# Extending Bell's beables to encompass dissipation, decoherence, and the quantum-to-classical transition through quantum trajectories

F. Lorenzen,[1] M. A. de Ponte,[2] and M. H. Y. Moussa[1]

[1]*Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13560-590 São Carlos, SP, Brazil*

[2]*Departamento de Física, Universidade Federal de São Carlos, Caixa Postal 676, 13565-905 São Carlos, SP, Brazil*

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In this paper, employing the Itô stochastic Schrödinger equation, we extend Bell's beable interpretation of quantum mechanics to encompass dissipation, decoherence, and the quantum-to-classical transition through quantum trajectories. For a particular choice of the source of stochasticity, the one leading to a dissipative Lindblad-type correction to the Hamiltonian dynamics, we find that the diffusive terms in Nelsons stochastic trajectories are naturally incorporated into Bohm's causal dynamics, yielding a unified Bohm-Nelson theory. In particular, by analyzing the interference between quantum trajectories, we clearly identify the decoherence time, as estimated from the quantum formalism. We also observe the quantum-to-classical transition in the convergence of the infinite ensemble of quantum trajectories to their classical counterparts. Finally, we show that our extended beables circumvent the problems in Bohm's causal dynamics regarding stationary states in quantum mechanics.

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## I. INTRODUCTION

The breakthrough brought about by the Copenhagen interpretation of quantum mechanics has been subject to many criticisms from its early days. Instead of being obstacles, such criticisms have contributed to broadening perspectives on the quantum mechanics program, as happened with the analysis provided by Einstein, Podolsky, and Rosen (EPR) [1] on the quantum mechanical description of physical reality. Within its spin version given by Bohm [2], the EPR argument provided the framework for the development of Bell's theorem [3] and the subsequent experiments [4] demonstrating nonlocality and thus the impossibility of a local realistic interpretation of quantum mechanics. All those striking proposals, devised in the last two decades, which rely on nonlocality—as quantum computation and communication [5], quantum cryptography [6], and teleportation [7], apart from entanglement degree measure [8], among other—owe to the EPR argument and Bell's theorem.

Apart from enlightening nonlocality and beyond the criticisms to the Copenhagen interpretation, alternative formulations have been developed to reconcile quantum mechanics with classical concepts. On this regard, the causal interpretation proposed by Bohm [9] and the stochastic quantization procedure by Nelson [10], comprehend deterministic theories which, ascribing (hidden) trajectories to quantum particles, provide the experimental statistics predicted by quantum mechanics. While in Bohm's approach an effective quantum potential derived from the quantum mechanical wave function is the key for the formulation of the causal trajectories, Nelson's completion relies on an underlying Brownian motion which substitutes the wave function by a diffusion mechanism. Although not attempting to embrace classical concepts, we finally mention Feynman's path integral formalism where the motion of a particle is described by a weighted sum over all possible trajectories connecting the end points [11].

Pursuing a realistic description of quantum mechanics that encompasses both the microworld and macroworld, Bell [3] proposed the *beable* interpretation as a counterpart to the orthodox *observable* approach of the Copenhagen school. This hidden-variable theory was intended to overcome limitations of the existing positional causal and stochastic approaches by Bohm [9] and Nelson [10], respectively. In this regard, Bell's theory assigns precise values to all physical properties, including discrete variables, even though we are *unable* to follow them; moreover, extensions to the relativistic limit seem to be straightforward. However, by focusing on the spatial distribution of fermion number, Bell still privileges the position representation, as did the previous approaches by Bohm [9] and Nelson [10].

In an earlier generalization of Bell's accomplishment, Vink [12] attributed a beable status to all observables, introducing the convenient assumption that on a sufficiently small scale—unreachable anyway—all quantities take discrete values on a lattice. In Vink's generalization, trajectories are associated with all physical quantities; when applied to position in the lattice continuum limit, it is shown to reproduce, with differing choices of manageable parameters, either the causal [9] or the stochastic [10] interpretation.

Our extension of Bell's beables was instigated by an earlier contribution by Santos and Escobar [13], carried out within Vink's formalism, in which the Schrödinger equation is replaced by its stochastic modification, derived for the continuous spontaneous localization (CSL) model [14]. After deducing a stochastic differential equation of motion, Santos and Escobar [13] pointed out the convergence between turbulence and the dynamics of the CSL model. Following the reasoning in Ref. [13], we also adopt the Itô stochastic modification of the Schrödinger equation; however, instead of using its CSL version, we leave open the choice of the stochastic source, thus generating a full extension of Bell's beables in which the Hamiltonian dynamics is corrected by an arbitrary Lindblad-like superoperator.

Aiming to introduce dissipation and decoherence into quantum trajectories, we next specify the stochastic source as the one prompting the dissipative Lindblad correction to

 

Hamiltonian dynamics. We show that, within our open beables, the diffusive terms, characteristic of Nelson's stochastic trajectories, are naturally incorporated into Bohm's ones, unifying the two interpretations. As a matter of fact, while Bohm's quantum potential stands out from the very formulation of Bells beables, through the quantum mechanical wave function, Nelson's diffusion mechanism follows from the introduced dissipative Lindblad dynamics, replacing the wave function by the density operator.

The trajectories resulting from such a unified Bohm-Nelson open model shed a new light on interesting aspects of dissipation, decoherence, and the quantum-to-classical transition. We first mention that our derived decaying trajectories fully account for the expected dissipative mechanisms we have introduced. Moreover, we verify that the decoherence time computed from the quantum formalism coincides exactly with the time where the interference between trajectories is dynamically suppressed. Therefore, the decoherence mechanism resulting from noise injection into the system—a major open problem in quantum-information theory—can also be observed from the quantum trajectories arising from the unified Bohm-Nelson approach here presented.

