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Dephasing of two interacting qubits under the influence of thermal reservoirs

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A phase relaxation process of interacting two qubits is studied by means of the quantum master equation, where each qubit is influenced by an independent thermal reservoir. The damping operator of the quantum master equation consists of two parts; one includes the effects of the interaction between the qubits and the other does not. It is shown that the interaction significantly affects the phase relaxation process. The decoherence of entanglement is investigated for the two qubits initially prepared in the X state.

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I. INTRODUCTION

A physical system is not closed in the real world and it is always influenced by a thermal reservoir which is a huge environmental system. The quantum-mechanical properties such as coherence and entanglement [1,2] are destructed by an interaction between a system and thermal reservoir. Hence the decoherence inevitably takes place in the real world and a physical system undergoes an irreversible time evolution or a relaxation process. Such an irreversible time evolution of a system can be formulated by various methods such as the projection operator method [3,4], the path-integral method [5,6], and the stochastic method [7]. The quantum master equations derived by means of the projection operator method [4] and the path-integral method [8] are very useful for investigating many physical systems influenced by thermal reservoirs in quantum optics and quantum information [4,9,10]. It is, however, difficult to derive the quantum master equations for physical systems, which consist of interacting subsystems under the influence of thermal reservoirs. As pointed out in Ref. [11], the effects of the interaction between subsystems on the relaxation process are essential for deriving the correct irreversible time evolutions of the systems [12,13]. Decoherence of interacting two-qubit system has recently been investigated by the quantum master equations with the damping operators, which include the effects of the interaction between the qubits [14–17]. In these studies, however, the interaction effects on the relaxation of the system are not so clear. Therefore in this paper, we would like to investigate the effects of the interaction between qubits on the phase relaxation process in detail

This paper is organized as follows. In Sec. II, we derive the quantum master equation for an interacting two-qubit system under the influence of independent thermal reservoirs, where the dephasing couplings between the qubits and thermal reservoirs are assumed. The damping operator of the quantum master equation is obtained in the form that makes clear the effects of the interaction between the qubits on the phase relaxation process of the system. In Sec. III, we obtain the nonequilibrium dynamics of two-qubit entanglement in the Markovian approximation, where the two qubits are initially prepared in the X state, which includes the Bell state, the Werner state, and the maximally entangled mixed state. In Sec. IV, we investigate the decay of entanglement for several two-qubit states. We will show that the interaction between the qubits significantly affects the decay of the entanglement. The results are compared with those obtained when the interaction effects on the phase relaxation process are ignored. In Sec. V, we provide concluding remarks.

II. QUANTUM MASTER EQUATION FOR DEPHASING

We consider a time evolution of interacting two qubits, referred to as qubit A and qubit B, where each qubit is influenced by an independent reservoir in the thermal equilibrium state. We suppose that the Hamiltonian $\hat{H}_Q$ of the two-qubit system is given by [18,19]

$$\hat{H}_Q = \hbar\omega(\hat{S}_A^z + \hat{S}_B^z) + \hbar g(\hat{S}_A^+\hat{S}_B^- + \hat{S}_A^-\hat{S}_B^+), \tag{1}$$

where $\hat{S}_A$ and $\hat{S}_A^{\pm}=\hat{S}_A^x \pm i\hat{S}_A^y$ ($\hat{S}_B$ and $\hat{S}_B^{\pm}=\hat{S}_B^x \pm i\hat{S}_B^y$) are spin-1/2 operators of the qubit A (the qubit B). The eigenstates of the two-qubit Hamiltonian $\hat{H}_Q$ are $|\Phi_0\rangle=|00\rangle$, $|\Phi_1\rangle=|11\rangle$, and $|\Psi_{\pm}\rangle=(|01\rangle \pm |10\rangle)/\sqrt{2}$, and the corresponding eigenvalues are given by $\hbar\omega$, $-\hbar\omega$, and $\pm\hbar g$, where $|0\rangle$ and $|1\rangle$ are the eigenstate of the z component of the spin-1/2 operator, such that $\hat{S}^z|0\rangle=\frac{1}{2}|0\rangle$ and $\hat{S}^z|1\rangle=-\frac{1}{2}|1\rangle$. Then the spin operators $\hat{S}_A^z$ and $\hat{S}_B^z$ are evolved by the Hamiltonian $\hat{H}_Q$ as follows:

$$\hat{S}_A^z(t) = e^{(it/\hbar)\hat{H}_Q}\hat{S}_A^z e^{-(it/\hbar)\hat{H}_Q} = \hat{S}_A^z - \frac{1}{2}(1-e^{2igt})|\Psi_+\rangle\langle\Psi_-| - \frac{1}{2}(1 - e^{-2igt})|\Psi_-\rangle\langle\Psi_+|, \tag{2}$$

$$\hat{S}_B^z(t) = e^{(it/\hbar)\hat{H}_Q}\hat{S}_B^z e^{-(it/\hbar)\hat{H}_Q} = \hat{S}_B^z + \frac{1}{2}(1-e^{2igt})|\Psi_+\rangle\langle\Psi_-| + \frac{1}{2}(1 - e^{-2igt})|\Psi_-\rangle\langle\Psi_+|, \tag{3}$$

which are derived in Appendix A. These equations are used to derive the quantum master equation of the interacting two-qubit system. The equality $\hat{S}_A^z(t)+\hat{S}_B^z(t)=\hat{S}_A^z+\hat{S}_A^z$ holds due to the commutation relation $[\hat{H}_Q,\hat{S}_A^z+\hat{S}_B^z]=0$.

