PHYSICAL REVIEW A **80**, 024302 (2009)

Composite pulses in NMR as nonadiabatic geometric quantum gates

Yukihiro Ota[1,*] and Yasushi Kondo[1,2]
[1]Research Center for Quantum Computing, Interdisciplinary Graduate School of Science and Engineering, Kinki University, 3-4-1 Kowakae, Higashi-Osaka 577-8502, Japan
[2]Department of Physics, Kinki University, 3-4-1 Kowakae, Higashi-Osaka 577-8502, Japan
(Received 14 March 2009; published 4 August 2009)

We show that some composite pulses widely employed in NMR experiments are regarded as nonadiabatic geometric quantum gates with Aharanov-Anandan phases. Thus, we reveal the presence of a fundamental issue on quantum mechanics behind a traditional technique. To examine the robustness of such composite pulses against fluctuations, we present a simple noise model in a two-level system. Then, we find that the composite pulses possesses purely geometrical nature even under a certain type of fluctuations.

DOI: 10.1103/PhysRevA.80.024302 PACS number(s): 03.67.Lx, 03.65.Vf, 82.56.Jn

Geometric phases have been attracting a lot of attention from the view point of the foundation of quantum mechanics and mathematical physics [1–4]. Recently, their application to quantum information processing is spotlighted [5,6], because they are expected to be robust against noise. However, the robustness of a geometric quantum gate (GQG), which is a quantum gate only using geometric phases, is not completely verified. Various examinations on this issue have been reported [7–12]. Blais and Tremblay [7] claimed that no advantage of the GQGs exists compared to the corresponding quantum gates with dynamical phases, while Zhu and Zanardi [8] showed that their nonadiabatic GQGs are robust against fluctuations in control parameters.

In this Brief Report, we show that some composite pulses widely employed in nuclear magnetic resonance (NMR) [13,14] to accomplish reliable operations is regarded as nonadiabatic GQGs based on an Aharonov-Anandan (AA) phase [15], and propose a simple noise model in a two-level system. Then, we classify fluctuations in terms of the robustness of the GQGs.

An AA phase appears under nonadiabatic cyclic time evolution of a quantum system [15]. We note that the generalization to the noncyclic case is given in Ref. [3,16]. Let us write the Bloch vector at $t(0\leq t\leq 1)$ as $\boldsymbol{n}(t)(\in\mathbb{R}^3)$. We denote a state vector given $\boldsymbol{n}(t)$ as $|\boldsymbol{n}(t)\rangle(\in\mathbb{C}^2)$. Namely, $\boldsymbol{n}(t)=\langle\boldsymbol{n}(t)|\boldsymbol{\sigma}|\boldsymbol{n}(t)\rangle$, where $\boldsymbol{\sigma}={}^t(\sigma_x,\sigma_y,\sigma_z)$. The symbol ^{t} means the transposition of a vector. Time evolution is described by the Schrödinger equation with the Hamiltonian $H(t)$. Note that $|\boldsymbol{n}(t)|=1$. Hereafter, we denote $\boldsymbol{n}(0)$ as $\boldsymbol{n}$. We take the natural unit system in which $\hbar=1$. Suppose that $|\boldsymbol{n}(1)\rangle=e^{i\gamma}|\boldsymbol{n}\rangle$ $(\gamma\in\mathbb{R})$: $\boldsymbol{n}(1)=\boldsymbol{n}$. The AA phase $\gamma_{\rm g}$ is defined as [15]

$$\gamma_{\rm g}=\gamma-\gamma_{\rm d}, \tag{1}$$

where

$$\gamma_{\rm d}=-\int_0^1\langle\boldsymbol{n}(t)|H(t)|\boldsymbol{n}(t)\rangle dt \tag{2}$$

is a dynamical phase.

