neq 1,2} = Y, t_i = XY$.

F. False qutrit-qubit channels

Here we give an example of CP analysis of a class of qutrit quantum channels that transform the qutrit density operator according to

$$\Phi: \rho \to \rho' = \begin{pmatrix} \rho_{2\times 2} & 0 \\ 0 & \rho_{33} \end{pmatrix}. \tag{58}$$

If $\rho_{33}=0$ the $\rho_{2\times 2}$ part resembles a qubit state, since it must be a positive and normalized 2×2 complex matrix. However, it turns out that no physical transformation will bring any qutrit state to state (58) with $\rho_{33}=0$. To see that let us take a transformation on the Bloch vector that has a form

$$\vec{n} \to \tilde{\Lambda} \vec{n} + \vec{\tilde{t}}, \tag{59}$$

where $\vec{\tilde{t}} = (t_1, t_2, t_3, 0, 0, 0, 0, t_8)$ and $\tilde{\Lambda}$ is a 8×8 real matrix with nonzero entries specified to be $\{\tilde{\Lambda}_{ij}\}_{i=1,2,3;j=1}^{8}$ (and the rest equal to zero). Such a form of transformation guarantees that the resulting density operator will have form (58).

The dynamical matrix of this channel happens to have a block-diagonal form

$$D_{false} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \tag{60}$$

where D_1 is a 6×6 and D_2 is a 3×3 matrix. It occurs as well that D_2 only depends on t_8 channel parameter; the rest of the parameters are embedded in the structure of D_1. What is important is the spectrum of D_{false}. It happens that for $t_8 > 1/6$ dynamical matrix will have at least one negative eigenvalue, since we have $\mathrm{Spec}(D_2) = \{1/3, 1/3, 1/3(1-6t_8)\}$. The desired situation in which channel Φ_{false} produces state (58) with $\rho_{33} = \frac{1}{2}(1-2t_8) = 0$ is impossible, since the value $t_8 = 1/2$ is not CP allowed.

We can take the subclass of the false qutrit-qubit channels to find out what is the geometry of the CP-allowed region in the channel parameter space. Let us choose a value of t_8 translation within the CP-allowed range, for example, $t_8 = 1/9$. Let us assume that the transformation is of the form

$$\Phi_6: \vec{n} \to (\tilde{\Lambda}_1 n_1, \tilde{\Lambda}_2 n_2, \tilde{\Lambda}_3 n_3, 0, 0, 0, 0, 1/9), \tag{61}$$

in other words, it is governed by three parameters: $\tilde{\Lambda}_1, \tilde{\Lambda}_2, \tilde{\Lambda}_3$. The CP-allowed region for channels given by Eq. (61) is characterized by the following polynomial, in which roots must be non-negative:

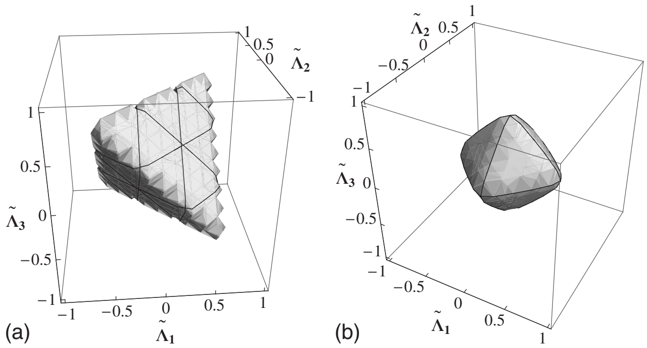

FIG. 7. The figures show (a) CP-allowed region for false qutrit-qubit channels given by Eq. (61) and (b) region of parameters $\tilde{\Lambda}_i$ for which false qutrit-qubit channels given by Eq. (61) satisfy PPT; channel parameters $\tilde{\Lambda}_i$ are dimensionless.

$$P_{\pm}(x) = -16 + 30(\tilde{\Lambda}_3 \pm \tilde{\Lambda}_2)^2 \mp 44\tilde{\Lambda}_1 - 30\tilde{\Lambda}_1^2 + [132 \pm 240\tilde{\Lambda}_1 + 81\tilde{\Lambda}_1^2 - 81(\tilde{\Lambda}_2 \pm \tilde{\Lambda}_3)^2]x + (-360 \mp 324\tilde{\Lambda}_1)x^2 + 324x^3 \qquad (62)$$

and is shown on Fig. 7(a). It resembles the well-known tetrahedron structure of CP-allowed region for damping parameters of qubit channel. Additionally to that, one can ask about the separability problem attached to this issue. In case of two-qubit states it is enough to check the partial transpose to answer the question of state separability. In case of qutrit states, it is not enough, however—one can analyze the region for which PPT is satisfied. For the bipartite state corresponding to Eq. (61) to satisfy PPT, one must have non-negative roots of polynomials $N_{\pm}(x)$,

$$N_{\pm}(x) = -16 + 30(\tilde{\Lambda}_3 \mp \tilde{\Lambda}_2)^2 \mp 44\tilde{\Lambda}_1 - 30\tilde{\Lambda}_1^2 + [132 \pm 240\tilde{\Lambda}_1 + 81\tilde{\Lambda}_1^2 - 81(\tilde{\Lambda}_2 \mp \tilde{\Lambda}_3)^2]x + (-360 \mp 324\tilde{\Lambda}_1)x^2 + 324x^3. \qquad (63)$$

