Perturbation theory for metastable states of the Dirac equation with quadratic vector interaction

Riccardo Giachetti*
Dipartimento di Fisica, Università di Firenze, 50019 Sesto Fiorentino, Italy
and Istituto Nazionale di Fisica Nucleare–Sezione di Firenze, 50019 Sesto Fiorentino, Italy

Vincenzo Grecchi†
Dipartimento di Matematica, Università di Bologna, 40126 Bologna, Italy
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The spectral problem of the Dirac equation in an external quadratic vector potential is considered using the methods of the perturbation theory. The cases in one spatial dimension and in three spatial dimensions with spherical symmetry are explicitly studied. The problem is singular and the perturbation series is asymptotic, so that the methods for dealing with divergent series must be used. Among these, the distributional Borel sum appears to be the most well suited tool to give answers and to describe the spectral properties of the system. A detailed investigation is made in one and in three space dimensions with a central potential. We present numerical results for the Dirac equation in one space dimension: these are obtained by determining the perturbation expansion and using the Padé approximants for calculating the distributional Borel transform. A complete agreement is found with previous nonperturbative results obtained by the numerical solution of the singular boundary value problem and the determination of the density of the states from the continuous spectrum.

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I. INTRODUCTION

It has been argued for a long time [1] that the Dirac equation with an unbounded potential in vector coupling has no discrete but only completely continuous spectrum. It was shown in [2,3] and subsequently confirmed in literature that precise conditions have to be imposed to the potential for this to be true. In two later papers [4,5], Titchmarsh proved that the Dirac equation with a linear vector potential satisfies the needed requirements and he studied the relativistic quantum mechanics of an electron in a constant (or piecewise constant) electric field. The first order of the asymptotic perturbation expansion was explicitly calculated due to the possibility of integrating the Dirac equation in one space dimension by means of hypergeometric functions, but hard technical difficulties prevented further analytical developments, as well as the extension to a higher power law for the interaction or to more general inhomogeneous electric fields. The Dirac equation with a vector potential that grows sufficiently fast at infinity is an incomplete dynamical problem. In classical terms, the particle can arrive at infinity in a finite time; in the operator language, we have (2,2) deficiency indices and the appropriate asymptotic boundary conditions must be defined. There is an infinite number of such boundary conditions of the self-adjoint type not very meaningful from a physical point of view. The most physical condition is the absence of sources at infinity, corresponding to the Gamow-Siegert conditions, which assume outgoing wave functions only: in this case, however, the Hamiltonian is not self-adjoint, the spectrum has complex eigenvalues, the dynamics is dissipative, and the eigenfunctions have the meaning of metastable states of the dissipative dynamics itself:

$$\|U_t\psi_n\|^2 = \|e^{-iE_nt/\hbar}\psi_n\|^2 = e^{-\Gamma_n t}\|\psi_n\|^2, \tag{1.1}$$

where $\Gamma_n = -2\,\mathrm{Im}(E_n/\hbar) > 0$ is the inverse of the mean lifetime of the metastable state ψ_n. Actually such states are stationary, if we neglect the decrease in the norm, so that they should be called metastationary; we prefer not to call them resonances, since they are indeed different from the usual resonances and have a direct physical meaning.

Thinking of the transition from the nonrelativistic to the relativistic system as a perturbation process in the small parameter $(1/c)$, the disappearance of the Schrödinger bound states connected with the confining potential defines a singular perturbation problem. A numerical nonperturbative investigation of the one-dimensional Dirac equation with linear and quadratic potential has recently appeared [6]: the purpose of that paper was to follow very closely the change in the spectrum when passing to the relativistic regime and to describe the spectral concentration [7,8] at finite values of $(1/c)$, or, in a more physical language, to determine the density of the states of the relativistic system. It was indeed produced a numerical evidence that the spectrum is completely continuous and that it is given by a sum of Breit-Wigner lines reducing to δ functions centered at the nonrelativistic eigenvalues for smaller and smaller values of the ratio of the interaction to the rest energy, so that the spectral measure becomes atomic as it should. In [6] the pair production rate was calculated from the linewidth of the lowest state finding a perfect agreement, for the linear potential and in the range of validity of the first perturbation order, with the results obtained from the imaginary part of the Schwinger effective action [9]: it should be noticed that the QED results on the pair production for fermions in noncon-

*giachetti@fi.infn.it
†grecchi@dm.unibo.it

stant electric fields are not yet so sound, although interesting proposals can be found in literature (see [10] for an up-to-date review). We also recall that models with power-law potentials, in three space dimensions and in spherical geometry, have been and are still being used in the study of composite systems, in order to determine the mass spectrum of meson families [11–13]. Mesons were first identified with bound states of a massive quark and antiquark and the nonobservation of free quarks suggested the presence of a very strong confining force. In model building the corresponding potential was generally assumed of a power-law form in the radial coordinate and the dynamical description was first given in terms of the Schrödinger equation. It has then been observed that the nonrelativistic approximation works much better for meson families composed of heavier quarks, while, to fit lighter mesons, a (possibly two-body) relativistic analysis becomes necessary [14–16].

In this paper we come back to the spectral properties of the Dirac equation with a quadratic vector potential in a perturbation framework completely different from that adopted in [6]. Indeed, since the quantitative and numerical results on this subject are rather new, we find it interesting to make a comparison of independent computational approaches and eventually to have a confirmation of the results. In particular, we use here the method of the *distributional Borel sum* (hereafter DBS), that has proved a very useful tool for dealing with physical systems of singular nature. Originating from a suggestion given by 't Hooft for double well problems [17], the DBS was studied in a series of papers [18–21] and it was successfully applied to $\lambda\Phi^4$ lattice fields with large coupling in [18], to the nonrelativistic Stark resonances in [20], and to the double well Schrödinger operators in [19,21]. To our knowledge, however, despite the very deep mathematical development of the subject, no explicit numerical calculations with the DBS have been made to date and no applications of it to the Dirac equation have been considered. In this sense our paper represents a test of the method for its possible practical uses. We will also show that the basic results of [20] can be transferred to our present context, although some peculiarities due to the relativistic nature of the problem will emerge and must be taken into account. We finally remark that the approach we develop here can prove to be unavoidable for dealing with many unbounded potentials. Indeed, as shown in [6], when the nonrelativistic limit presents an unbounded attractive potential $V(r)$ in the Schrödinger equation, the Dirac equation has also a "relativistic" repulsive term $V^2(r)/c$ that dominates asymptotically and that is responsible for the singular change from the discrete spectrum to the completely continuous one. Power-law potentials with exponents between zero and unity give rise to perturbation problems that can be dealt nicely by means of the Borel sum. However, in cases of physical interest, the exponent can become greater than unity and the use of the DBS becomes necessary. We have chosen to deal with a quadratic potential for several reasons. In the first place we wanted to prove the effectiveness of the DBS on the unexplored ground of its numerical applications and to compare its results with those on the spectral density found in [6] for the same type of potential. The comparison is even more interesting due to a kind of complementarity of the two approaches, in the sense that each one of them is more efficient in a different range of the values of the coupling constant (lower values for the DBS, somewhat higher for the spectral density), so to allow for the choice of the most appropriate numerical scheme in a specific situation. Moreover, although the DBS method can be generalized to different unbounded potentials, the calculation of large order perturbation terms can pose nontrivial practical problems lacking a suitable recurrence relation. A quadratic vector potential has recently been used in [22] for evaluating the pair production in an inhomogeneous electric field via instantons, in an adiabatic WKB approximation: a comparison with ascertained numerical results obtained in a different framework can help for a better understanding of the validity of that approximation. A special mention must finally be addressed to the fresh and expanding subject of the graphene physics, where the electronic properties are described by the Dirac equation and the influence of impurities is introduced by means of a potential in vector coupling [23,24]: the treatment we give could provide the appropriate analog for those situations that in a nonrelativistic framework are well approximated by harmonic oscillators.