Next, suppose $\boldsymbol{n}_+$ and $\boldsymbol{n}_-$ are two Bloch vectors satisfying (a) $\boldsymbol{n}_+\cdot\boldsymbol{n}_-=-1$ (i.e., $\langle\boldsymbol{n}_+|\boldsymbol{n}_-\rangle=0$) and (b) $\boldsymbol{n}_\pm(1)=\boldsymbol{n}_\pm$ (i.e., there exist $\gamma_\pm\in\mathbb{R}$ such that $|\boldsymbol{n}_\pm(1)\rangle=e^{i\gamma_\pm}|\boldsymbol{n}_\pm\rangle$. An arbitrary quantum state $|\boldsymbol{n}\rangle$ is expressed by $|\boldsymbol{n}\rangle=a_+|\boldsymbol{n}_+\rangle+a_-|\boldsymbol{n}_-\rangle$, where $a_\pm=\langle\boldsymbol{n}_\pm|\boldsymbol{n}\rangle$. We call $\boldsymbol{n}_\pm$ basis Bloch vector corresponding to $H(t)$. The initial state $|\boldsymbol{n}\rangle$ is transformed into the final state $|\boldsymbol{n}(1)\rangle=a_+e^{i\gamma_+}|\boldsymbol{n}_+\rangle+a_-e^{i\gamma_-}|\boldsymbol{n}_-\rangle$. Thus, the time evolution operator U at $t=1$ generated by $H(t)$ $(t\in[0,1])$ is rewritten as

$$U=e^{i\gamma_+}|\boldsymbol{n}_+\rangle\langle\boldsymbol{n}_+|+e^{i\gamma_-}|\boldsymbol{n}_-\rangle\langle\boldsymbol{n}_-|. \tag{3}$$

Equation (3) becomes a quantum gate with a geometric phase, when the dynamical component of $\gamma_\pm$ is vanishing.

Let us focus on the Hamiltonian for a one-qubit system,

$$H(t)=\frac{1}{2}\omega(t)\boldsymbol{m}(t)\cdot\boldsymbol{\sigma}\quad(0\leq t\leq 1), \tag{4}$$

which is inspired by a NMR Hamiltonian. In the case of NMR, $\omega(t)$ and $\boldsymbol{m}(t)$ are the amplitude of and the unit vector parallel to a magnetic field, respectively. The dynamical phase vanishes when $\boldsymbol{m}(t)\cdot\boldsymbol{n}(t)=0$ [17]. We note that the integrand in Eq. (2) is rewritten as $\langle\boldsymbol{n}(t)|H(t)|\boldsymbol{n}(t)\rangle=(\omega(t)/4)\text{tr}[(\boldsymbol{m}(t)\cdot\boldsymbol{\sigma})(\boldsymbol{n}(t)\cdot\boldsymbol{\sigma})]=(\omega(t)/2)\boldsymbol{m}(t)\cdot\boldsymbol{n}(t)$, where we use $\text{tr}[H(t)]=0$ and $\text{tr}(\sigma_i\sigma_j)=2\delta_{ij}$. This condition has been widely used in the experiments on nonadiabatic GQGs [6].

A series of pulses, $90_x180_y90_x$ has been widely employed in the field of NMR for wide band decoupling [13,14], where β_k denotes a spin rotation by the angle β in degree around k axis. This is called composite pulse and corresponds to the unitary operator $e^{-i\pi\sigma_x/4}e^{-i\pi\sigma_y/2}e^{-i\pi\sigma_x/4}$, which is equal to $e^{-i\pi\sigma_y/2}$. This is generated by the Hamiltonian

$$H(t)=\pi\boldsymbol{m}(t)\cdot\boldsymbol{\sigma}\quad(0\leq t\leq 1), \tag{5}$$

where

*Present address: CCSE, Japan Atomic Energy Agency, 6-9-3 Higashi-Ueno, Tokyo 110-0015, Japan and CREST(JST), 4-1-8 Honcho, Kawaguchi, Saitama, 332-0012, Japan.