Figure 7(b) shows the region for which both CP and PPT conditions are satisfied for channels (61).

G. Two-qutrit states and affine transformations of qutrit Bloch vectors

As already said, the dynamical matrix D_Φ corresponds to a density matrix via $\rho_\Phi = \frac{1}{N} D_\Phi$. The latter, in our case ($N=3$) describes a class of two-qutrit states that can be parametrized by $\{\Lambda_i, t_i\}^{CPM}$. The two qutrit state space is being investigated, especially the so-called magic simplex which can be considered an analog of the magic tetrahedron of bipartite qubits [15]. The magic simplex of bipartite qutrits is only embedded in the space of all bipartite qutrits. As an example, it does not contain a state given by the density operator

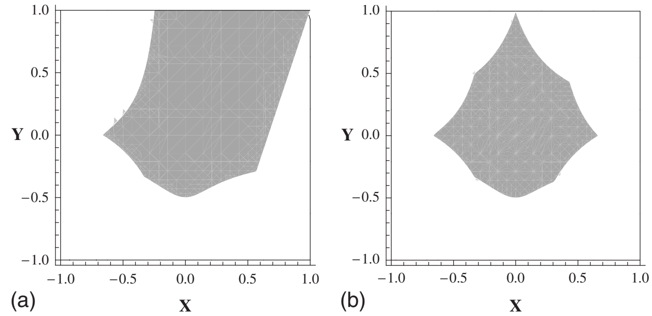

FIG. 8. (a) CP-allowed region for a damping qutrit channel parametrized with $\Lambda_3 = X, \Lambda_8 = Y, \Lambda_{i\neq 3,8} = XY$ and (b) region of parameters X, Y for which the same channel satisfies PPT; X, Y are dimensionless.

$$\rho = \frac{1}{3}(|\Psi\rangle\langle\Psi| + 2|\Phi\rangle\langle\Phi|), \qquad (64)$$

where $|\Psi\rangle = |0,0\rangle$ and $|\Psi\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + |2,2\rangle)$. Interestingly, this state can be obtained from $\rho_\Phi = \frac{1}{N} D_\Phi(\mathrm{N}=3)$ with a proper choice of parameters: $\Lambda_{3,6,7,8} = 1$ and the rest equal to 0. A damping channel with such parameter values will transform any qutrit Bloch vector according to

$$\vec{n} \to \vec{n}' = \{0,0,n_3,0,0,n_6,n_7,n_8\}. \qquad (65)$$

Geometrically, the channel projects the Bloch vector on to the 3–6–7–8 subspace and the other components of $\vec{n}$ are lost. It is therefore a phase flip type channel [6].

H. Qutrit entanglement breaking channels

For bipartite qutrit states there is no sufficient and necessary condition for entanglement detection. Therefore, one cannot perform the same entanglement breaking channel analysis as before, in case of qutrit-qubit channels. However, one can still ask when the two qutrit states, or qutrit channels, satisfy PPT condition.

First, a qutrit channel that transforms qutrit Bloch vector according to Eq. (36) satisfies PPT when the characteristic polynomial of the corresponding dynamical matrix D_Φ is symmetric with respect to the sign change of $\Lambda_{2,5,7}, t_{2,5,7}$. PPT means evaluating non-negativity of partially transposed dynamical matrix D_Φ. In the general case, the easiest way is to analyze the characteristic polynomial. One obtains seven nonlinear constraints on damping and translation channel parameters. In some specific cases these might simplify to linear inequalities defining hyperplanes. The simplest general nonlinear constraint for PPT has the same form as Eq. (55), since this inequality has the symmetric property mentioned above. On Fig. 8 we show an example of a damping channel with $\Lambda_3 = X, \Lambda_8 = Y, \Lambda_{i\neq 3,8} = XY$ for which we have evaluated CP-allowed region [Fig. 8(a)] and on the top of that we show parameter values for which this channel satisfies PPT [Fig. 8(b)].