The plan of the paper is as follows. In the next section, for the sake of completeness and because the knowledge of the DBS is not so widely diffused, we give a brief description of the method and we state more properly the spectral problem. In Sec. III we study the general conditions for applying the DBS to the Dirac equation and we discuss the possible strategies for the one-dimensional and the three-dimensional problems. Finally, in Sec. IV, we present the numerical treatment for the one-dimensional case. Using a perturbation parameter proportional to the ratio of the interaction to the rest mass energy, we calculate the perturbation series of the energy of the fundamental state up to a large order and we then construct its Borel transform determining its asymptotic behavior. We then approximate the Borel transform by Padé approximants. Although, in a strict mathematical sense, this is not a completely rigorous procedure since it is not proved that the Borel transform is a Stieltjes function, however, an idea of the accuracy of the approximation can be obtained by comparing the position of the poles of the Padé approximants with the location suggested by the asymptotic form of the perturbation series: indeed, as shown in Sec. IV, we find the Padé poles exactly where the asymptotic behavior indicates they should be. We finally check the stability of the poles with the increasing orders of the Padé approximants implying the stability of the imaginary part of the perturbed energy. The latter is then calculated and it is found a perfect agreement with the results of [6], confirming them and proving the correctness and the effectiveness of both methods also in explicit numerical calculations.

II. SPECTRAL PROBLEM

We consider the $(3+1)$-dim Dirac equation interacting by means of a central vector potential $(0, V(r))$. It is well known that the use of the spherical spinors [25] easily leads to the diagonalization of the angular momentum and to the reduction of the Dirac equation to the system of the two first-order differential equations

$$\frac{1}{c}[W+mc^2-V(r)]X_1(r)-\hbar\left(\frac{d}{dr}+\frac{\kappa}{r}\right)X_2(r)=0,$$

$$\hbar\left(\frac{d}{dr}-\frac{\kappa}{r}\right)X_1(r)+\frac{1}{c}[W-mc^2-V(r)]X_2(r)=0. \quad (2.1)$$

In Eq. (2.1) m is the mass and W is the energy of the particle, while $X_1(r)=rg(r)$ and $X_2(r)=rf(r)$, where $f(r)$ and $g(r)$ are, respectively, the "large" and the "small" spherical component of the spinor. Finally, the parameter κ accounts for the angular momentum and the parity: $\kappa=-(\ell+1)$ for $j=\ell+1/2$ and $\kappa=\ell$ for $j=\ell-1/2$ [25], so that, in three space dimensions, κ can assume all integer values except zero. If, instead, we let $\kappa=0$ in Eq. (2.1) and we change r in x, with $-\infty<x<+\infty$, we obtain the one space-dimensional Dirac equation that has been discussed in [6]. We now assume a quadratic potential

$$V(r)=(1/2)m\omega^2r^2.$$

We rescale the system by introducing the dimensionless variables

$$y=r\left(\frac{m\omega}{\hbar}\right)^{1/2},\quad \beta=\frac{1}{2}\left(\frac{\hbar\omega}{mc^2}\right)^{1/2},\quad E=\frac{2}{\hbar\omega}(W-mc^2)$$

and we define the unknown functions

$$\Phi_1(r)=2^{-1/2}[X_1(r)+iX_2(r)],$$

$$\Phi_2(r)=-i2^{-1/2}[X_1(r)-iX_2(r)].$$

System (2.1) then becomes

$$\frac{d}{dy}\Phi_1(y)-i\left[\frac{1}{2\beta}+\beta(E-y^2)\right]\Phi_1(y)+\left(\frac{1}{2\beta}-\frac{i\kappa}{y}\right)\Phi_2(y)=0,$$

$$\frac{d}{dy}\Phi_2(y)+\left(\frac{1}{2\beta}+\frac{i\kappa}{y}\right)\Phi_1(y)+i\left[\frac{1}{2\beta}+\beta(E-y^2)\right]\Phi_2(y)=0.$$

By eliminating $\Phi_2(y)$ we find the second-order equation

$$\frac{d^2}{dy^2}\Phi_1(y)-\frac{2i\kappa\beta}{y(y-2i\kappa\beta)}\frac{d}{dy}\Phi_1(y)+\left[E-y^2-\frac{\kappa^2}{y^2}+2i\beta y+\beta^2(E-y^2)^2-\frac{\kappa}{y}\frac{1+2\beta^2(E-y^2)}{(y-2i\kappa\beta)}\right]\Phi_1(y)=0. \quad (2.2)$$

Besides infinity, this equation, with $y\in[0,+\infty)$, presents the obvious singularity at the origin, absent in the one-dimensional case with $\kappa=0$ and $y\in(-\infty,+\infty)$. The nonrelativistic limit for $\beta\to0$,

$$\frac{d^2}{dy^2}\Phi_1(y)+\left[E-y^2-\frac{\ell(\ell+1)}{y^2}\right]\Phi_1(y)=0, \quad (2.3)$$

reproduces the usual Schrödinger equation, with orbital angular momentum ℓ, in a quadratic potential. The properties of the equation with $\beta\neq0$, however, are very different from those of the nonrelativistic counterpart, since Eq. (2.2), even neglecting the presence of an imaginary term, presents an asymptotic oscillatory behavior that prevents the existence of any normalizable solution. Therefore, as we said, the perturbation expansion from the nonrelativistic system turns out to be singular.