Moreover, as an initial pure density operator is drawn into a complete statistical mixture by the dissipation-fluctuation dynamics described by the Lindblad operator, we also show that the infinite ensemble of trajectories converges to a finite number of classical counterparts. In fact, considering two-state superpositions of harmonic oscillator states, we verify that all the ensemble of trajectories reduces to the two classical paths associated to each state of the superposition, which compose the diagonal elements of a maximally mixed density operator. In short, our extended open beables are shown to encompass dissipation, decoherence, and the quantum-to-classical transition within a unified Bohm-Nelson dissipative hidden-variable model.

We mention that in our analysis we take into account a particular quantum system, the harmonic oscillator, and some of its best-known states: the coherent, number, and squeezed states, apart from a superposition of coherent states representing a "Schrödinger catlike" state. The coherent and squeezed states are considered to address the dissipative mechanism we have introduced in Bell's beables, allowing us to conclude that our decaying trajectories accounts for the behaviors expected from quantum mechanics. The squeezed states are also used to address the squeezing of the variances of the quadrature operators, apart from the decoherence mechanism through quantum trajectories. By its turn, the number state is considered to illustrate the fact that our extended beables circumvent the problems in Bohm's approach regarding stationary states in quantum mechanics [15]. Finally, the Schrödinger catlike states enable us to readdress the decoherence suppression from the perspective of a superposition state, apart from the quantum-to-classical transition through our decaying trajectories.

## II. EXTENDED (OPEN) BEABLES

In our generalized beable approach we replace the Schrödinger equation, as in [3], with its stochastic Itô version,

$$d|\psi\rangle = \left( -\frac{i}{\hbar} H dt + \mathbf{O} \cdot d\mathbf{\Lambda} - \frac{1}{2}\gamma \mathbf{O}^{\dagger} \cdot \mathbf{O} dt \right) |\psi\rangle, \qquad (1)$$

where $\mathbf{O} \equiv \{\mathcal{O}_i\}$ is a set of operators on the Hilbert space of the system, while $\mathbf{\Lambda} \equiv \{\Lambda_i\}$ is a set of random operators characterized by a real Wiener process of strength $\gamma$, satisfying $\overline{d\Lambda_i}=0$, $\overline{d\Lambda_i d\Lambda_j}=\gamma\delta_{ij}dt$. As pointed out above, in Ref. [13] Santos and Escobar started from a particular choice of the Wiener process, the one considered in the CSL proposal of a unified description of microscopic and macroscopic systems.

Using the Itô calculus to derive the evolution of the system density operator from Eq. (1), we obtain

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H,\rho] + \gamma \mathbf{O}\rho(t) \cdot \mathbf{O}^{\dagger} - \frac{\gamma}{2}\{\mathbf{O}^{\dagger} \cdot \mathbf{O}, \rho(t)\}, \qquad (2)$$

generating the probability density $\mathcal{P}_n(t)=\langle \varphi_n|\rho(t)|\varphi_n\rangle =\rho_{nn}(t)$ to measure the system in the eigenstate $|\varphi_n\rangle$ of a quantity satisfying $\Phi|\varphi_n\rangle = n|\varphi_n\rangle$. The continuity equation in the $\Phi$ representation is thus given by

$$\hbar \frac{\partial P_n(t)}{\partial t} = \sum_m J_{nm}, \qquad (3)$$

where the last two terms on the right-hand side (rhs) of the source matrix,

$$\begin{aligned} J_{nm} = {} & 2\,\mathrm{Im}[\langle \varphi_n|H|\varphi_m\rangle\langle\varphi_m|\rho(t)|\varphi_n\rangle] \\ & + \gamma \sum_l \{\langle\varphi_n|\mathbf{O}|\varphi_l\rangle\langle\varphi_l|\rho(t)|\varphi_m\rangle\langle\varphi_m|\mathbf{O}^{\dagger}|\varphi_n\rangle \\ & - \mathrm{Re}[\langle\varphi_n|\mathbf{O}^{\dagger}|\varphi_m\rangle\langle\varphi_m|\mathbf{O}|\varphi_l\rangle\langle\varphi_l|\rho|\varphi_n\rangle]\}, \end{aligned}$$

which are added to Bell's beables, follow from the stochastic elements in Eq. (1). As the classical counterpart to the continuity equation [Eq. (3)] is given by the master equation

$$\frac{\partial P_n(t)}{\partial t} = \sum_m (T_{nm}P_m - T_{mn}P_n),$$

where $T_{nm}dt$ is the transition probability governing the jumps from state $\varphi_m$ to $\varphi_n$, Bell's shortcut from quantum observables to beables follows from the solution of $T_{nm}$ derived by equating

$$\frac{1}{\hbar} J_{nm} = T_{nm}P_m - T_{mn}P_n. \qquad (4)$$

A particular simplified solution to the mixed quantum-classical equation [Eq. (4)] for a general complex source matrix $J_{nm}$ emerging from the stochastic ingredient is given by

$$T_{nm} = \begin{cases} \dfrac{J_{nm}+J_{nm}^*}{2\hbar P_m}, & J_{nm} \geq 0 \\ 0, & J_{nm} \leq 0. \end{cases}$$

We stress that the summation $\Sigma_m J_{nm}$ gives a real quantity, as well as the source matrix following from the original beable interpretation.