VII. SUMMARY

In the paper we have analyzed the complete positivity (CP) constraints on channel parameters. We have investigated general qutrit-qubit channels and qutrit channels which have a form of affine transformation on qutrit Bloch vectors. We were working with the Choi-Sudarshan dynamical matrix that allows to evaluate the complete positivity of any transformation. Our method was to evaluate positivity of dynamical matrix by direct calculation of eigenvalues or analysis of the characteristic polynomial.

Our results show that for qutrit-qubit channels, the CP constraints are no longer strictly linear—there are nonlinearities appearing. For some types of these channels, however, one might obtain linear inequalities on channels' parameters (defining hyperplanes), but in general—nonlinear. Not surprisingly, when working with qutrit channels, we encounter the same effect. In our work we have repeated the results for damping qutrit channels that were obtained in [14] but with a slightly different method. On top of that, we have presented the Kraus representation for this type of channels. For damping qutrit channels there are seven linear inequalities for channels' parameters and two nonlinear. Furthermore, we have analyzed complete positivity of channels that move any qutrit Bloch vector to a specific vector (channels based only on translation). For these channels we have shown the CP constraints, which reveal again nonlinear structures in the CP-parameter space. Followed by that, we have analyzed the combined action of damping and shifting of qutrit Bloch vectors with respect to complete positivity. As an interesting example, we chose here a special class of qutrit channels, *false* qutrit-qubit channels, which when analyzed reveal similar CP properties to qubit channels.

Knowing the Choi-Jamiołkowski-Sudarshan isomorphism between the channels and states, we have investigated the entanglement issues related to the channels' classes we have worked with. Especially in case of qutrit-qubit channels, when we know the sufficient and necessary condition for entanglement detection (PPT Peres-Horodeccy criterion), we have analyzed the entanglement breaking channels. Not every qutrit-qubit channel is entanglement breaking—we have presented a class of channels that are not entanglement breaking. In case of bipartite qutrit states, although PPT condition is not enough, we have discussed the *PPT qutrit channels*. We also show that the class of qutrit channels we work with corresponds to a class of bipartite qutrit states that cannot be fully described by the magic simplex states. We are working on establishing the relation between these two.

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APPENDIX A

In Sec. V we have presented CP conditions for qutrit-qubit damping channels. The general case includes many channel parameters Λ_{ij}—therefore it is arduous to present all the inequalities. We have shown already one of them—quadratic in channels parameters, here we present the inequality which has cubic terms in Λ_{ij},

$$3\sum_{ij}\Lambda_{ij}^2+\frac{1}{4}F(\Lambda_{ij})>10, \tag{A1}$$

where

$$F(\Lambda_{ij})=F(\Lambda_{11},\ldots,\Lambda_{18},\Lambda_{21},\ldots,\Lambda_{28},\Lambda_{31},\ldots,\Lambda_{38}),$$

and

$$\begin{aligned}F(\Lambda_{ij})\equiv{}&3\sqrt{3}\Lambda_{31}(2\Lambda_{13}\Lambda_{22}+\Lambda_{17}\Lambda_{24}-\Lambda_{16}\Lambda_{25}+\Lambda_{15}\Lambda_{26}-\Lambda_{14}\Lambda_{27})+3\sqrt{3}\Lambda_{32}(-2\Lambda_{13}\Lambda_{21}+\Lambda_{16}\Lambda_{24}+\Lambda_{17}\Lambda_{25}-\Lambda_{14}\Lambda_{26}\\&-\Lambda_{15}\Lambda_{27})+3\sqrt{3}\Lambda_{33}(\Lambda_{15}\Lambda_{24}-\Lambda_{14}\Lambda_{25}-\Lambda_{17}\Lambda_{26}+\Lambda_{16}\Lambda_{27})+3\sqrt{3}\Lambda_{34}(-\Lambda_{17}\Lambda_{21}-\Lambda_{16}\Lambda_{22}-\Lambda_{15}\Lambda_{23}+\Lambda_{13}\Lambda_{25}\\&+\sqrt{3}\Lambda_{18}\Lambda_{25}-\sqrt{3}\Lambda_{15}\Lambda_{28})+3\sqrt{3}\Lambda_{34}(-\Lambda_{17}\Lambda_{21}-\Lambda_{16}\Lambda_{22}-\Lambda_{15}\Lambda_{23}+\Lambda_{13}\Lambda_{25}+\sqrt{3}\Lambda_{18}\Lambda_{25}-\sqrt{3}\Lambda_{15}\Lambda_{28})\\&+3\sqrt{3}\Lambda_{35}(\Lambda_{16}\Lambda_{21}-\Lambda_{17}\Lambda_{22}+\Lambda_{14}\Lambda_{23}-\Lambda_{13}\Lambda_{24}-\sqrt{3}\Lambda_{18}\Lambda_{24}+\sqrt{3}\Lambda_{14}\Lambda_{28})+3\sqrt{3}\Lambda_{36}(-\Lambda_{15}\Lambda_{21}+\Lambda_{14}\Lambda_{22}+\Lambda_{17}\Lambda_{23}\\&-\Lambda_{13}\Lambda_{27}+\sqrt{3}\Lambda_{18}\Lambda_{27}-\sqrt{3}\Lambda_{17}\Lambda_{28})+3\sqrt{3}\Lambda_{37}(\Lambda_{14}\Lambda_{21}+\Lambda_{15}\Lambda_{22}-\Lambda_{16}\Lambda_{23}+\Lambda_{13}\Lambda_{26}-\sqrt{3}\Lambda_{18}\Lambda_{26}+\sqrt{3}\Lambda_{16}\Lambda_{28})\\&+3\sqrt{3}\Lambda_{11}(2\Lambda_{23}\Lambda_{32}-2\Lambda_{22}\Lambda_{33}+\Lambda_{27}\Lambda_{34}-\Lambda_{26}\Lambda_{35}+\Lambda_{25}\Lambda_{36}-\Lambda_{24}\Lambda_{37})-3\sqrt{3}\Lambda_{12}(2\Lambda_{23}\Lambda_{31}-2\Lambda_{21}\Lambda_{33}-\Lambda_{26}\Lambda_{34}\\&-\Lambda_{27}\Lambda_{35}+\Lambda_{24}\Lambda_{36}+\Lambda_{25}\Lambda_{37})+9\Lambda_{38}(\Lambda_{15}\Lambda_{24}-\Lambda_{14}\Lambda_{25}+\Lambda_{17}\Lambda_{26}-\Lambda_{16}\Lambda_{27}).\end{aligned} \tag{A2}$$