We suppose that each qubit undergoes a phase relaxation process or dephasing, which is caused by an independent thermal reservoir. In this case, the interaction Hamiltonian between the two qubits and thermal reservoirs is assumed to be

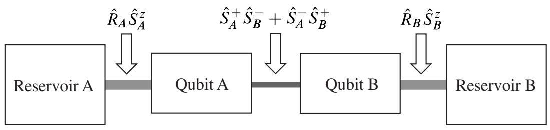

FIG. 1. The schematic representation of the interacting qubits under the influence of the individual thermal reservoirs, where $\hat{S}^z_{A,B}$ and $\hat{S}^{\pm}_{A,B}$ are spin-1/2 operators and $\hat{R}_{A,B}$ is a Hermitian operator of the thermal reservoir. The interaction between the qubits affects the relaxation process.

$$\hat{H}_{QR} = \hbar\lambda_A \hat{S}_A^z \hat{R}_A + \hbar\lambda_A \hat{S}_B^z \hat{R}_B, \tag{4}$$

where $\hat{R}_A$ and $\hat{R}_B$ are some Hermitian operators of the thermal reservoirs. The system that we consider is depicted in Fig. 1. We denote the density operator of the total system as $\hat{W}(t)$, the time-evolution of which is determined by the Liouville-von Neumann equation,

$$\frac{\partial}{\partial t}\hat{W}(t) = -\frac{i}{\hbar}[\hat{H}_Q + \hat{H}_R + \hat{H}_{QR}, \hat{W}(t)], \tag{5}$$

where $\hat{H}_R$ represents the Hamiltonian of the two independent thermal reservoirs. In the interaction picture, we have

$$\frac{\partial}{\partial t}\hat{W}^{\rm int}(t) = -\frac{i}{\hbar}[\hat{H}^{\rm int}_{QR}(t), \hat{W}^{\rm int}(t)]. \tag{6}$$

In this equation, we set the operators $\hat{W}^{\rm int}(t) = e^{(it/\hbar)(\hat{H}_Q+\hat{H}_R)}\hat{W}(t)e^{-(it/\hbar)(\hat{H}_Q+\hat{H}_R)}$ and $\hat{H}^{\rm int}_{QR}(t) = e^{(it/\hbar)(\hat{H}_Q+\hat{H}_R)}\hat{H}_{QR}e^{-(it/\hbar)(\hat{H}_Q+\hat{H}_R)}$. Furthermore we assume that the system and thermal reservoir are uncoupled at the initial time and the strength of the system reservoir interaction is sufficiently weak. Then, applying the projection operator method, we can derive the time-convolutionless quantum master equation for the reduced density operator $\hat{W}^{\rm int}_Q(t) = {\rm Tr}_R\, \hat{W}^{\rm int}(t)$ of the two-qubit system, up to the second order with respect to the system-reservoir interaction [3,4],

$$\begin{aligned}\frac{\partial}{\partial t}\hat{W}^{\rm int}_Q(t) &= -\frac{1}{\hbar^2}\int_0^t dt'\ {\rm Tr}_R[\hat{H}^{\rm int}_{QR}(t),[\hat{H}^{\rm int}_{QR}(t'), \hat{W}^{\rm int}_Q(t)\otimes\hat{\rho}_R]] \\ &= \hat{\mathcal{L}}^{\rm int}_A(t)\hat{W}^{\rm int}_Q(t) + \hat{\mathcal{L}}^{\rm int}_B(t)\hat{W}^{\rm int}_Q(t),\end{aligned} \tag{7}$$

where $\hat{\rho}_R$ is the equilibrium state of the thermal reservoirs and ${\rm Tr}_R$ stands for the trace operation over the reservoir Hilbert spaces. In this equation, the damping operators $\hat{\mathcal{L}}^{\rm int}_A(t)$ and $\hat{\mathcal{L}}^{\rm int}_B(t)$ are given by

$$\begin{aligned}\hat{\mathcal{L}}^{\rm int}_A(t)\hat{W}^{\rm int}_Q(t) = &-\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t)\hat{R}_A(t')\rangle\hat{S}^z_A(t)\hat{S}^z_A(t')\hat{W}^{\rm int}_Q(t) \\ &+\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t')\hat{R}_A(t)\rangle\hat{S}^z_A(t)\hat{W}^{\rm int}_Q(t)\hat{S}^z_A(t') \\ &+\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t)\hat{R}_A(t')\rangle\hat{S}^z_A(t')\hat{W}^{\rm int}_Q(t)\hat{S}^z_A(t) \\ &-\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t')\hat{R}_A(t)\rangle\hat{W}^{\rm int}_Q(t)\hat{S}^z_A(t')\hat{S}^z_A(t),\end{aligned} \tag{8}$$

$$\begin{aligned}\hat{\mathcal{L}}^{\rm int}_B(t)\hat{W}^{\rm int}_Q(t) = &-\lambda_B^2\int_0^t dt'\langle\hat{R}_B(t)\hat{R}_B(t')\rangle\hat{S}^z_B(t)\hat{S}^z_B(t')\hat{W}^{\rm int}_Q(t) \\ &+\lambda_B^2\int_0^t dt'\langle\hat{R}_B(t')\hat{R}_B(t)\rangle\hat{S}^z_B(t)\hat{W}^{\rm int}_Q(t)\hat{S}^z_B(t') \\ &+\lambda_B^2\int_0^t dt'\langle\hat{R}_B(t)\hat{R}_B(t')\rangle\hat{S}^z_B(t')\hat{W}^{\rm int}_Q(t)\hat{S}^z_B(t) \\ &-\lambda_B^2\int_0^t dt'\langle\hat{R}_B(t')\hat{R}_B(t)\rangle\hat{W}^{\rm int}_Q(t)\hat{S}^z_B(t')\hat{S}^z_B(t),\end{aligned} \tag{9}$$

where $\hat{S}^z_A(t)$ and $\hat{S}^z_B(t)$ are given by Eqs. (2) and (3), and $\langle\cdots\rangle = {\rm Tr}(\cdots\hat{\rho}_R)$. Furthermore, in deriving Eq. (7), we have assumed that the equality $\langle\hat{R}_A(t)\rangle = \langle\hat{R}_B(t)\rangle = 0$ holds.

In the Schrödinger picture, the time-convolutionless quantum master equation of the interacting two-qubit system becomes

$$\frac{\partial}{\partial t}\hat{W}_Q(t) = -\frac{i}{\hbar}[\hat{H}_Q, \hat{W}_Q(t)] + \hat{\mathcal{L}}_A(t)\hat{W}_Q(t) + \hat{\mathcal{L}}_B(t)\hat{W}_Q(t), \tag{10}$$

with

$$\begin{aligned}\hat{\mathcal{L}}_A(t)\hat{W}_Q(t) = &-\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t')\hat{R}_A(0)\rangle\hat{S}^z_A\hat{S}^z_A(-t')\hat{W}_Q(t) \\ &+\lambda_A^2\int_0^t dt'\langle\hat{R}_A(0)\hat{R}_A(t')\rangle\hat{S}^z_A\hat{W}_Q(t)\hat{S}^z_A(-t') \\ &+\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t')\hat{R}_A(0)\rangle\hat{S}^z_A(-t')\hat{W}_Q(t)\hat{S}^z_A \\ &-\lambda_A^2\int_0^t dt'\langle\hat{R}_A(0)\hat{R}_A(t')\rangle\hat{W}_Q(t)\hat{S}^z_A(-t')\hat{S}^z_A,\end{aligned} \tag{11}$$