In Vink's extension of the theory, where all the degrees of freedom must be discrete and finite, the position is restricted to sites of a lattice which, in the one-dimensional case, becomes $x_n = n\varepsilon$, $n$ being integers and $\varepsilon$ being the lattice distance. In the continuous limit $\varepsilon \to 0$, smooth wave functions in the positional representations, $\psi(x,t) = R(x,t)e^{iS(x,t)/\hbar}$, can be expanded to first order in $\varepsilon$ (with $|\varphi_n\rangle = |x_n\rangle$) as

$$\psi(x_m \pm \varepsilon, t) \simeq \psi(x_m, t) \pm \varepsilon \left\{ \Delta R(x_m, t) + \frac{i}{\hbar} R(x_m, t) \Delta S(x_m, t) \right\} e^{iS(x_m,t)/\hbar},$$

leading to both source and transition matrices as expansions to dominant order in $\varepsilon$. We have defined the derivative $\Delta F(x,t) = [F(x+\varepsilon,t) - F(x,t)]/\varepsilon$.

We end up this section by noticing that when neglecting the coupling strength $\gamma$ we recover exactly the formalism of Bell's beables. Therefore, as already mentioned, our contribution consists of the extension of Bell's beables considering, instead of the Schrödinger equation, its Itô stochastic form.

## III. OPEN BEABLES APPLIED TO THE HARMONIC OSCILLATOR

Regarding the source of stochasticity, we choose the one leading to the well-known dissipative Lindblad-type correction to Hamiltonian dynamics [16]. For the particular case of the harmonic oscillator, governed by the discretized Hamiltonian,

$$H_{mn} = -\frac{\hbar^2}{2M\varepsilon^2}(\delta_{m,n+1} + \delta_{m,n-1} - 2\delta_{m,n}) + \frac{M\omega^2 x_n^2}{2}\delta_{m,n},$$

our choice is carried out under the assumption that the whole set $\{\mathcal{O}_i\}$ reduces to the annihilation operator

$$a_{nm} = \frac{1}{\sqrt{2\hbar M\omega}}\left( M\omega x_m \delta_{n,m} + \hbar \frac{\delta_{n+1,m} - \delta_{n,m}}{\varepsilon} \right).$$

Applying the above development for the open beables to the harmonic oscillator we obtain, up to zero order in $\varepsilon$, the real part of the source matrix,

$$\begin{aligned}\mathrm{Re}(J_{mn}) = {} & \frac{\hbar}{M\varepsilon}[\Delta S(\varepsilon n)P_n \delta_{n+1,m} - \Delta S(\varepsilon n)P_n \delta_{n-1,m}] \\ & + \frac{\gamma\hbar^2}{2M\omega\varepsilon^2}\mathrm{Re}[(\rho_{m+1,n} - \rho_{m,n})\delta_{n-1,m} - (\rho_{n+1,m} \\ & - \rho_{n,m})\delta_{n+1,m}],\end{aligned} \tag{5}$$

where the Kronecker deltas $\delta_{m+1,n}$ and $\delta_{m-1,n}$ allow both forward and backward steps between the very closely spaced lattice sites. Whereas the first two terms on the rhs of Eq. (5) conform to Hamiltonian dynamics, the last two terms refer to the stochastic ingredient.

To obtain the quantum trajectories, we compute, up to the time interval $dt$ and first order in $\varepsilon$, the system position $x(t) \simeq x(t+dt) + \varepsilon\Sigma_m(m-n)T_{mn}dt$, which results, for the forward movement, in the quantum equation of motion,

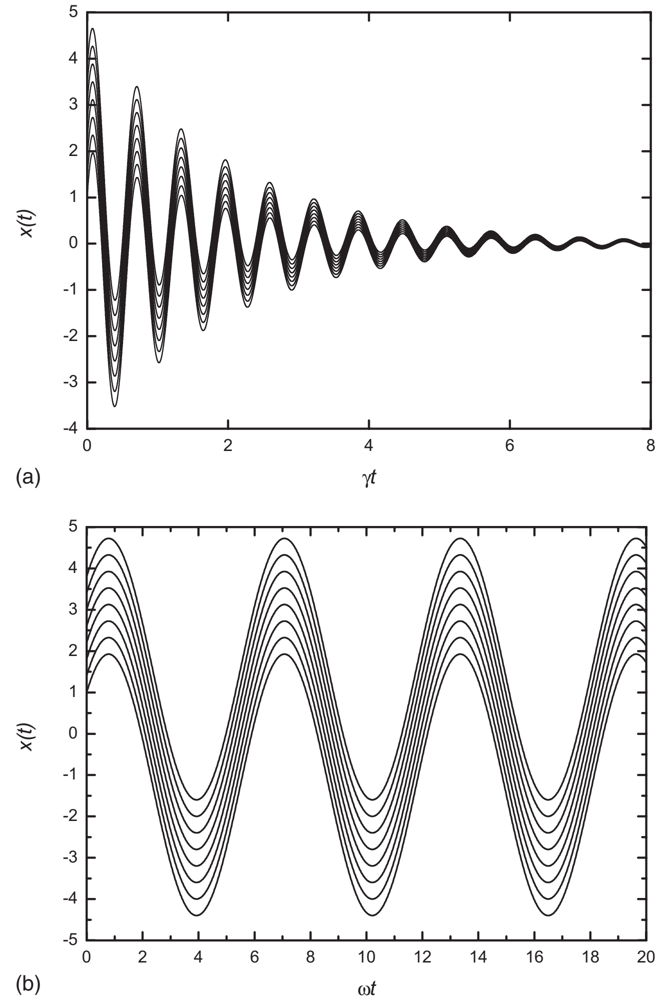


FIG. 1. Quantum trajectories $x(t)$ plotted (in units of $\sqrt{\hbar/M\omega}$) plotted against the scaled times (a) $\gamma t$ and (b) $\omega t$ for the dissipative and nondissipative harmonic oscillators prepared in the coherent state $|\alpha_0\rangle$, with $|\alpha_0|^2 = 5$. We set $\omega/\gamma = 10$ for the dissipative case.