We can transform the domain of definition of differential equation (2.2) with a map U defined by

$$\Phi(y)=U(\Phi_1(y))=\left(1-\frac{2i\kappa\beta}{y}\right)^{-1/2}\Phi_1(y). \quad (2.4)$$

The transformed differential equation becomes then

$$\frac{d^2}{dy^2}\Phi(y)+\left\{E-y^2-\frac{\kappa(\kappa+1)}{y^2}+2i\beta\left[y-\frac{\kappa(\kappa+1)}{(y-2i\kappa\beta)y^2}\right]+\beta^2\left[(y^2-E)^2+\frac{2\kappa(y^2-E)}{(y-2i\kappa\beta)y}+\frac{3\kappa^2}{(y-2i\kappa\beta)^2y^2}\right]\right\}\Phi(y)=0. \quad (2.5)$$

Observing that $\beta>0$, we find it useful to introduce the parameter $g=\mp i\beta$ according to whether $\kappa>0$ or $\kappa<0$, respectively. With such a choice of the signs, when $g>0$, no additional singularities besides the origin and the infinity appear at finite values of y and the transformation U is unitary. The equation to be studied can finally be written as

$$H_g\Phi(y)=0,\quad H_g\equiv-\frac{d^2}{dy^2}+V(E,g,\kappa,y) \quad (2.6)$$

whose "potential" is given by

$$V(E,g,\kappa,y)=V_0(E,\kappa,y)+V_1(E,\kappa,g,y), \quad (2.7)$$

where

$$V_0=y^2-E+\frac{|\kappa|(|\kappa|\pm1)}{y^2},$$

$$V_1=\pm2g\left[y-\frac{|\kappa|(|\kappa|\pm1)}{(y+2|\kappa|g)y^2}\right]+g^2\left[(y^2-E)^2\pm\frac{2|\kappa|(y^2-E)}{(y+2|\kappa|g)y}+\frac{3|\kappa|^2}{(y+2|\kappa|g)^2y^2}\right].$$

The double signs correspond again to $\kappa>0$ and $\kappa<0$, respectively, while the dimensionless energy E, entering $V(E,g,\kappa,y)$ in a polynomial expression, can be considered, for the moment, a parameter that will be determined during the calculation. It is common sense—and it can also be proved rigorously—that Eq. (2.6) with positive g admits bound states: starting from these we can then try to make an analytic continuation in the complex plane of the coupling constant g from the real positive to the positive or negative imaginary axis in order to recover the values of the parameters of the initial problem. The procedure, however, presents some delicate points. Indeed the analytic continuation of the eigenvalue of the complete equation would be immediate if the eigenvalue itself was given as a convergent series expansion in the coupling constant g. Unfortunately this is not the case: the perturbation expansion in g is asymptotic and has zero radius of convergence [26]. Since finally our original problem has exactly $g^2<0$, the system we are dealing with is

therefore close to an unstable quartic oscillator, but for some nonpolynomial terms for which an additional discussion is in order: all the considerations we developed in Sec. I about the incompleteness, self-adjoint extensions, and boundary conditions at infinity thus apply.

Let us now recall that the study of the asymptotic expansions for treating perturbation problems in quantum mechanics was systematically undertaken since the beginning of the 70s and the main concept allowing to deal with divergent series was the Borel summability [8,27–29]. Given a formal series

$$S = \sum_{n=0}^{\infty} a_n z^n \tag{2.8}$$

we define its Borel transform as

$$B(u) = \sum_{n=0}^{\infty} [a_n/\Gamma(n+1+\nu)]u^n, \tag{2.9}$$

where ν is a fixed parameter independent of n that is usually chosen in relation to the asymptotic behavior of the coefficients $\{a_n\}$, but whose choice is otherwise irrelevant. If then $B(u)$ has a nonvanishing radius of convergence, admits an analytic continuation to a neighborhood of the positive real axis, and if the integral

$$f(z) = z^{-1}\int_0^{\infty} du(u/z)^{\nu}\exp(-u/z)B(u) \tag{2.10}$$

converges in a so-called "Nevanlinna disk" $C_R=\{\mathrm{Re}(z^{-1}) > R^{-1}, R>0\}$, we say that $f(z)$ is the sum of S in C_R. In some favorable cases, where stronger properties hold, more convenient summation methods have also been proposed as, for example, the Stieltjes sum [30]: this can be defined when the classical Stieltjes moment problem has a solution and the result can be conveniently expressed in terms of a convergent Padé approximation. There are many cases, however, in which the Borel summability and, *a fortiori*, the Stieltjes summability cannot be applied: notably, this occurs when the Borel transform develops some singular point along the positive real axis, so that integral (2.10) is not defined. The DBS originates from the need of finding a more flexible method of summation that could be able to avoid the difficulties connected with the emergence of the singularities. The idea is to follow closely the way used in the discussion of the Riemann-Hilbert boundary value problem with data assigned on the positive real axis. More precisely, we reduce the requirement of the analytic continuation of $B(u)$ in a neighborhood of the positive real axis to the assumption of the existence of the analytic continuation of $B(u)$ in the intersection of a neighborhood of the positive real axis with the upper complex half plane, so that the boundary value $B(u+i0)$ is well defined for any $u>0$. The integral yielding $f(z)$ is then replaced by

$$\phi(z) = z^{-1}\int_0^{\infty} B(u+i0)\rho(u/z)du, \tag{2.11}$$

where the measure $\rho(u)du$ is finite and has positive moments $\mu_k=\int_0^{\infty}u^k\rho(u)du$. The function $\phi(z)$ defined by integral (2.11) is analytic in the upper half plane and it is called the *upper sum*. In particular for $\mu_k=\Gamma(k+1+\nu)$ we have $\rho(u)=u^{\nu}\exp(-u)$ and Eq. (2.11) reproduces the relationship between boundary values of analytic functions and many distributions defined by the inverse Laplace transform [31]. Since then $B(u-i0)=\overline{B(u+i0)}$ we also define the *lower sum*

$$\overline{\phi(\bar{z})} = z^{-1}\int_0^{\infty} \overline{B(u+i0)}\rho(u/z)du$$

and we consider the real and the imaginary parts of $\phi(z)$:

$$f(z) = \frac{1}{2}[\phi(z) + \overline{\phi(\bar{z})}] = \mathrm{Re}(\phi(z)),$$

$$d(z) = \frac{1}{2}[\phi(z) - \overline{\phi(\bar{z})}] = \mathrm{Im}(\phi(z)). \tag{2.12}$$

It is then natural to assume $f(z)$ as the DBS of the series, while the discontinuity along the positive real axis, $d(z)$, is uniquely determined and it has a zero asymptotic power series expansion. In [32] $d(z)$ was related to the Schwinger effective action.

III. APPLICATION OF THE DBS

In this section we study the large order perturbation approach to Eq. (2.6) in one and in three space dimensions and we stress some physical relations between angular momentum and perturbation expansion. For the sake of completeness we will recall some useful facts related to general and well known properties of operators: greater details can be found in [7,8,33,34]. These notions can be collected in two different main subjects. The first concerns the analytic properties of the operator family related to the coupling constant of the perturbation part. This is essential if we want to explore the changes in the spectrum of the operators during the analytic continuation process. The second one deals with the estimates of the asymptotic growth of the perturbation series, which imply the existence of appropriate summation mechanisms.