$$\begin{aligned}\hat{\mathcal{L}}_B(t)\hat{W}_Q(t) = &-\lambda_B^2\int_0^t dt'\langle\hat{R}_B(t')\hat{R}_B(0)\rangle\hat{S}^z_B\hat{S}^z_B(-t')\hat{W}_Q(t) \\ &+\lambda_B^2\int_0^t dt'\langle\hat{R}_B(0)\hat{R}_B(t')\rangle\hat{S}^z_B\hat{W}_Q(t)\hat{S}^z_B(-t') \\ &+\lambda_B^2\int_0^t dt'\langle\hat{R}_B(t')\hat{R}_B(0)\rangle\hat{S}^z_B(-t')\hat{W}_Q(t)\hat{S}^z_B \\ &-\lambda_B^2\int_0^t dt'\langle\hat{R}_B(0)\hat{R}_B(t')\rangle\hat{W}_Q(t)\hat{S}^z_B(-t')\hat{S}^z_B,\end{aligned} \tag{12}$$

where the derivation is provided in Appendix B. Substituting Eqs. (2) and (3) into these equations and using the relations $\hat{S}_A^z|\Psi_\pm\rangle = \frac{1}{2}|\Psi_\mp\rangle$ and $\hat{S}_B^z|\Psi_\pm\rangle = -\frac{1}{2}|\Psi_\mp\rangle$, we can derive the explicit forms of the damping operators $\hat{\mathcal{L}}_A(t)$ and $\hat{\mathcal{L}}_B(t)$. After a straightforward calculation, we finally obtain the time-convolutionless quantum master equation of the interacting two-qubit system,

$$\frac{\partial}{\partial t}\hat{W}_Q(t) = -\frac{i}{\hbar}[\hat{H}_Q, \hat{W}_Q(t)] + \hat{\mathcal{L}}_Q^{(0)}(t)\hat{W}_Q(t) + \hat{\mathcal{L}}_Q^{(1)}(t)\hat{W}_Q(t), \tag{13}$$

where the damping operators $\hat{\mathcal{L}}_Q^{(0)}(t)$ and $\hat{\mathcal{L}}_Q^{(1)}(t)$ are given by

$$\hat{\mathcal{L}}_Q^{(0)}(t)\hat{W}_Q(t) = \phi(t)[\hat{S}_A^z\hat{W}_Q(t), \hat{S}_A^z] + \phi^*(t)[\hat{S}_A^z, \hat{W}_Q(t)\hat{S}_A^z] + \phi(t) \times[\hat{S}_B^z\hat{W}_Q(t), \hat{S}_B^z] + \phi^*(t)[\hat{S}_B^z, \hat{W}_Q(t)\hat{S}_B^z], \tag{14}$$

and

$$\begin{aligned}\hat{\mathcal{L}}_Q^{(1)}(t)\hat{W}_Q(t) = &-\psi_-(t)[\hat{S}_A^z|\Psi_-\rangle\langle\Psi_-|\hat{W}_Q(t), \hat{S}_A^z] - \psi_-^*(t) \\ &\times[\hat{S}_A^z, \hat{W}_Q(t)|\Psi_-\rangle\langle\Psi_-|\hat{S}_A^z] - \psi_+(t)[\hat{S}_A^z|\Psi_+\rangle \\ &\times\langle\Psi_+|\hat{W}_Q(t), \hat{S}_A^z] - \psi_+^*(t)[\hat{S}_A^z, \hat{W}_Q(t)|\Psi_+\rangle \\ &\times\langle\Psi_+|\hat{S}_A^z] - \psi_-(t)[\hat{S}_B^z|\Psi_-\rangle\langle\Psi_-|\hat{W}_Q(t), \hat{S}_B^z] \\ &- \psi_-^*(t)[\hat{S}_B^z, \hat{W}_Q(t)|\Psi_-\rangle\langle\Psi_-|\hat{S}_B^z] - \psi_+(t)[\hat{S}_B^z|\Psi_+\rangle \\ &\times\langle\Psi_+|\hat{W}_Q(t), \hat{S}_B^z] - \psi_+^*(t)[\hat{S}_B^z, \hat{W}_Q(t)|\Psi_+\rangle \\ &\times\langle\Psi_+|\hat{S}_B^z].\end{aligned} \tag{15}$$

In these equations, the functions $\phi(t)$ and $\psi_\pm(t)$ are given by

$$\phi(t) = \lambda^2\int_0^t dt'\langle\hat{R}(t')\hat{R}(0)\rangle, \tag{16}$$

$$\psi_\pm(t) = \lambda^2\int_0^t dt'\langle\hat{R}(t')\hat{R}(0)\rangle(1 - e^{\pm 2igt'}). \tag{17}$$

Here we have assumed that the two independent thermal reservoirs interacting with the qubits have the same properties and thus we have dropped the subscripts A and B. The damping operator $\hat{\mathcal{L}}_Q^{(1)}(t)$ includes the effects of the interaction between the two qubits while $\hat{\mathcal{L}}_Q^{(0)}(t)$ does not. If we ignore the interaction effects on the phase relaxation process of the system, the damping operator $\hat{\mathcal{L}}_Q^{(1)}(t)$ does not appear in the quantum master equation. In fact, we have $\lim_{g\to 0}\hat{\mathcal{L}}_Q^{(1)}(t) = 0$.