$$x(t) \simeq x(t+dt) + \varepsilon \sum_m (m-n) \left\{ \frac{1}{M\varepsilon}\Delta S(\varepsilon n)\delta_{n+1,m} - \frac{\gamma\hbar}{2M\omega\varepsilon^2}\frac{1}{P_n}\mathrm{Re}(\rho_{n+1,m} - \rho_{n,m})\delta_{n+1,m} \right\} dt.$$

Expanding the density operators $\rho_{n+1,m}$ and $\rho_{n,m}$ to first order in $\varepsilon$, we thus obtain

$$\frac{dx}{dt} = \frac{1}{M}\frac{\partial S(x,t)}{\partial x} + \frac{\gamma\hbar}{2M\omega}\frac{1}{R(x,t)}\frac{\partial R(x,t)}{\partial x}, \tag{6}$$

where the dissipation-fluctuation source furnishes Nelson's stochastic trajectories, whereas its absence ($\gamma = 0$) reduces Eq. (6) to Bohm's causal dynamics. We stress that the white noise fluctuation which is also present in Nelson's approach—apart from the diffusion constant $\mathcal{D} = \gamma\hbar/2M\omega$—can be straightforwardly accounted for within our extended beables. To this end, we only have to enlarge the Wiener process to $\overline{d\Lambda_i} = 0$, $\overline{d\Lambda_i d\Lambda_j} = \gamma\delta_{ij}dt + \lambda(x,t)\delta_{ij}\delta(x)\delta(t)d\eta(t)$, taking into account a back-

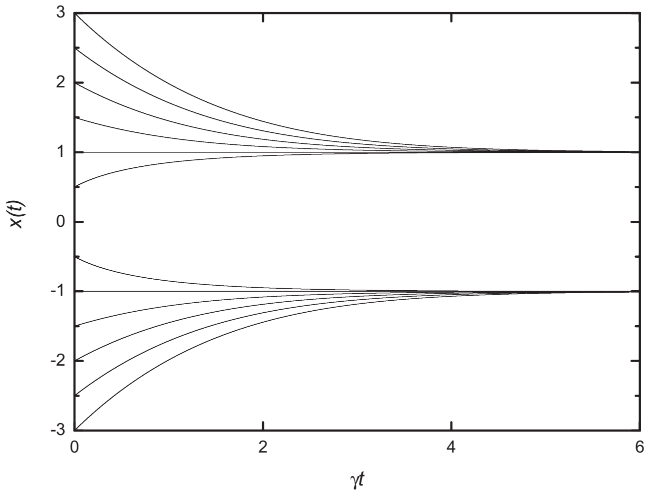


FIG. 2. Trajectories $x(t)$ (in units of $\sqrt{\hbar/M\omega}$) against $\gamma t$ for the dissipative harmonic oscillator prepared in the Fock state $|1\rangle$, with $\omega/\gamma=10$.

ground fluctuation in space and time of state-dependent strength $\lambda(x,t)$, with $d\eta(t)$ describing the fluctuations where $\langle d\eta^2\rangle=2dt$ [12]. For the cases to be analyzed below—where the harmonic oscillator is assumed to be prepared in a coherent state or in a superposition of them—this background static, with constant $\lambda=\sqrt{\gamma}$, adds to the rhs of Eq. (6) the desired correction $\sqrt{\mathcal{D}}d\eta/2dt$.

A general quantum equation of motion associated with the mixed state, following from the same steps leading from Eq. (5) to Eq. (6), is then

$$\frac{dx}{dt}=\frac{\hbar}{M}\frac{1}{\mathrm{Re}\,\rho(x,x;t)}\left[\mathrm{Im}\left(\frac{\partial}{\partial x}\rho(x,x';t)\right)+\frac{\gamma}{2\omega}\mathrm{Re}\left(\frac{\partial}{\partial x}\rho(x,x';t)\right)\right]_{x'=x}. \tag{7}$$

## IV. COHERENT, NUMBER, AND SQUEEZED STATES

Before addressing a Schrödinger catlike state, we consider the coherent, number, and squeezed states of the harmonic oscillator aiming to analyze dissipation and decoherence through quantum trajectories.