The differential equation defined in Eq. (2.6) and dependent upon a parameter g has nice properties for $g=0$ in the sense that we are able to determine exactly its discrete spectrum. The natural question to be posed is whether an eigenvalue μ of $H_0=H_{g=0}$ with a certain (algebraic) multiplicity M gives rise to nearby eigenvalues $\mu_1(g),\ldots,\mu_s(g)$ with total multiplicity M when the perturbation is switched on. If this is the case, the eigenvalue μ is said to be stable. We just recall that the discussion of this type of problems is simple enough when the perturbation potential $V_1(E,\kappa,g,y)$ is H_0 bounded, namely, when the domain of definition $\mathcal{D}(V_1)$ contains $\mathcal{D}(H_0)$, since this property yields $\|V_1\phi\| \le a\|H_0\phi\| + b\|\phi\|$, for constant a,b and for any $\phi \in \mathcal{D}(H_0)$. From the previous inequality, as in the proof of the Kato-Rellich theorem, we can easily deduce that $\mathcal{D}(H_g)$ is independent of g and that, for any $\phi \in \mathcal{D}(H_g)$, $H_g\phi$ is an entire analytic function of g. In

this case the perturbation scheme is said to be regular. A slightly generalized strategy must be adopted when the perturbation is not H_0 bounded and especially when the behavior of the whole system depends dramatically upon the sign of the coupling constant or its square as in our specific problem. The idea is to extract the fundamental properties of the relatively bounded case by means of the notion of analytic family of operators introduced by Kato [7]: this term denotes a collection of operators $\{H_g\}_{g\in R}$ dependent upon a coupling constant g taking values in a region R of the complex plane, where each H_g has a nonempty resolvent and is defined on a domain $\mathcal{D}$ independent of g; moreover, for each $\phi\in\mathcal{D}$, $H_g\phi$ is required to be strongly analytic in g. Although weaker than relative boundedness, this analyticity property is still sufficient to study the continuation of the eigenvalues in a region of the complex plane of g and it can be proved that for ξ in an appropriate open region of the resolvent set of H_{g_0} the resolvent $(\xi-H_g)^{-1}$ is analytic in g when $|g-g_0|$ is sufficiently small. The stability of discrete and nondegenerate eigenvalues immediately follows. For isolated eigenvalues of multiplicity n there exist ℓ families of eigenvalues $\lambda_i(g)$ admitting Puiseux expansions in terms of g^{1/q_i}, for integers $\{q_i\}_{i=1,\ell}$, with total multiplicity n. We are therefore left with the need to prove that a family of operators is analytic: the technique commonly used for this purpose is to establish appropriate quadratic estimates that allow us to deduce that each operator of the family is closed and that the perturbation term is small—in the sense of Kato [7]—with respect to any member of the family, so that the domains are indeed independent of the coupling constant. This argument can be formalized as follows [8]: if T_0 and T_1 are closed operators with $\mathcal{D}(T_0)\cap\mathcal{D}(T_1)$ dense, then T_0+T_1 is closed if the inequality

$$(T_0+T_1)^*(T_0+T_1)\geq a(T_0^*T_0+T_1^*T_1)+b \tag{3.1}$$

is satisfied for some constants a,b. The one-dimensional quartic oscillator with a Hamiltonian $H=p^2+x^2+g^2x^4$ describes a situation very close to the Dirac equation in one space dimension we want to discuss and its specific estimate [33] can be formulated as follows: for $a<1-|\mathrm{Re}(g^2)|/|g^2|$ and for all $\phi\in\mathcal{D}(p^2)\cap\mathcal{D}(x^4)$ there exists b such that $a[\|(p^2+x^2)\phi\|^2+|g|^4\|x^4\phi\|^2]\leq\|(p^2+x^2+g^2x^4)\phi\|^2+b\|\phi\|^2$. We finally observe that this estimate implies the Herglotz property of $\lambda_i(g^2)$, namely, $\mathrm{Im}(\lambda_i(g^2))>0$ for $\mathrm{Im}(g^2)>0$, which gives precise information about the analytic structure of $\lambda_i(g^2)$ and shows that $g=0$ cannot be an isolated singular point, but it must be a limit point of singularities of the eigenvalues. We will now discuss the application of these general facts to the Dirac equation in a quadratic vector potential.

A. Case in one space dimension

Let us consider first the one-dimensional problem. If we put $\kappa=0$ in Eqs. (2.6) and (2.7), we have the differential equation

$$[p^2+(1-2Eg^2)y^2+2gy+g^2y^4]\Phi(y)=E(1-g^2E)\Phi(y) \tag{3.2}$$

defined for $-\infty<y<\infty$, where $p=-i(d/dy)$. As we said, in order to apply the general theory, we need a quadratic estimate analogous to that of the anharmonic oscillator. A less cumbersome notation is obtained if we rescale the variables by a dilation $y\to\alpha y$, $p\to p/\alpha$ and then we choose

$$\lambda=E\alpha^2(1-Eg^2),\quad \sigma=2g\alpha^3,$$

$$\alpha=(1-2Eg^2)^{-1/4}. \tag{3.3}$$

The inverse relations are

$$E=2\lambda(1+\lambda\sigma^2)^{1/4}[1+(1+\lambda\sigma^2)^{1/2}]^{-1},$$

$$g=\frac{\sigma}{2}(1+\lambda\sigma^2)^{1/8}, \tag{3.4}$$

and by means of Eq. (3.4) it is possible to reconstruct the function $E(g)$ from $\lambda(\sigma)$. In the new parameterization (3.3), Eq. (3.2) becomes

$$\left[p^2+y^2+\sigma y+\frac{\sigma^2}{4}y^4\right]\Phi(y)=\lambda\Phi(y) \tag{3.5}$$

and the quadratic estimate has to be done for the operator on the left-hand side of Eq. (3.5). For $\phi\in\mathcal{D}(p^2)\cap\mathcal{D}(y^4)$ and $\mathrm{Im}(\sigma)\neq 0$ we consider the closed operators

$$T_0=p^2+y^2+\mathrm{Re}(\sigma)y,$$

$$T_1=i\,\mathrm{Im}(\sigma)y+\frac{\sigma^2}{4}y^4. \tag{3.6}$$

Everything is known about T_0 since it is just the Hamiltonian of a harmonic oscillator in the displaced variable $[y+\mathrm{Re}(\sigma)/2]$. A direct calculation given in the Appendix proves quadratic estimate (3.1).

It will be shown in Sec. IV that for parity reasons only even powers of the coupling constant σ contribute to the perturbation expansion of the eigenvalues. As a consequence of Eq. (A1) we can state that on the domain $\mathcal{D}(p^2)\cap\mathcal{D}(x^4)$ and with T_0,T_1 given in Eq. (3.6), the operators $H_\sigma=T_0+T_1$ form a holomorphic family with compact resolvents for σ^2 in the complex plane cut along the negative axis. We thus get a perturbation series for each eigenvalue

$$\lambda(z)=\sum_{n=0}^{\infty}a_nz^n \tag{3.7}$$

with $z=\sigma^2$. For Eq. (3.7) there exists a Borel transform (2.9). The corresponding sum is given by

$$\lambda(z)=z^{-1-\nu}\int_0^\infty u^\nu\exp(-u/z)B(u+i0)du. \tag{3.8}$$

B. Case in three space dimensions

We next consider the three-dimensional problem assuming $\kappa=-1$ in order to simplify the notation: indeed the treat-

ment is qualitatively the same for any other allowed value of κ. Equations (2.6) and (2.7), then, specifies to

$$H_g\Phi(y) \equiv \left\{ p^2 + y^2 - 2gy + g^2 \left[(y^2 - E)^2 - \frac{2(y^2 - E)}{y(y+2g)} + \frac{3}{y^2(y+2g)^2} \right] \right\} \Phi(y) = E\Phi(y) \tag{3.9}$$

and since g is proportional to c^{-1} the limit $g \to 0$ correctly reproduces Schrödinger equation (2.3) with angular momentum $\ell = 0$.