III. NONEQUILIBRIUM DYNAMICS OF ENTANGLEMENT

Using the quantum master Eq. (13) with Eqs. (14) and (15), we can investigate the nonequilibrium dynamics of two-qubit entanglement in the phase relaxation process. For this purpose, we suppose that the two qubits are initially prepared in the X state [20],

$$\hat{W}_Q(0) = \begin{pmatrix} a & 0 & 0 & x \\ 0 & b & y & 0 \\ 0 & y^* & c & 0 \\ x^* & 0 & 0 & d \end{pmatrix}, \tag{18}$$

where we have used the standard base $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Since $\mathrm{Tr}\,\hat{W}_Q(t) = 1$ and $\hat{W}_Q(0) > 0$, the matrix elements a, b, c, d are non-negative parameters satisfying the relations $a+b+c+d=1$, $\sqrt{ad} \geq |x|$, and $\sqrt{bc} \geq |y|$ since the normalization and positivity of $\hat{W}_Q(0)$. The X state $\hat{W}_Q(0)$ can be expressed in terms of the eigenstates $|\Phi_0\rangle$, $|\Phi_1\rangle$, and $|\Psi_\pm\rangle$ of the Hamiltonian $\hat{H}_Q$ as

$$\begin{aligned}\hat{W}_Q(0) = {}& a|\Phi_0\rangle\langle\Phi_0| + d|\Phi_1\rangle\langle\Phi_1| + x|\Phi_0\rangle\langle\Phi_1| + x^*|\Phi_1\rangle\langle\Phi_0| \\ &+ \frac{1}{2}(b + c + y + y^*)|\Psi_+\rangle\langle\Psi_+| + \frac{1}{2}(b + c - y - y^*)|\Psi_-\rangle \\ &\times\langle\Psi_-| + \frac{1}{2}(b - c - y + y^*)|\Psi_+\rangle\langle\Psi_-| + \frac{1}{2}(b - c + y \\ &- y^*)|\Psi_-\rangle\langle\Psi_+|.\end{aligned} \tag{19}$$

It is easy to see that the form of the X state is invariant during the time evolution generated by the quantum master Eq. (13). Thus we can express the two-qubit state $\hat{W}_Q(t)$ at time t as

$$\begin{aligned}\hat{W}_Q(t) = {}& A(t)|\Phi_0\rangle\langle\Phi_0| + D(t)|\Phi_1\rangle\langle\Phi_1| + X(t)|\Phi_0\rangle\langle\Phi_1| + X^*(t) \\ &\times|\Phi_1\rangle\langle\Phi_0| + B(t)|\Psi_+\rangle\langle\Psi_+| + C(t)|\Psi_-\rangle\langle\Psi_-| + Y(t) \\ &\times|\Psi_+\rangle\langle\Psi_-| + Y^*(t)|\Psi_-\rangle\langle\Psi_+|,\end{aligned} \tag{20}$$

with $A(0)=a$, $B(0)=(1/2)(b+c+y+y^*)$, $C(0)=(1/2)(b+c-y-y^*)$, $D(0)=d$, $X(0)=x$, and $Y(0)=(1/2)(b-c-y+y^*)$. Substituting Eq. (20) into the quantum master Eq. (13), we can derive the differential equations for the parameters $A(t)$, $B(t)$, $C(t)$, $D(t)$, $X(t)$, and $Y(t)$,

$$\dot{A}(t) = 0, \tag{21}$$

$$\dot{B}(t) = -\frac{1}{2}[\phi_+(t) + \phi_+^*(t)]B(t) + \frac{1}{2}[\phi_-(t) + \phi_-^*(t)]C(t), \tag{22}$$

$$\dot{C}(t) = -\frac{1}{2}[\phi_-(t) + \phi_-^*(t)]C(t) + \frac{1}{2}[\phi_+(t) + \phi_+^*(t)]B(t), \tag{23}$$

$$\dot{D}(t) = 0, \tag{24}$$

$$\dot{X}(t) = [-2i\omega - \phi(t) - \phi^*(t)]X(t), \tag{25}$$

$$\dot{Y}(t) = \left\{-2ig - \frac{1}{2}[\phi_+ + \phi_-^*(t)]\right\}Y(t) + \frac{1}{2}[\phi_-(t) + \phi_+^*(t)]Y^*(t), \tag{26}$$

where we set $\phi_\pm(t) = \phi(t) \pm \psi_\pm(t)$, that is,

$$\phi_{\pm}(t) = \lambda^2 \int_0^t dt' \langle \hat{R}(t')\hat{R}(0) > e^{\pm 2igt'}. \quad (27)$$

In the rest of this paper, we assume that the Markovian approximation [4] can be applied. In this case, the time-dependent functions $\phi(t)$ and $\phi_{\pm}(t)$ are replaced with $\phi(\infty)$ and $\phi_{\pm}(\infty)$. Furthermore, we assume that the correlation function of the thermal reservoir decays exponentially with time t, that is, $\langle \hat{R}(t)\hat{R}(0)\rangle = \Delta^2 e^{-t/\tau_c}$, where we denote the relaxation time as τ_c. Then $\phi(\infty)$ and $\phi_{\pm}(\infty)$ are given by

$$\phi(\infty) = \Delta^2 \tau_c \equiv \frac{1}{T_{\rm ph}}, \quad (28)$$

$$\phi_{\pm}(\infty) = \Delta^2 \tau_c \frac{1 \pm 2ig\tau_c}{1+(2g\tau_c)^2} = \frac{1}{T_{\rm ph}}\left(\frac{1 \pm 2i\alpha}{1+4\alpha^2}\right), \quad (29)$$

where we set $\alpha = g\tau_c$. The parameter $T_{\rm ph}$ represents the dephasing time of the qubit. Substituting Eqs. (28) and (29) into Eqs. (21)–(26) and solving the equations, we finally obtain the two-qubit state $\hat{W}_Q(t)$ at time t,

$$\hat{W}_Q(t) = \begin{pmatrix} a(t) & 0 & 0 & x(t) \\ 0 & b(t) & y(t) & 0 \\ 0 & y^*(t) & c(t) & 0 \\ x^*(t) & 0 & 0 & d(t) \end{pmatrix}, \quad (30)$$

where the matrix elements $a(t)$, $b(t)$, $c(t)$, $x(t)$, and $y(t)$ are given by

$$a(t) = a(0), \quad (31)$$

$$d(t) = d(0), \quad (32)$$

$$b(t) = \frac{1}{2}[b(0)+c(0)] + \frac{1}{2}[b(0)-c(0)]e^{-ut}\left(\cosh dt + \frac{u}{d}\sinh dt\right) + [y(0)-y^*(0)]\left(\frac{ig}{d}\right)e^{-ut}\sinh dt, \quad (33)$$

$$c(t) = \frac{1}{2}[b(0)+c(0)] - \frac{1}{2}[b(0)-c(0)]e^{-ut}\left(\cosh dt + \frac{u}{d}\sinh dt\right) - [y(0)-y^*(0)]\left(\frac{ig}{d}\right)e^{-ut}\sinh dt, \quad (34)$$

$$x(t) = x(0)e^{-2i\omega t - 2t/T_{\rm ph}}, \quad (35)$$

$$y(t) = \frac{1}{2}[y(0)+y^*(0)]e^{-2ut} + \frac{1}{2}[y(0)-y^*(0)]e^{-ut}\left(\cosh dt - \frac{u}{d}\sinh dt\right) + [b(0)-c(0)]\left[\frac{i(g+v)}{d}\right]e^{-ut}\sinh dt. \quad (36)$$