### A. Coherent state: Noise-free dissipative dynamics

Starting with the coherent state $|\alpha_0\rangle$, the dissipative master equation following from Eq. (2) leads to the solution $\langle x|\alpha(t)\rangle=R(x,t)e^{iS(x,t)/\hbar}$, where the functions

$$R(x,t)=\left(\frac{M\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\left(\sqrt{\frac{M\omega}{2\hbar}}x-\mathrm{Re}[\alpha(t)]\right)^2\right],$$

$$S(x,t)=\mathrm{Im}[\alpha(t)]\left(\hbar\sqrt{\frac{2M\omega}{\hbar}}x-\mathrm{Re}[\alpha(t)]\right),$$

depend implicitly on time $t$, through the evolving coherent state $\alpha(t)=\alpha_0 e^{-(\gamma/2+i\omega)t}$, and explicitly on position $x$. The associated equation of motion,

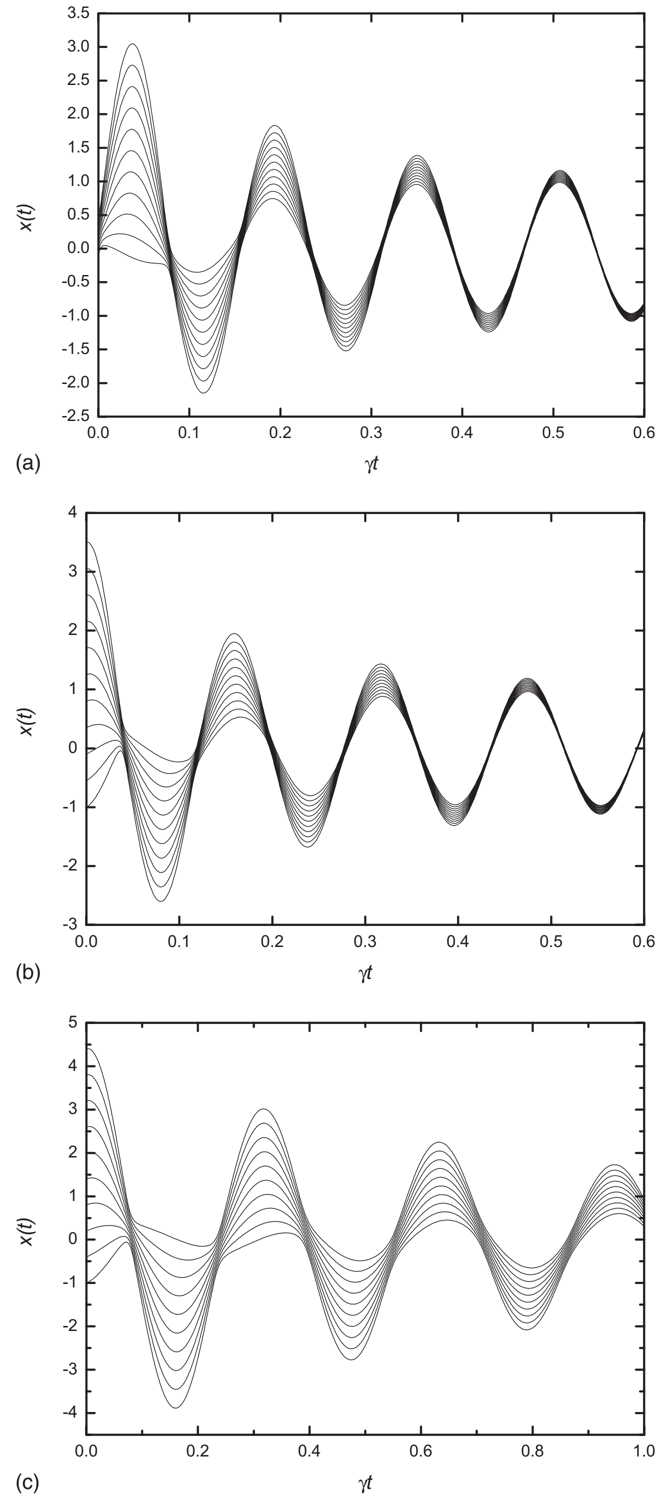


FIG. 3. Trajectories $x(t)$ (in units of $\sqrt{\hbar/M\omega}$) against $\gamma t$ for the dissipative harmonic oscillator prepared in the (a) position and (b) momentum squeezed states, with $r=1$, $|\alpha_0|^2=1$, and $\omega/\gamma=40$.

$$\frac{dx}{dt}=-\frac{\gamma}{2}x+\sqrt{\frac{2\hbar}{M\omega}}\left[\omega\alpha_I(t)+\frac{\gamma}{2}\alpha_R(t)\right],$$

prompts the trajectories $x(t)$ vs $\gamma t$ drawn in Fig. 1(a) for unit constants $\hbar=1$ and $M=1$, together with $|\alpha_0|^2=5$ and