With respect to Eq. (3.2), Eq. (3.9) contains nonpolynomial terms proportional to g^2. The rational terms have vanishing denominators when both g and y tend to zero and the simultaneous limit is not well defined. The expansion in g of the perturbation potential leads to higher and higher divergences at the origin:

$$V_1 \simeq -2gy + g^2\{y^{-4}[3 + 2Ey^2 + (E^2 - 2)y^4 - 2Ey^6 + y^8] - gy^{-5}[4(3 + Ey^2 - y^4)] + O(g^2)\}.$$

These terms make rather problematic the possibility of achieving a reasonable quadratic estimate. On the other hand, the expansion in y in the neighborhood of the origin gives, up to $O(y)$,

$$V_1 \simeq \frac{3}{4y^2} - \frac{3 - 4Eg^2}{4gy} + \frac{9 - 8Eg^2 + 16E^2g^4}{16g^2}. \tag{3.10}$$

As the quadratic divergence at the origin is due to the angular momentum of the system, the term $(3/4)y^{-2}$ shows that, as a consequence of relativity, we are dealing with a spin (1/2) fermion and that the spin is the only angular momentum left when $\kappa = -1$. Besides these physical observations, however, the expansion cannot be used perturbatively, not even for searching the local solutions at the origin. A different approach has therefore to be searched. We thus find it necessary to fix the final value of the parameter g and to put this final value g_* in the singular terms: we say that we *freeze* the parameter at its final value wherever it is needed. We then write the equation

$$\left(\left[p^2 + y^2 + \frac{3}{4y^2} \right] - \left\{ 2gy + g^2 \left[\frac{3}{4g_*^2} \frac{y + 4g}{(y + 2g_*)^2 y} - (y^2 - E)^2 + 2\frac{y^2 - E}{(y + 2g_*)y} \right] \right\} \right) \Phi(y) = E\Phi(y) \tag{3.11}$$

that coincides with Eq. (3.9) when $g_* = g$. We denote, respectively, by T_0 and T_1 the operators in the first and in the second square bracket; we then observe that with g frozen at any nonvanishing g_* the factors $(y+2g_*)^{-1}$ are bounded in $0 \le y \le +\infty$ and do not need further estimates. Therefore, the two terms of T_1 diverging like y^{-1} at the origin and presenting an at most constant behavior at infinity are small in the sense of Kato with respect to $y^2 + (3/4)y^{-2}$ in T_0, so that they can be neglected in the estimate. Since it is well known that $p^2 + y^2 + j(j+1)y^{-2}$ is a closed operator on the maximal domain of the functions satisfying the condition $\Phi(y) \simeq y^{j+1}$ at $y = 0$, which correspond to the regular solutions of differential equation (3.9)—and therefore T_0 is closed on the maximal domain with $\Phi(y) \simeq y^{3/2}$ at the origin—by means of a calculation analogous to Eq. (A1) we conclude that $T_0 + T_1$ is closed and that the set $\{H_g\}$, whose elements are specified in Eq. (3.9), form an analytic family of operators with compact resolvent. We can thus conclude that for the three-dimensional case too we get a perturbation series like Eq. (3.7) with $z = \sigma^2$ for which Borel transform (2.9) and its inverse (3.8) are well defined.

It is evident that the concrete calculation of the perturbation expansion is much more difficult in three dimensions than in one. In practice, expansion (3.10) could suggest as preferable the use of a variational method by defining the isospectral dilated operator

$$H_\eta(g,E) = \eta^{-2}\left(\left[p^2 + y^2 + \frac{3}{4y^2} \right] + \left[(\eta^4 - 1)y^2 - \eta\frac{3(3\eta y + 4g)}{4(\eta y + 2g)^2 y} - 2g\eta^3 y + g^2\eta^2 \times (\eta^2 y^2 - E)^2 + 2g^2\eta\frac{E - \eta^2 y^2}{(\eta y + 2g)y} \right] \right),$$

where $g = -i|g|$, $\eta = \exp(i\vartheta)$, and $0 < \vartheta < \pi/6$. We could then approximate the first eigenvalues restricting the Hamiltonian on the linear space of the first n eigenvectors of $p^2 + y^2 + (3/4)y^{-2}$. We leave this problem for future investigations.

A final comment on the perturbation treatment of the Dirac equation in two space dimensions can also be made. As it is well known, in this case the algebra of the α and β matrices admits a 2-dim representation in terms of the Pauli matrices σ_i and the stationary Dirac equation in the presence of a central vector potential $(0, V(r))$ can be written in the form [35]

$$\left[c\sum_{i=1}^{2} (\sigma_i p_i) + \sigma_3 mc^2 + V(r) \right] w = Ew, \tag{3.12}$$

where $w = {}^t[w_1(x,y), w_2(x,y)]$ is a two component spinor and p_i are the momentum operators. Equation (3.12) is separable in spherical coordinates with the same procedure used in [25] for the 3-dim equation. By diagonalizing the third component of the angular momentum and denoting by j the corresponding eigenvalue, we easily get a system of two coupled first-order differential equations in two radial functions $a(r)$ and $b(r)$ corresponding to the spinor components; defining then two rescaled functions $X_1(r) = r^{1/2}b(r)$ and $X_2(r) = -ir^{1/2}a(r)$, the resulting system is coincident with Eq. (2.1), where κ is allowed to assume the values $-j$. The same perturbation treatment given in the previous item (b) applies therefore in two spatial dimensions also.

IV. NUMERICAL DEVELOPMENTS

In this last section we present the numerical calculations concerning the perturbation treatment of Eq. (3.5) yielding the determination of $\lambda(\sigma)$ from which we can obtain $E(g)$ by

inverting definitions (3.3). We take p^2+y^2 as unperturbed operator and we write the perturbation part in the form $\sigma U_1+\sigma^2 U_2$, where $U_1=y$ and $U_2=y^4/4$. We introduce the usual creation and destruction operators

$$a = 2^{-1/2}(d/dy + y), \quad a^\dagger = 2^{-1/2}(-d/dy + y),$$

$$[a,a^\dagger] = 1,$$

so that

$$p^2 + y^2 = 2a^\dagger a + 1, \quad y = 2^{-1/2}(a^\dagger + a),$$

$$y^4 = 4^{-1}(a^{\dagger 4} + 4a^{\dagger 3}a + 6a^{\dagger 2}a^2 + 4a^\dagger a^3 + a^4 + 6a^{\dagger 2} + 12a^\dagger a + 6a^2 + 3).$$