In these equation, we set $d = \sqrt{u^2 - 4g(g+v)}$ and

$$u = \frac{1}{T_{\rm ph}}\left(\frac{1}{1+4\alpha^2}\right), \quad v = \frac{1}{T_{\rm ph}}\left(\frac{2\alpha}{1+4\alpha^2}\right). \quad (37)$$

When we ignore the effects of the interaction between the qubits on the phase relaxation process, the matrix elements $b(t)$, $c(t)$, and $y(t)$ are replaced with

$$b(t) = \frac{1}{2}[b(0)+c(0)] + \frac{1}{2}[b(0)-c(0)]e^{-u_0 t}\left(\cosh d_0 t + \frac{u_0}{d_0}\sinh d_0 t\right) + [y(0)-y^*(0)]\left(\frac{ig}{d_0}\right)e^{-u_0 t}\sinh d_0 t, \quad (38)$$

$$c(t) = \frac{1}{2}[b(0)+c(0)] - \frac{1}{2}[b(0)-c(0)]e^{-u_0 t}\left(\cosh d_0 t + \frac{u_0}{d_0}\sinh d_0 t\right) - [y(0)-y^*(0)]\left(\frac{ig}{d_0}\right)e^{-u_0 t}\sinh d_0 t, \quad (39)$$

$$y(t) = \frac{1}{2}[y(0)+y^*(0)]e^{-2u_0 t} + \frac{1}{2}[y(0)-y^*(0)]e^{-u_0 t}\left(\cosh d_0 t - \frac{u_0}{d_0}\sinh d_0 t\right) + [b(0)-c(0)]\left(\frac{ig}{d_0}\right)e^{-u_0 t}\sinh d_0 t, \quad (40)$$

and the others remain unchanged. Here we set $u_0 = 1/T_{\rm ph}$ and $d_0 = \sqrt{u_0^2 - 4g^2}$. The entanglement of two qubits can be measured by means of the concurrence [21]. For the X state $\hat{W}_Q(t)$, we obtain the concurrence $C(t)$ [20],

$$C(t) = 2\max[0, |x(t)| - \sqrt{b(t)c(t)}, |y(t)| - \sqrt{a(t)d(t)}]. \quad (41)$$

Substituting Eqs. (31)–(36) into this equation, we find the nonequilibrium dynamics of the two-qubit entanglement during the time-evolution generated by the quantum master Eq. (13).

IV. DECAY OF ENTANGLEMENT

In this section, using the result obtained in the previous section, we investigate the decay of entanglement for several two-qubit states. We first consider the decay of the partially entangled pure states $|\Phi\rangle = \alpha|00\rangle + \beta|11\rangle$ and $|\Psi\rangle = \alpha|01\rangle + \beta|10\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$, which exhibit the quite different behaviors in the relaxation process [22,23]. It is easy to see that the effects of the interaction between the qubits do not appear in the time-evolution of the entanglement when the two qubits are initially prepared in the state $|\Phi\rangle$. In fact, the concurrence is given by $C(t) = 2|\alpha\beta|e^{-2t/T_{\rm ph}}$. The relaxation time of the concurrence is half of the phase relaxation time. On the other hand, when the two qubits are initially in the state $|\Psi\rangle$, the concurrence $C(t)$ at time t is calculated from Eqs. (31)–(36) and (41),

$$C(t) = 2|y(t)|, \quad (42)$$

with

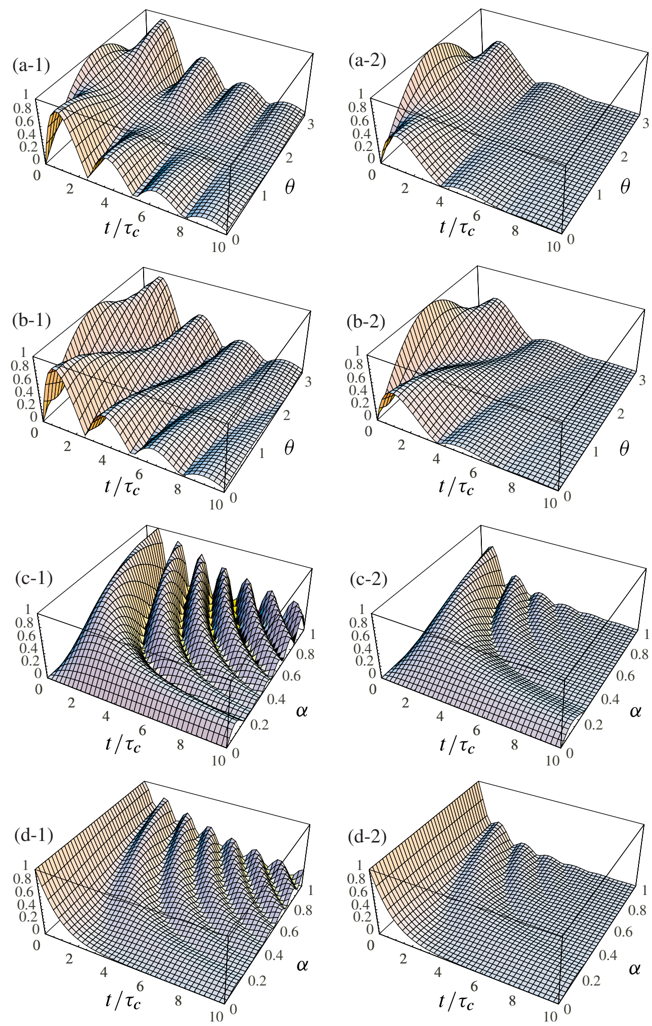

FIG. 2. (Color online) The time evolution of the concurrence $C(t)$ of the two qubits initially prepared in $|\Psi\rangle$, where we set (a-1) $\alpha=0.5$, $\phi=0.0$, (b-1) $\alpha=0.5$, $\phi=\pi/4$, (c-1) $\theta=0.0$, $\phi=\pi/4$, and (d-1) $\theta=\pi/2$, $\phi=\pi/4$. The figures (a-2), (b-2), (c-2), and (d-2) show the concurrence when the interaction effects on the relaxation are ignored, where the values of the parameters are the same as those for (a-1), (b-1), (c-1), and (d-1). In all the figures, we set $T_{\mathrm{ph}}/\tau_c=2.0$.