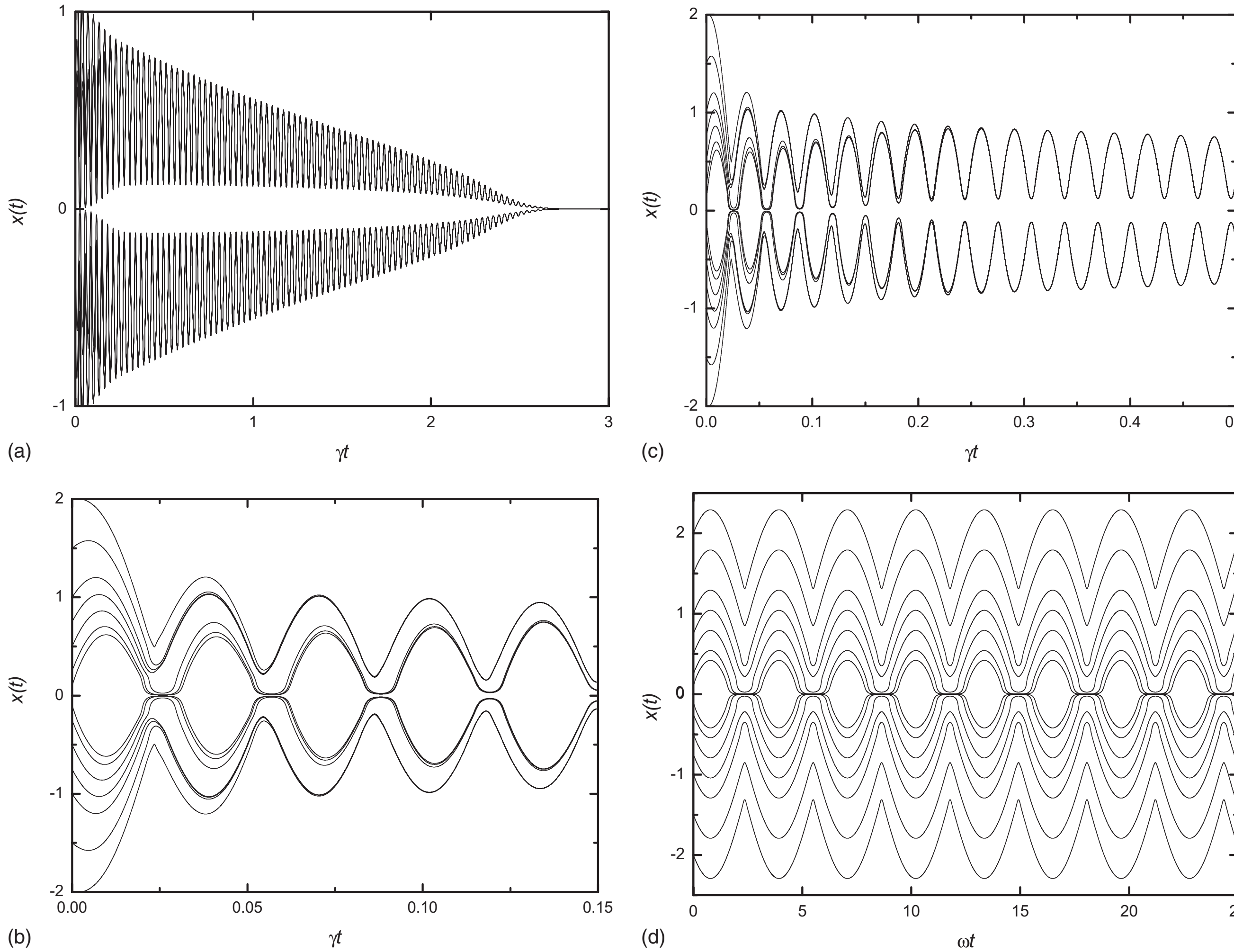


FIG. 4. In (a) we plot the quantum trajectories $x(t)$ (in units of $\sqrt{\hbar/M\omega}$) against the scaled time $\gamma t$ for the dissipative harmonic oscillator prepared in the superposition state $|\psi\rangle = \mathcal{N}(|\alpha_0\rangle + |-\alpha_0\rangle)$, with $|\alpha_0|^2=5$, $\omega/\gamma=10^2$. In (b) we focus on a small time scale $\gamma t \ll 1$ to draw attention to the decoherence process, while in (c) we look a little further than $\gamma t \ll 1$ to visualize the quantum-to-classical transition. Finally, in (d) we plot the trajectories $x(t)\times\omega t$ associated with a nondissipative Schrödinger catlike state, assuming the same parameters as in Fig. 4(a) except $\gamma=0$.

$\omega/\gamma=10$. As expected, the ensembles of trajectories associated with the initial positions distributed according to $|\alpha_0(x)|^2$ are constrained by the dissipative mechanisms to the vacuum equilibrium state, differently from what happens for the case where $\gamma=0$. In fact, in Fig. 1(b) we plot $x(t)$ vs $\omega t$, assuming that a dissipation-free oscillator is again prepared in the coherent state $|\alpha_0\rangle$, with the same parameters as in Fig. 1(a). A comparison between Figs. 1(a) and 1(b) reveals, apart from the fact that the amplitude of the oscillating trajectories remains unaltered for $\gamma=0$, an in-phase evolution of the decaying trajectories for $\gamma\neq 0$. We thus conclude that the quantum trajectories derived from our extended beables corroborate the well-known and remarkable property that, for an absolute zero reservoir, a dissipative system prepared in a coherent state remains pure despite losing excitation [17,18], i.e., $\alpha(t)=\alpha_0 e^{-(\gamma/2+i\omega)t}$. In other words, the decaying quantum trajectories depicted in Fig. 1(a) illustrate the fact that, at absolute zero, a coherent state evolves coherently under a noiseless dissipative dynamics, i.e., without the fluctuations that usually go with the dissipation.

### B. Number state: Decay of initially stationary states

Next we consider the number state to address the fact that, although the Bohmian mechanics ascribes no motion to a quantum system in an stationary state [15], the unified dissipative Bohm-Nelson version deduced here unequivocally reveals the evolution of an initially stationary state toward the vacuum. Assuming that a harmonic oscillator is prepared in the Fock state $|1\rangle$, the trajectories governed by the equation

$$\frac{dx}{dt} = \frac{\gamma}{2}\left(\frac{\hbar}{M\omega}\frac{1}{x} - x\right)$$

are plotted in Fig. 2, obtained with $\hbar = M = \omega/\gamma = 1$. We observe that the ensembles of trajectories, distributed according to the initial values $|\langle x|1\rangle|^2$, all converge, as expected, to both maximum probability densities prescribed by quantum mechanics, at $x=\pm 1$ for the choice of parameters made above. Therefore, our extended beables provide a satisfactory scenario for the trajectories associated with stationary states in quantum mechanics, a controversial point in Bohm-