In the standard Dirac notation, the solutions of the eigenvalue equation for the unperturbed operator, $(2a^\dagger a+1)|\psi\rangle = \lambda|\psi\rangle$, are the usual occupation number states $|n\rangle$, expressed in terms of Hermite functions, where $\lambda=\lambda_n\equiv 2n+1$ with integer n. For later use we recall the matrix elements of y, y^2, and y^4 with respect to pairs of such states, namely,

$$\langle k|y|n\rangle = 2^{-1/2}(\sqrt{n+1}\,\delta_{k,n+1} + \sqrt{n}\,\delta_{k,n-1}),$$

$$\begin{aligned}\langle k|y^4|n\rangle = 2^{-2}\{&\sqrt{(n+4)(n+3)(n+2)(n+1)}\,\delta_{k,n+4} \\ &+ (4n+6)\sqrt{(n+2)(n+1)}\,\delta_{k,n+2} + [6n^2+3(2n+1)] \\ &\times \delta_{k,n} + (4n-2)\sqrt{n(n-1)}\,\delta_{k,n-2} \\ &+ \sqrt{n(n-1)(n-2)(n-3)}\,\delta_{k,n-4}\},\end{aligned} \tag{4.1}$$

where $\delta_{i,j}$ is the usual Kronecker symbol, taking values 1 or 0 according to whether the indices are equal or different. Since the Hermite functions have the parity of n, we see from Eq. (4.1) that $\langle k|U_1|n\rangle$ is vanishing when the integers k and n have the same parity and $\langle k|U_2|n\rangle$ is vanishing when the parity is opposite. We now set up the standard perturbation framework by defining

$$\lambda_n = (2n+1) + \sum_{i=1}^{\infty} g^i \lambda_n^{(i)},$$

$$|\psi_n\rangle = |n\rangle + \sum_{i=1}^{\infty} g^i |\psi_n^{(i)}\rangle.$$

As usual, the global phase of $|\psi_n\rangle$ can be chosen in such a way to have $\langle n|\psi_n^{(1)}\rangle=0$. A straightforward calculation leads then to the first- and second-order quantities

$$\lambda_n^{(1)} = 0, \quad \lambda_n^{(2)} = 2^{-4}[6n(n+1)-1],$$

$$|\psi_n^{(1)}\rangle = -(2\sqrt{2})^{-1}(\sqrt{n+1}|n+1\rangle - \sqrt{n}|n-1\rangle),$$

$$\begin{aligned}|\psi_n^{(2)}\rangle = &-\frac{\sqrt{(n+4)(n+3)(n+2)(n+1)}}{128}|n+4\rangle \\ &+ \frac{(2n+1)\sqrt{(n+2)(n-1)}}{32}|n+2\rangle \\ &+ \frac{(2n+1)\sqrt{n(n-1)}}{32}|n-2\rangle \\ &+ \frac{\sqrt{n(n-1)(n-2)(n-3)}}{128}|n-4\rangle\end{aligned} \tag{4.2}$$

and to the recurrence relation, for $i>2$,

$$\begin{aligned}&[(p^2+y^2)-(2n+1)]|\psi_n^{(i)}\rangle + U_1|\psi_n^{(i-1)}\rangle + U_2|\psi_n^{(i-2)}\rangle \\ &- \sum_{s=1}^{i-1} \lambda_n^{(s)}|\psi_n^{(i-s)}\rangle - \lambda_n^{(i)}|n\rangle = 0.\end{aligned} \tag{4.3}$$

We now prove by induction that for any $m\geq 0$ we have the following: (a) $\lambda_n^{(2m+1)}=0$; (b) $|\psi_n^{(2m+1)}\rangle$ has the opposite parity of n; (c) $|\psi_n^{(2m+2)}\rangle$ has the same parity of n.

Relations (4.2) give the initial step of the inductive argument. Suppose now the properties we require are true for $i \leq 2m$. The scalar product of Eq. (4.3) by $|n\rangle$ gives

$$\lambda_n^{(i)} = \langle n|U_1|\psi_n^{(i-1)}\rangle + \langle n|U_2|\psi_n^{(i-2)}\rangle \tag{4.4}$$

and we first consider $i=2m+1$. By the inductive hypothesis both the matrix elements of U_1 and U_2 are vanishing. The last sum is also vanishing, since $\lambda_n^{(s)}=0$ for odd s and $\langle n|\psi_n^{(2m+1-s)}\rangle=0$ for even s. As a consequence $\lambda_n^{(2m+1)}=0$. Looking then at the perturbation contributions to the states, we have

$$|\psi_n^{(i)}\rangle = \sum_{k=0}^{\infty} |k\rangle\langle k|\psi_n^{(i)}\rangle, \tag{4.5}$$

where the scalar product of Eq. (4.3) by $|k\rangle$ provides the relation, for $k\neq n$,

$$\begin{aligned}\langle k|\psi_n^{(i)}\rangle = (2n-2k)^{-1}\Bigg(&\langle k|U_1|\psi_n^{(i-1)}\rangle + \langle k|U_2|\psi_n^{(i-2)}\rangle \\ &- \sum_{s=1}^{i-1} \lambda_n^{(s)}\langle k|\psi_n^{(i-s)}\rangle\Bigg)\end{aligned} \tag{4.6}$$

from which the parity properties are easily deduced. The induction is therefore complete and it implies, in particular, that the perturbation expansion of the eigenvalues is in σ^2 rather than in σ.

By using Eqs. (4.1)–(4.6) we can set up a recursion scheme by which the expansions of the eigenvalues are determined. With the help of some computer algebra we have calculated the first 94 coefficients of the perturbation expansion in σ of the lowest unperturbed eigenvalue and of the fundamental state in terms of rational numbers, namely, with infinite precision. Starting from there we have continued the expansion with a FORTRAN code in floating point up to 250th order, thus obtaining a contribution to the eigenvalue up to order 125 in σ^2. The coefficients of the asymptotic series in

TABLE I. Stabilization of the lowest poles of the Borel transform $B(u)$ given in Eq. (2.9) with the order of the Padé approximants. The value of ν is taken to be $-1/2$.

[58,59]	[59,58]	[59,59]	[59,60]	[60,59]	[60,60]
2.67561	2.67562	2.67571	2.67575	2.67576	2.67572
2.71528	2.71529	2.71583	2.71611	2.71612	2.71589
2.79311	2.79314	2.79480	2.79563	2.79564	2.79497
2.92226	2.92235	2.92678	2.92899	2.92901	2.92722
3.12725	3.12752	3.13965	3.14582	3.14589	3.14082
3.45803	3.45883	3.49492	3.51474	3.51500	3.49830
4.03675	4.03958	4.17104	4.26976	4.27146	4.18431

σ^2 have alternate signs and, on the basis of the numbers we have calculated, we find the evidence for an asymptotic behavior

$$a_n \approx \frac{\pi}{2}\left(1-\frac{\sqrt{6}}{\pi^{3/2}}\right)(-1)^n\left(\frac{3}{8}\right)^n \Gamma(n+1/2). \tag{4.7}$$

To be more precise, we have defined the sequence $\{\breve{a}_n\}$ where $\breve{a}_n$ is given by the ratio of the coefficient a_n of the initial divergent series divided by the right-hand side of Eq. (4.7). We have then used the Shanks transformation $A_n=(\breve{a}_{n+1}\breve{a}_{n-1}-\breve{a}_n^2)/(\breve{a}_{n+1}+\breve{a}_{n-1}-2\breve{a}_n)$ [36], in order to improve the convergence of $\{\breve{a}_n\}$ and we have found $A_{125}=1.000\,09$, where the rounding numerical error can be estimated on the order of 10^{-4}.