$$y(t)=\frac{1}{2}e^{-2ut}\cos\phi\sin\theta-\frac{i}{2}e^{-ut}\left(\cosh dt-\frac{u}{d}\sinh dt\right)\sin\phi\sin\theta+\frac{i(g+v)}{d}e^{-ut}\sinh dt\cos\theta, \tag{43}$$

where we set $\alpha=\cos(\theta/2)$ and $\beta=e^{i\phi}\sin(\theta/2)$. When the interaction effects are ignored, the parameters u, v, and d in Eq. (43) are replaced with u_0, 0, and d_0. The concurrence $C(t)$ given by Eq. (42) is plotted in Fig. 2. The figure shows that the interaction effects on the phase relaxation process not only reduce the decay of the concurrence but also make short the period of the oscillatory behavior of the entanglement.

We next consider the Werner state $\hat{W}_F=F|\Psi_+\rangle\langle\Psi_+|+\frac{1}{3}(1-F)(\hat{1}-|\Psi_+\rangle\langle\Psi_+|)$ as the initial two-qubit state [24], where $|\Psi_+\rangle=(|01\rangle+|10\rangle)/\sqrt{2}$. We assume that the Werner state is entangled at the initial time, that is, $F>\frac{1}{2}$. In this case, using Eqs. (31)–(36) and (41), we obtain the concurrence $C(t)$,

$$C(t)=2\max\left[0,\frac{4F-1}{6}e^{-2ut}-\frac{1-F}{3}\right], \tag{44}$$

which shows that the entanglement sudden death [25,26] occurs at

$$T_{\mathrm{ESD}}=\frac{1}{2}T_{\mathrm{ph}}(1+4\alpha^2)\ln\left(\frac{4F-1}{2-2F}\right). \tag{45}$$

The survival time of the entanglement becomes $(1+4\alpha^2)$ times of that obtained in the case that we ignore the interaction effects on the phase relaxation process. Therefore the interaction between the qubits suppresses the decay of the entanglement.

We suppose that the two qubits are initially prepared in the following state [18,19]:

$$\hat{W}_Q(0)=\frac{1}{3}\begin{pmatrix} a & 0 & 0 & 0\\ 0 & 1 & e^{i\phi} & 0\\ 0 & e^{-i\phi} & 1 & 0\\ 0 & 0 & 0 & 1-a\end{pmatrix}, \tag{46}$$

where $1\geq a\geq 0$. In this case, Eqs. (31)–(36) and (41) provide the concurrence $C(t)$ of the two-qubit state $\hat{W}_Q(t)$,

$$C(t)=\max[0,\mathcal{C}(t)], \tag{47}$$

with

$$\mathcal{C}(t)=\frac{2}{3}e^{-ut}\sqrt{e^{-2ut}\cos^2\phi+\left(\cosh dt-\frac{u}{d}\sinh dt\right)^2\sin^2\phi}-\sqrt{a(1-a)}. \tag{48}$$

The concurrence $C(t)$ given by Eq. (47) is plotted in Fig. 3. The figure shows that the effects of the interaction between the two qubits on the phase relaxation process significantly reduce the decay of the entanglement. When we ignore the interaction effects, the concurrence given by Eq. (47) becomes equal to that obtained in Refs. [18,19].

We finally consider the maximally entangled mixed state (MEMS) as the initial state of the two qubits [27,28], which has the maximum amount of entanglement with a given mixedness. One of the MEMSs is given by

$$\hat{W}_Q(0)=\begin{cases}\hat{\rho}^{(\mathrm{I})}_{\mathrm{MEMS}}, & \left(\frac{2}{3}\leq r\leq 1\right)\\ \hat{\rho}^{(\mathrm{II})}_{\mathrm{MEMS}}, & \left(0\leq r\leq \frac{2}{3}\right),\end{cases} \tag{49}$$

with

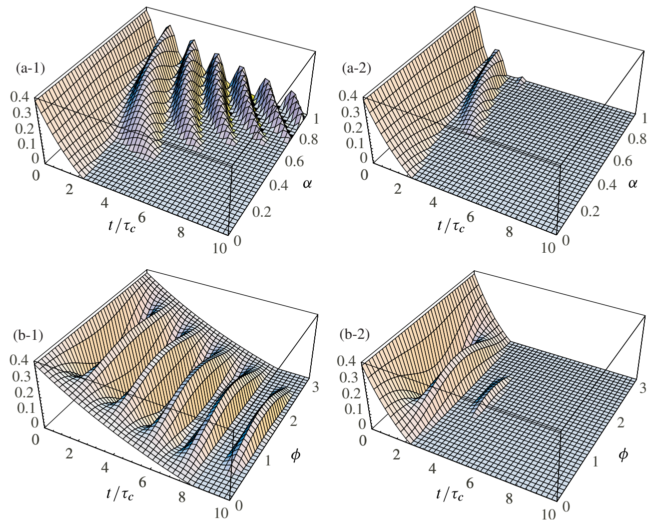

FIG. 3. (Color online) The time evolution of the concurrence $C(t)$, where we set (a-1) $T_{\mathrm{ph}}/\tau_c=5.0$, $\phi=\pi/4$, $a=0.2$ and (b-1) $T_{\mathrm{ph}}/\tau_c=5.0$, $\alpha=0.8$, $a=0.2$. The figures (a-2) and (b-2) show the concurrence when the interaction effects on the relaxation are ignored, where the values of the parameters are the same as those for (a-1) and (b-1).

$$\hat{\rho}_{\mathrm{MEMS}}^{(\mathrm{I})}=\begin{pmatrix} \frac{r}{2} & 0 & 0 & \frac{r}{2} \\ 0 & 1-r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r}{2} & 0 & 0 & \frac{r}{2} \end{pmatrix}, \tag{50}$$

$$\hat{\rho}_{\mathrm{MEMS}}^{(\mathrm{II})}=\begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{r}{2} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r}{2} & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{51}$$