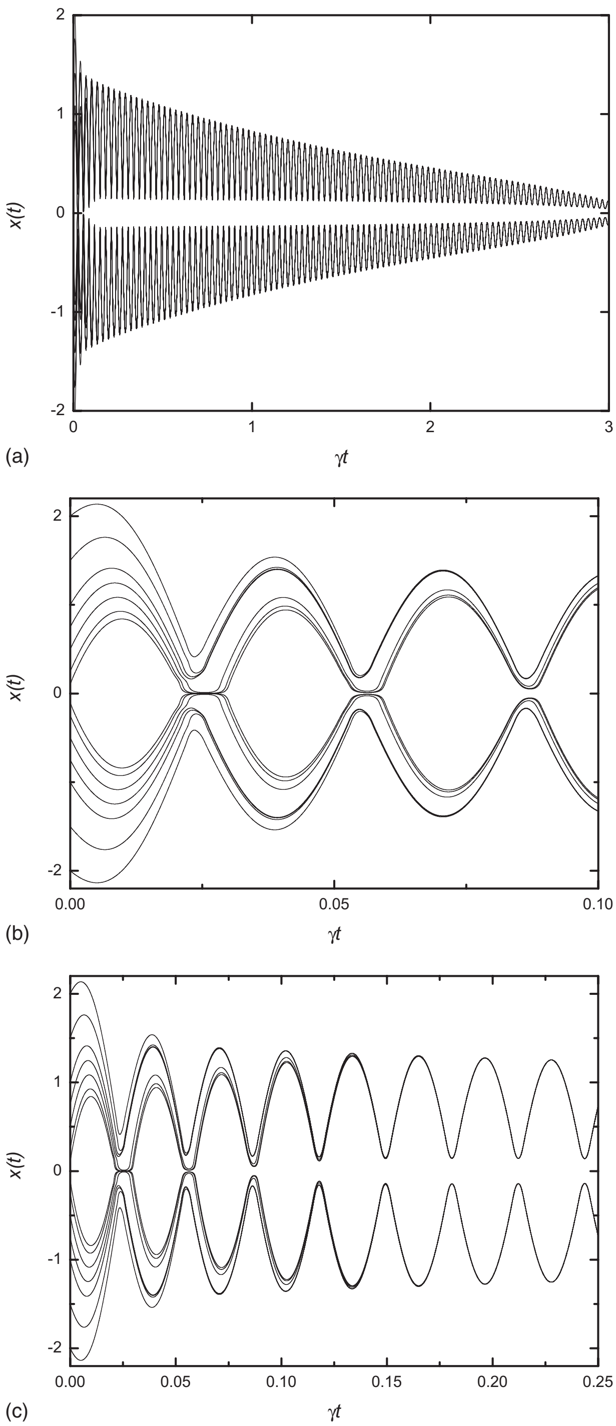


FIG. 5. In (a) we plot the quantum trajectories $x(t)$ (in units of $\sqrt{\hbar/M\omega}$) against the scaled time $\gamma t$ for the dissipative harmonic oscillator prepared in the superposition state $|\psi\rangle=\mathcal{N}(|\alpha_0\rangle+|-\alpha_0\rangle)$, with $|\alpha_0|^2=10$, and $\omega/\gamma=10^2$. In (b) we focus on a small time scale $\gamma t \ll 1$ to draw attention to the decoherence process, while in (c) we look a little further than $\gamma t \ll 1$ to visualize the quantum-to-classical transition.

ian mechanics. Putting it differently, whereas the Bohmian approach is unable to assign trajectories to quantum systems in stationary states, within our open beables an initially stationary state undergoes decaying trajectories converging to the expected quantum mechanics predictions.

### C. Squeezed states: Variances and decoherence

To address variances and decoherence, we assume that the harmonic oscillator is prepared in the displaced squeezed state $|\alpha,\zeta\rangle=D(\alpha)S(\zeta)|0\rangle$, where $D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a)$ and $S(\zeta)=\exp[(\zeta a^{\dagger 2}-\zeta^* a^2)/2]$ are the displaced and squeezed operators. The complex number $\alpha=|\alpha|e^{i\phi}$ is related to the position and momentum coordinates by $\alpha=(x_\alpha+ip_\alpha)/\sqrt{2}$, whereas $\zeta=re^{i\theta}$ defines the squeezing factor $r$ and direction $\theta$ in-phase space [19]. Computing the density operator from the Glauber-Sudarshan $P$ function, we obtain the result

$$\rho(x,x';t)=\sqrt{\frac{M\omega}{\pi\hbar f_-(t)}}\exp\Bigg\{\frac{g^2(x,x';t)}{2f_-(t)} - \frac{M\omega}{4\hbar}(x-x')^2[1+f_+(t)]-i\sqrt{\frac{2M\omega}{\hbar}}e^{-\gamma t/2}(x-x') \times\mathrm{Im}[\alpha(e^{i\omega t}\cosh r+e^{-i(\theta+\omega t)}\sinh r)]\Bigg\},$$

where

$$f_\pm(t)=1+e^{-\gamma t}[2\sinh^2 r\pm\cos(2\omega t+\theta)\sinh(2r)],$$

$$g(x,x';t)=-\sqrt{\frac{M\omega}{2\hbar}} \times[(x-x')e^{-\gamma t}\sin(2\omega t+\theta)\sinh(2r)+i(x+x')] +2ie^{-\gamma t/2}\,\mathrm{Re}[\alpha(e^{i\omega t}\cosh r-e^{-i(\theta+\omega t)}\sinh r)].$$

Using Eq. (7), we present in Figs. 3(a) and 3(b) the trajectories associated with the position and momentum squeezed states, with $\theta=0$ and $\theta=\pi$, respectively, and $r=1$. We have assumed $\hbar=M=1$, $|\alpha_0|^2=1$, and the ratio $\omega/\gamma=40$. As seen in both figures, the initial distribution of positions, giving by $\rho(x;0)$, is naturally quite distinct for $\theta=0$ and $\pi$, being, as expected, squeezed in the first case and stretched in the second, relative to the coherent state case ($r=0$). As a matter of fact, $\theta=0$ ($\pi$) prompts a state squeezed in position (momentum), with the variance $\Delta x_\alpha=(\langle x_\alpha^2\rangle-\langle x_\alpha\rangle^2)^{1/2}$ ($\Delta p_\alpha$) being less than the vacuum fluctuation.