Some comments on the general features of the perturbation series and of its Borel transform are in order. For positive values of σ^2 the series is Borel summable. For negative σ^2 (namely for imaginary σ, which is the case we are interested in), all the terms of the series acquire equal signs and the series itself becomes Borel summable only in the distributional sense. We will therefore assume the asymptotic series with all positive terms and thus consider $z=\sigma^2=4\Omega_*>0$ in the Borel antitransformation integral (3.8): the meaning of Ω_* will be clarified below. Dropping $(-1)^n$ in the asymptotic behavior (4.7) it is clear that the Borel transform develops singularities on the positive real axis, the first of which has to be expected in the neighborhood of 8/3. The shift in the energy of the state, given by $f(z)$ in Eq. (2.12), can be plainly calculated by taking the principal part of Eq. (3.8) with respect to the poles on the real axis.

The imaginary part of the perturbed energy, given by $d(z)$ in Eq. (2.12), has a much more interesting physical meaning: indeed $2d(z)$ gives the decay constant Γ of the state itself as in Eq. (1.1) and, in an interpretation connected with a second quantization framework, to the particle-antiparticle production rate $w^f=2\ \mathrm{Im}\ \mathcal{L}_{\mathrm{eff}}$, where $\mathcal{L}_{\mathrm{eff}}$ is the effective Lagrangian of the system [6,9]. The imaginary part will be computed by summing the contributions of the positive real poles to integral (3.8) calculated along the integration path encircling those poles in the upper complex half plane. An efficient way to make the calculation is to use a Padé approximation for the Borel transform $B(u)$ of the perturbation series. Since $B(u)$ is expressed by a series expansion up to order 125, we can take Padé approximants of rather high order and to look at the stabilization of the values of the poles. The results for the seven lowest poles are summarized in Table I.

From these numbers it appears, first of all, that the locations of the lowest poles are in the neighborhood of 8/3 as deduced from Eq. (4.7): this was to be expected and indicates the consistency of the approximation, as we said in Sec. I. We then see that their values tend to stabilize with the increasing order of the Padé, the lower poles obviously faster than the upper ones, for which approximants of higher order would be required. Therefore, recalling that no further information can be deduced by the Padé construction when the sum of the degrees of numerator and denominator exceeds the order of the series, a considerably larger number of perturbation terms should be calculated in order to stabilize the higher poles. This means, in turn, that the calculations of the real and imaginary parts of the perturbed energy give much more precise results for small values of z, for which the

TABLE II. The imaginary part of λ given in Eq. (3.8), with $\nu=-1/2$, for different orders of the Padé approximants.

Ω	[57,57]	[58,58]	[59,59]	[60,60]
0.05	0.251588×10^{-5}	0.251584×10^{-5}	0.251588×10^{-5}	0.251586×10^{-5}
0.08	0.321991×10^{-3}	0.321360×10^{-3}	0.321933×10^{-3}	0.322124×10^{-3}
0.10	0.162741×10^{-2}	0.161602×10^{-2}	0.162615×10^{-2}	0.163092×10^{-2}
0.12	0.483648×10^{-1}	0.476170×10^{-1}	0.482741×10^{-1}	0.486376×10^{-1}
0.15	0.146167×10^{-1}	0.141402×10^{-1}	0.145544×10^{-1}	0.148164×10^{-1}
0.20	0.456068×10^{-1}	0.426354×10^{-1}	0.451921×10^{-1}	0.470098×10^{-1}
0.25	0.925871×10^{-1}	0.837260×10^{-1}	0.913049×10^{-1}	0.970549×10^{-1}

TABLE III. Stability of the results for Im(λ) with varying values of the constant ν. Use has been made of the Padé approximant of order [60, 60].

Ω	$\nu=-1/2$	$\nu=0$	$\nu=1/2$	$\nu=1$
0.05	0.251586×10^{-5}	0.251586×10^{-5}	0.251586×10^{-5}	0.251586×10^{-5}
0.08	0.322124×10^{-3}	0.322124×10^{-3}	0.322099×10^{-3}	0.322113×10^{-3}
0.10	0.163092×10^{-2}	0.163309×10^{-2}	0.163333×10^{-2}	0.163335×10^{-2}
0.12	0.486376×10^{-1}	0.488771×10^{-1}	0.489565×10^{-1}	0.489141×10^{-1}
0.15	0.148164×10^{-1}	0.150256×10^{-1}	0.151061×10^{-1}	0.150456×10^{-1}
0.20	0.470098×10^{-1}	0.485809×10^{-1}	0.487832×10^{-1}	0.484381×10^{-1}
0.25	0.970549×10^{-1}	0.101983	0.100359	0.100484

contribution of the lowest poles is largely dominant: a natural fact in a perturbation framework, which can be clearly observed in Table II, where, for different values of Ω, we have reported the numerical data for the imaginary part of λ given in Eq. (3.8), namely,

$$\mathrm{Im}(\lambda) = -\frac{\pi}{(4\Omega_*)^{1+\nu}} \sum_{\{\text{poles } p_i\}} p_i^{\nu}\, \mathrm{Res}_{p_i}(P_{[m,n]}(B(u))) \times \exp\left(-\frac{p_i}{4\Omega_*}\right) \tag{4.8}$$

with $\nu=-1/2$. In Eq. (4.8) $B(u)$ is the Borel transform (2.9) of the perturbation series and $P_{[m,n]}$ is the $[m,n]$ Padé approximant of $B(u)$ with poles $\{p_i\}$. By Res_{p_i} we have indicated, as usual, the residue at the pole p_i. Finally, the value of Ω_* has been constructed by using formulas (3.3) with $\Omega_* = \Omega(1+2E\Omega)^{-3/2}$ and $\Omega=|g|^2$ is the same parameter used in [6]. We strongly stress that while in Eq. (3.3), with imaginary g and σ, $\lambda=E$ to the order $|g|^2$, the relationship between σ and g is not so straightforward, as the correction due to the term α^6 is large and cannot by no means be neglected: on the contrary, it proves fundamental to establish the correct relationship between the perturbation data and the data calculated in [6], taking the zeroth order $\lambda=1$ inside α. We also report, in Table III, a further check of stability of the results: the numerical values give imaginary part (4.8) calculated for different values of the constant ν and using the [60, 60] Padé approximant.