The concurrence of the state $\hat{W}_Q(0)$ is given by $C(0)=r$. In this case, the matrix elements $a(t)$, $b(t)$, $c(t)$, $d(t)$, $x(t)$, and $y(t)$ of the X state $\hat{W}_Q(t)$ are calculated from Eqs. (31)–(36),

$$a(t)=d(t)=\frac{r}{2}, \tag{52}$$

$$b(t)=\frac{1-r}{2}\left[1+e^{-ut}\left(\cosh dt+\frac{u}{d}\sinh dt\right)\right], \tag{53}$$

$$c(t)=\frac{1-r}{2}\left[1-e^{-ut}\left(\cosh dt+\frac{u}{d}\sinh dt\right)\right], \tag{54}$$

$$x(t)=\frac{r}{2}e^{-2i\omega t-2t/T_{\mathrm{ph}}}, \tag{55}$$

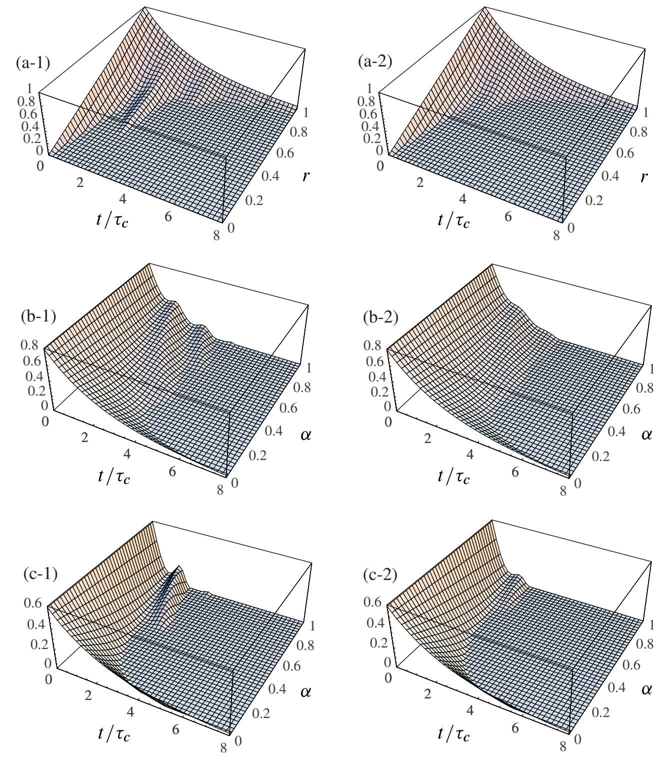

FIG. 4. (Color online) The time evolution of the concurrence $C(t)$, where we set (a-1) $T_{\mathrm{ph}}/\tau_c=5.0$, $\alpha=0.8$, (b-1) $T_{\mathrm{ph}}/\tau_c=5.0$, $r=0.8$, and (c-1) $T_{\mathrm{ph}}/\tau_c=5.0$, $r=0.6$. The figures (a-2), (b-2), and (c-2) show the concurrence when the interaction effects on the relaxation are ignored, where the values of the parameters are the same as those for (a-1), (b-1), and (c-1).

$$y(t)=(1-r)\left[\frac{i(g+v)}{d}\right]e^{-ut}\sinh dt, \tag{56}$$

for $\frac{2}{3}\leq r\leq 1$ and

$$a(t)=d(t)=\frac{1}{3}, \tag{57}$$

$$b(t)=\frac{1}{6}\left[1+e^{-ut}\left(\cosh dt+\frac{u}{d}\sinh dt\right)\right], \tag{58}$$

$$c(t)=\frac{1}{6}\left[1-e^{-ut}\left(\cosh dt+\frac{u}{d}\sinh dt\right)\right], \tag{59}$$

$$x(t)=\frac{r}{2}e^{-2i\omega t-2t/T_{\mathrm{ph}}}, \tag{60}$$

$$y(t)=\frac{1}{3}\left[\frac{i(g+v)}{d}\right]e^{-ut}\sinh dt, \tag{61}$$

for $0\leq r\leq\frac{2}{3}$. Substituting these equations into Eq. (41), we obtain the concurrence $C(t)$, which is plotted in Fig. 4. It is found from the figure that ignoring the effects of the qubit interaction on the phase relaxation process enhances the decay of the entanglement.

V. CONCLUDING REMARKS

In this paper, using the quantum master equation derived by the projection operator method, we have investigated the dephasing of interacting two-qubit system under the influence of independent thermal reservoirs. In particular, we have paid attention to the effects of the interaction between the qubits on the phase relaxation process of the two-qubit system. For this purpose, we have derived the quantum master equation with the damping operator which consists of the two parts; one includes the interaction effects and the other does not. Then we have applied this result for investigating the nonequilibrium dynamics of the two qubits initially prepared in the X state, which includes the partially entangled pure state, the Werner state, and the maximally entangled mixed state. These states are important in the quantum information processing. Calculating the concurrence, we have examined the decay of the two-qubit entanglement. The results have been compared with those obtained when we have ignored the effects of the two-qubit interaction on the damping operator of the quantum master equation. We have found that the interaction effects make short the period of the oscillatory behavior of the two-qubit entanglement as well as reduce the decay of it. Therefore we have explicitly shown the fact that the effects of the interaction between subsystems on the relaxation process are essential for investigating the decoherence of quantum systems Ref. [11]. Here it is should be noted that not only an interaction between subsystems but also an interaction with an external field affects the relaxation process of the relevant system. For example, an external field which is used to control qubits inevitably influences the decoherence of them. Hence the investigation of the interaction effects on the decoherence is very important in quantum information theory as well as nonequilibrium quantum statistical mechanics. In this paper, we have assumed that each qubit interacts with the independent thermal reservoir. It may be important to consider the case that two interacting qubits are influenced by a common thermal reservoir. This will be done in a future publication.