Another crucial feature of both figures is the destruction of interference between the trajectories associated with the initial position distribution $\rho(x;0)$. It is expected and evident from Figs. 3(a) and 3(b) that the interference between the trajectories is more pronounced as they get closer to each other and mutually deforming each other, around the origin, i.e., for $\gamma t \ll 1$. As time goes on, the interference mechanism between trajectories is suppressed, leading to the loss of coherence which occurs around the decoherence time of both squeezed states, given by

$$\tau_D=\frac{1}{4\sinh^2 r},$$

as computed by a technique developed in Ref. [20]. In agreement with Figs. 3(a) and 3(b), the coherence loss occurs around $\tau_D\simeq 0.2$ for $r=1$.

Therefore, our decaying trajectories illustrates, beyond the squeezing of the variances of the quadrature operators, the decoherence mechanism as the loss of interference between trajectories. Moreover, the decaying trajectories enable us to estimate the decoherence time, in good accordance with the results derived from quantum mechanics.

### D. Schrödinger catlike state: Decoherence and the quantum-to-classical transition

We finally consider the case of a harmonic oscillator prepared in a superposition of two equally excited coherent states $|\psi\rangle = \mathcal{N}(|\alpha_0\rangle + |-\alpha_0\rangle)$, which is dragged into a statistical mixture described in the position representation by

$$\rho(x,x';t) = \mathcal{N}^2 \sum_{j,k=1}^{2} e^{-2|\alpha_0|^2(1-e^{-\gamma t})(1-\delta_{jk})} \times \langle x|(-1)^j \alpha(t)\rangle\langle(-1)^k \alpha(t)|x'\rangle. \tag{8}$$

The equation of motion [Eq. (7)] provides the trajectories $x(t)$ vs $\gamma t$ associated with the density matrix [Eq. (8)], which are drawn in Figs. 4(a) and 5(a) for $|\alpha_0|^2 = 5$ and 10, respectively. We again take $\hbar$ and $M$ to be unity, with $\omega/\gamma = 10$. In Figs. 4(b) and 5(b) we focus on a small time scale ($\gamma t \ll 1$) to show clearly that the scaled decoherence time $\gamma\tau_D$ of the catlike state $|\psi\rangle$, computed from the quantum mechanics formalism to be around $(2|\alpha_0|^2)^{-1}$ [21], is manifested in the behavior of the trajectories. In fact, it is exactly within the scaled times of 0.1 and 0.05, associated with $|\alpha_0|^2 = 5$ and 10, respectively, which we observe interference between the trajectories, causing them to deviate substantially from their paths. Moreover, around these scaled times the two ensembles of trajectories, each associated with a component of the superposition, begin to draw apart from each other. Finally, in Figs. 4(c) and 5(c), we move on to a slightly greater time scale than $\gamma\tau_D \ll 1$ to focus on the quantum-to-classical transition, as the reservoir drives the pure state superposition into a statistical mixture. In fact, in both figures, all the trajectories generated from the distribution $|\psi|^2$ of the initial positions—linked to interference and, thereby, quantum statistics—converge to their two classical counterparts.

For comparison with the decaying motion in Fig. 4(a), we plot in Fig. 4(d) the trajectories $x(t)$ vs $\omega t$, following from Eq. (6) with the same parameters as in Fig. 4(a), except $\gamma = 0$. As expected, in this case the interference between the trajectories associated with each component of the superposition remains unaffected.

## V. CONCLUDING REMARKS

We have further extended Bell's beables interpretation, within Vink's first generalization of the theory, to visualize in the quantum trajectories the phenomena of dissipation, decoherence, and quantum-to-classical transition. Instead of starting from the Schrödinger equation, as Bell did, we worked with its Itô stochastic form and, for a particular choice of the source of stochasticity—the one leading to a dissipative Lindblad correction to the Hamiltonian dynamics—we have unified Bohm's and Nelson's approaches. Evidently, other choices of the stochastic source, not addressed here, may provide other still more general equations of motion accounting, for example, for temperature effects.

Our extended Bell's beables enabled us to observe clearly the loss of quantum interference between trajectories exactly around the decoherence time computed from quantum theory. We also observe the convergence of all trajectories to their classical counterparts as the pure density operator is driven into a complete statistical mixture. Therefore, all quantum features related to the quantum-to-classical transition are incorporated into our unified Bohm-Nelson dissipative hidden-variable model. The extended open beables also provide trajectories for states which reduce, in the limit $\gamma \to 0$, to the stationary states of quantum mechanics, thus circumventing a controversial point in Bohm's formalism. We finally suggest that our analysis could provide insights into the classical counterparts of other interesting quantum phenomena, not treated here, related to open quantum systems, such as dissipative tunneling and entanglement sudden death.

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