We will conclude the section by comparing the results here obtained with those presented in [6]. The methods used to get them, as we said, are completely different. On the one hand, we have a perturbation approach that becomes more and more precise when the perturbation parameter decreases and needs many further terms for intermediate values of that parameter: the imaginary part of the energy is directly calculated by using the Padé approximants to invert the Borel transform in the distributional sense and integrating along the appropriate path in the complex plane. On the other hand, the determination of the spectral concentration is obtained by a numerical integration of a differential equation that presents some accuracy problem for very low values of the coupling constant and works better at higher values: the imaginary part is now deduced from the half-width of the Breit-Wigner lines that fit the continuous spectrum. This makes the two methods complementary and puts rather narrow bounds to the range of the Ω values in which the comparison makes sense: a reasonable interval for the present data could be assumed to be $0.1\leq\Omega\leq0.25$. Within these bounds Fig. 1 shows that the agreement is complete proving the effectiveness of both the DBS and the method of [6] for solving numerical problems.

V. CONCLUSIONS

In this paper we have presented an application of the singular perturbation theory to the Dirac equation in an external vector potential with quadratic growth, in one space dimension and in three space dimensions with spherical symmetry. We have studied possible solution strategies depending upon the analytical properties of the asymptotic series produced by the perturbation treatment and we have proved that the distributional Borel sum is the most suited approach for studying the problem. In one space dimension we have used this tool to get explicit numerical values of the imaginary part of the complex eigenvalue that comes from the perturbation of the ground state finding a complete agreement with the spectral concentration obtained in [6] in a competing and complementary way. We have thus presented an effective method to obtain numerical results for not explicitly solvable models that have been treated in different approximations, our case, in particular, corresponding to the fermion pair production rate in an inhomogeneous electric field [10,22]. We finally observe that the great interest in diverging perturbation series and their use in numerical calculations is probably due to the analogy with the quantum field theory [26,37]: this is also

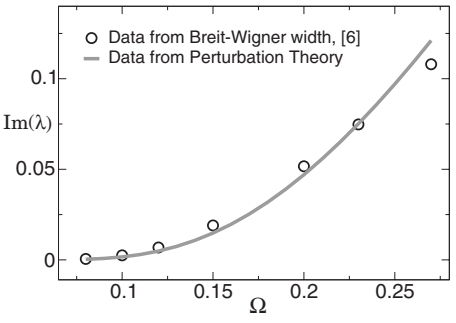

FIG. 1. Imaginary part of the perturbed lowest eigenvalue. Dotted line: data from [6]. Solid line: perturbation data.

one of the most important reasons for which anharmonic oscillators are still actively investigated in nonrelativistic quantum mechanics. One recent example is given by the family of $\mathcal{PT}$-symmetric Hamiltonians, indexed by a positive number N, studied in [38]. In fact our paper treats the relativistic analog of the member of that family with $N=2$. Due to the peculiar properties of the Dirac equation and its interpretation difficulties as a single-particle equation, we find it very interesting, both from a mathematical and a physical point of view, to consider the relativistic version of the members of that family with higher values of N, which would be one step forward in the direction of quantum fields.

APPENDIX: QUADRATIC ESTIMATE

In this appendix we give the proof of quadratic inequality (3.1) with T_0 and T_1 given in Eq. (3.6). We have

$$
\begin{aligned}
(T_0+T_1)^*(T_0+T_1) &= [p^2+y^2+\mathrm{Re}(\sigma)y]^2+\left[i\,\mathrm{Im}(\sigma)y+\frac{\sigma^2}{4}y^4\right]\left[-i\,\mathrm{Im}(\sigma)y+\frac{\bar{\sigma}^2}{4}y^4\right]+[p^2+y^2+\mathrm{Re}(\sigma)y]\left[i\,\mathrm{Im}(\sigma)y+\frac{\sigma^2}{4}y^4\right]\\
&\quad+\left[-i\,\mathrm{Im}(\sigma)y+\frac{\bar{\sigma}^2}{4}y^4\right][p^2+y^2+\mathrm{Re}(\sigma)y]\\
&=\left[1-\frac{|\mathrm{Re}(\sigma^2)|}{|\sigma|^2}\right][p^2+y^2+\mathrm{Re}(\sigma)y]^2+\left[\mathrm{Im}(\sigma)y^2+\frac{1}{4}\mathrm{Im}(\sigma)\mathrm{Im}(\sigma^2)y^5+\frac{|\sigma|^4}{16}y^8\right]\\
&\quad+\frac{|\mathrm{Re}(\sigma^2)|}{|\sigma|^2}\left\{\left[p^2+y^2+\mathrm{Re}(\sigma)y\pm\frac{|\sigma|^2}{4}y^4\right]^2-\frac{|\sigma|^4}{16}y^8\right\}+\mathrm{Im}(\sigma)p+\mathrm{Im}(\sigma^2)(py^3+y^3p).
\end{aligned}
$$

For $0<a<1-|\mathrm{Re}(\sigma^2)|/|\sigma|^2$ and for a certain $R>0$ we then have

$$
\begin{aligned}
(T_0+T_1)^*(T_0+T_1) &\geq a\left\{[p^2+y^2+\mathrm{Re}(\sigma)y]^2+\left[\mathrm{Im}(\sigma)y^2+\frac{1}{4}\mathrm{Im}(\sigma)\mathrm{Im}(\sigma^2)y^5+\frac{|\sigma|^4}{16}y^8\right]\right\}+\frac{|\mathrm{Re}(\sigma^2)|}{|\sigma|^2}\left[\mathrm{Im}(\sigma)y^2\right.\\
&\quad\left.+\frac{1}{4}\mathrm{Im}(\sigma)\mathrm{Im}(\sigma^2)y^5\right]+R\left\{[p^2+y^2+\mathrm{Re}(\sigma)y]^2+\frac{|\sigma|^4}{16}y^8\right\}+\frac{\mathrm{Im}(\sigma)}{2}(p\pm1)^2-\frac{\mathrm{Im}(\sigma)}{2}\\
&\quad\times(p^2+1)+|\mathrm{Im}(\sigma^2)|(p\pm y^3)^2-|\mathrm{Im}(\sigma^2)|(p^2+y^6)\\
&\geq a(T_0^*T_0+T_1^*T_1)-b+R\left\{[p^2+y^2+\mathrm{Re}(\sigma)y]^2-|\mathrm{Im}(\sigma^2)|p^2-\frac{\mathrm{Im}(\sigma)}{2}(p^2+1)+\frac{b}{2}\right\}\\
&\quad+\left\{\frac{|\mathrm{Re}(\sigma^2)|}{|\sigma|^2}\left[\frac{1}{4}\mathrm{Im}(\sigma)\mathrm{Im}(\sigma^2)y^5+\mathrm{Im}(\sigma)y^2\right]+R\frac{|\sigma|^4}{16}y^8-|\mathrm{Im}(\sigma^2)|y^6+\frac{b}{2}\right\}. \qquad \text{(A1)}
\end{aligned}
$$

In the last inequality we have neglected two positive terms and we have chosen a number $b>0$ in such a way to make positive the last two square brackets. In spite of the explicit calculation, one could also have argued that the linear term σy is small in the sense of Kato with respect to the leading term of the perturbation $(\sigma^2/4)y^4$: consequently, it gives a negligible contribution to the quadratic estimate which therefore reduces to that of the anharmonic oscillator. We have chosen this shortcut to deal with the quadratic estimates of the three-dimensional case.

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