APPENDIX A: DERIVATION OF EQS. (2) and (3)

When the spin operators $\hat{S}_A^z$ and $\hat{S}_B^z$ act on vectors belonging to the tensor product Hilbert space of the two qubits A and B, they read $\hat{S}_A^z \otimes \hat{1}_B$ and $\hat{1}_A \otimes \hat{S}_B^z$, where $\hat{1}_A$ and $\hat{1}_B$ are identity operators of the qubits. Noting that $\hat{S}^z = \frac{1}{2}|0\rangle\langle 0| - \frac{1}{2}|1\rangle\langle 1|$ and $\hat{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$, we can express the spin operators $\hat{S}_A^z$ and $\hat{S}_B^z$ in terms of the eigenstates $|\Phi_0\rangle$, $|\Phi_1\rangle$, $|\Psi_+\rangle$, and $|\Psi_-\rangle$ of the two-qubit Hamiltonian $\hat{H}_Q$,

$$\hat{S}_A^z = \frac{1}{2}(|\Phi_0\rangle\langle\Phi_0| - |\Phi_1\rangle\langle\Phi_1| + |\Psi_+\rangle\langle\Psi_-| + |\Psi_-\rangle\langle\Psi_+|), \tag{A1}$$

$$\hat{S}_B^z = \frac{1}{2}(|\Phi_0\rangle\langle\Phi_0| - |\Phi_1\rangle\langle\Phi_1| - |\Psi_+\rangle\langle\Psi_-| - |\Psi_-\rangle\langle\Psi_+|). \tag{A2}$$

Then we can derive Eqs. (2) and (3) as follows:

$$\hat{S}_A^z(t) = e^{(it/\hbar)\hat{H}_Q}\hat{S}_A^z e^{-(it/\hbar)\hat{H}_Q} = \frac{1}{2}(|\Phi_0\rangle\langle\Phi_0| - |\Phi_1\rangle\langle\Phi_1| + e^{2igt}|\Psi_+\rangle\langle\Psi_-| + e^{-2igt}|\Psi_-\rangle\langle\Psi_+|) = \hat{S}_A^z - \frac{1}{2}(1-e^{2igt}) \times |\Psi_+\rangle\langle\Psi_-| - \frac{1}{2}(1-e^{-2igt})|\Psi_-\rangle\langle\Psi_+|, \tag{A3}$$

$$\hat{S}_B^z(t) = e^{(it/\hbar)\hat{H}_Q}\hat{S}_B^z e^{-(it/\hbar)\hat{H}_Q} = \frac{1}{2}(|\Phi_0\rangle\langle\Phi_0| - |\Phi_1\rangle\langle\Phi_1| - e^{2igt}|\Psi_+\rangle\langle\Psi_-| - e^{-2igt}|\Psi_-\rangle\langle\Psi_+|) = \hat{S}_B^z + \frac{1}{2}(1-e^{2igt}) \times |\Psi_+\rangle\langle\Psi_-| + \frac{1}{2}(1-e^{-2igt})|\Psi_-\rangle\langle\Psi_+|. \tag{A4}$$

APPENDIX B: DERIVATION OF EQS. (11) and (12)

In this appendix, we derive Eqs. (11) and (12) from Eqs. (8) and (9). The damping operators $\hat{\mathcal{L}}_A(t)$ and $\hat{\mathcal{L}}_B(t)$ in the Schrödinger picture are related to $\hat{\mathcal{L}}_A^{\text{int}}(t)$ and $\hat{\mathcal{L}}_B^{\text{int}}(t)$ in the interaction picture by

$$\hat{\mathcal{L}}_A(t)\hat{W}_Q(t) = e^{-(it/\hbar)\hat{H}_Q}\{\hat{\mathcal{L}}_A^{\text{int}}(t)[e^{(it/\hbar)\hat{H}_Q}\hat{W}_Q(t)e^{-(it/\hbar)\hat{H}_Q}]\}e^{(it/\hbar)\hat{H}_Q} = e^{-(it/\hbar)\hat{H}_Q}\{\hat{\mathcal{L}}_A^{\text{int}}(t)\hat{W}_Q^{\text{int}}(t)\}e^{(it/\hbar)\hat{H}_Q}, \tag{B1}$$

$$\hat{\mathcal{L}}_B(t)\hat{W}_Q(t) = e^{-(it/\hbar)\hat{H}_Q}\{\hat{\mathcal{L}}_B^{\text{int}}(t)[e^{(it/\hbar)\hat{H}_Q}\hat{W}_Q(t)e^{-(it/\hbar)\hat{H}_Q}]\}e^{(it/\hbar)\hat{H}_Q} = e^{-(it/\hbar)\hat{H}_Q}\{\hat{\mathcal{L}}_B^{\text{int}}(t)\hat{W}_Q^{\text{int}}(t)\}e^{(it/\hbar)\hat{H}_Q}. \tag{B2}$$

Substituting Eq. (8) into Eq. (B1), we obtain

$$\begin{aligned}\hat{\mathcal{L}}_A(t)\hat{W}_Q(t) = &-\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t)\hat{R}_A(t')\rangle\hat{S}_A^z\hat{S}_A^z(t'-t)\hat{W}_Q(t) \\ &+\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t')\hat{R}_A(t)\rangle\hat{S}_A^z\hat{W}_Q(t)\hat{S}_A^z(t'-t) \\ &+\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t)\hat{R}_A(t')\rangle\hat{S}_A^z(t'-t)\hat{W}_Q(t)\hat{S}_A^z \\ &-\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t')\hat{R}_A(t)\rangle\hat{W}_Q(t)\hat{S}_A^z(t'-t)\hat{S}_A^z = \\ &-\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t-t')\hat{R}_A(0)\rangle\hat{S}_A^z\hat{S}_A^z(t'-t)\hat{W}_Q(t) \\ &+\lambda_A^2\int_0^t dt'\langle\hat{R}_A(0)\hat{R}_A(t-t')\rangle\hat{S}_A^z\hat{W}_Q(t)\hat{S}_A^z(t'-t) \\ &+\lambda_A^2\int_0^t dt'\langle\hat{R}_A(t-t')\hat{R}_A(0)\rangle\hat{S}_A^z(t'-t)\hat{W}_Q(t)\hat{S}_A^z \\ &-\lambda_A^2\int_0^t dt'\langle\hat{R}_A(0)\hat{R}_A(t-t')\rangle\hat{W}_Q(t)\hat{S}_A^z(t'-t)\hat{S}_A^z.\end{aligned} \tag{B3}$$

In the second equality, we have used fact that $\langle\hat{R}_A(t)\hat{R}_A(t')\rangle = \langle\hat{R}_A(t-t')\hat{R}_A(0)\rangle$ and $\langle\hat{R}_A(t')\hat{R}_A(t)\rangle = \langle\hat{R}_A(0)\hat{R}_A(t-t')\rangle$ are satisfied since the reservoir is in the thermal equilibrium state. Thus changing the integration variable from t' to $t-t'$ in Eq. (B3), we obtain Eq. (11). In the same way, we can derive Eq. (12) from Eq. (B2).

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