In an analogous way, we have analyzed qutrit-qubit channels which apart from damping action allow shifting (Sec. V B). We have shown before a quadratic inequality for channel parameters Λ_{ij},l_i; here we show the cubic inequality

$$3\sum_{ij}\Lambda_{ij}^2+\frac{1}{4}F(\Lambda_{ij})+6\sum_{i=1}^{3}l_i^2>10. \tag{A3}$$

APPENDIX B

In the discussion of the damping qutrit channels we presented a set of conditions [Eqs. (42) and (43)] that come from imposing positivity condition on dynamical matrix D_Φ. They correspond to eigenvalues of the matrix and when calculated explicitly, three of them have sophisticated form. In qutrit case, the dynamical matrix D_Φ is nine dimensional. We already showed six eigenvalues in the form of inequalities. The remaining three correspond to finding roots of polynomial of the third order. The polynomial $P(x)=Ax^3+Bx^2+Cx+D$ which we analyze in order to obtain the rest of CP conditions (eigenvalues $d_{7,8,9}$) has coefficients

$$A = 8, \quad B = -24(1 + \Lambda_3 + \Lambda_8),$$

$$C = 18[(\Lambda_3 + \Lambda_8)^2 - (\Lambda_1 + \Lambda_2)^2 - (\Lambda_4 + \Lambda_5)^2 - (\Lambda_6 + \Lambda_7)^2] + 24(1 + 2\Lambda_3 + 2\Lambda_8 + \Lambda_3\Lambda_8),$$

$$D = -8 - 18[(\Lambda_3 + \Lambda_8)^2 - (\Lambda_1 + \Lambda_2)^2 - (\Lambda_4 + \Lambda_5)^2 - (\Lambda_6 + \Lambda_7)^2] + 27\Lambda_3[(\Lambda_4 + \Lambda_5)^2 + (\Lambda_6 + \Lambda_7)^2] - 54(\Lambda_1 + \Lambda_2) \times(\Lambda_4 + \Lambda_5)(\Lambda_6 + \Lambda_7) + 9\Lambda_8[4(\Lambda_1 + \Lambda_2)^2 + (\Lambda_4 + \Lambda_5)^2 + (\Lambda_6 + \Lambda_7)^2] - 24(\Lambda_8 + \Lambda_3 + \Lambda_3\Lambda_8) - 4\Lambda_8(\Lambda_3 + \Lambda_8)^2 - 32\Lambda_8\Lambda_3^2 - 20\Lambda_8^2\Lambda_3.$$

The roots can be of course found explicitly, but we do not present them here. What is interesting, for some specific parameter values, number of real roots can be reduced (and therefore number of CP inequalities). To deduce that one has to analyze function

$$f(a,b,c) = 3^{-6}(3b - a^2)^3 + 54^{-2}(9ab - 27c - 2a^3)^2$$

and evaluate it at $f(B/8, C/8, D/8)$ (where B, C, D are polynomial coefficients given above). For the dynamical matrix D_Φ this function is always nonpositive (and indicates therefore real roots). When $f(B/8, C/8, D/8)=0$ the polynomial has three real roots and at least two are equal—then the number of CP conditions is reduced.

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