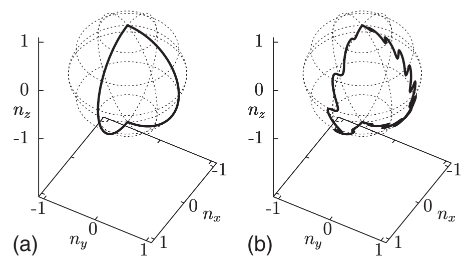

FIG. 1. Temporal behavior of the basis Bloch vector ${}^t(0,1,0)$ during the composite pulse $90_x180_y90_x$. (a) without and (b) with fluctuations in the control parameters. The fluctuations are given by Eq. (15), where $f_0=g_0=0.1$ and $\xi=\eta=5$.

$$\boldsymbol{m}(t)=\begin{cases}{}^t(1,0,0) & (0\le t\le 1/4)\\ {}^t(0,1,0) & (1/4\le t\le 3/4)\\ {}^t(1,0,0) & (3/4\le t\le 1)\end{cases}.$$

Hereafter, we will denote $t_0=0$, $t_1=1/4$, $t_2=3/4$, and $t_3=1$. Various types of composite pulses have been proposed [13,14], and their usages have been also discussed in the context of NMR quantum computing [18].

Let us examine the time evolution generated by Hamiltonian (5) from the view point of nonadiabatic GQGs. We choose $\boldsymbol{n}_\pm={}^t(0,\pm 1,0)$, where $\boldsymbol{n}_+\cdot\boldsymbol{n}_-=-1$. Then, we have the explicit formula

$$\boldsymbol{n}_\pm(t)=\pm\begin{pmatrix}\sin\theta(t)\sin\phi(t)\\ -\sin\theta(t)\cos\phi(t)\\ \cos\theta(t)\end{pmatrix},\tag{6}$$

where

$$\theta(t)=2\pi t-\frac{\pi}{2},\quad \phi(t)=\begin{cases}\pi/2 & (t_1\le t\le t_2)\\ 0 & (\text{otherwise})\end{cases}.$$

The temporal behavior of $\boldsymbol{n}_+$ on the Bloch sphere is shown in Fig. 1(a). The trajectory $\boldsymbol{n}_+$ is closed. It means that $|\boldsymbol{n}_+(1)\rangle=e^{i\gamma_+}|\boldsymbol{n}_+\rangle$. We find that $|\boldsymbol{n}_\pm(1)\rangle=e^{\mp i\pi/2}|\boldsymbol{n}_\pm\rangle$ via solving the Schrödinger equation. We note that π is a solid angle surrounded by the trajectory $\boldsymbol{n}_+(t)$. We also find that $\boldsymbol{m}(t)\cdot\boldsymbol{n}_\pm(t)=0$ at any $t\in[0,1]$, and thus the dynamical component is vanishing. Accordingly, we obtain the nonadiabatic GQG, $U=e^{-i\pi/2}|\boldsymbol{n}_+\rangle\langle\boldsymbol{n}_+|+e^{i\pi/2}|\boldsymbol{n}_-\rangle\langle\boldsymbol{n}_-|=e^{-i\pi\sigma_y/2}$. One of the most commonly employed composite pulses turns out a nonadiabatic GQG [19].

We will classify fluctuations in terms of robustness of the composite pulse $90_x180_y90_x$. A noise model will be proposed based on a fluctuated closed curve on the Bloch sphere. We examine the situation in which the radio-frequency (rf) amplitude and phase, and the resonance off-set are temporary fluctuated around their aimed values. The fluctuated curve is given by

$$\tilde{\boldsymbol{n}}_\pm(t)=\pm\begin{pmatrix}\sin[\theta(t)+f(t)]\sin[\phi(t)+g(t)]\\ -\sin[\theta(t)+f(t)]\cos[\phi(t)+g(t)]\\ \cos[\theta(t)+f(t)]\end{pmatrix},\tag{7}$$

where we assume that $f(t)$ and $g(t)$ are continuous and smooth in [0,1] [20] and satisfy

$$f(t_0)=g(t_0)=0,\quad f(t_3)=g(t_3)=0.\tag{8}$$

We will discuss the relevance of $f(t)$ and $g(t)$ to fluctuations below. The trajectory $\tilde{\boldsymbol{n}}_\pm(t)$ is closed under the assumption (8), as shown in Fig. 1(b). Thus, we have

$$|\tilde{\boldsymbol{n}}_\pm(1)\rangle=e^{i\tilde{\gamma}_\pm}|\tilde{\boldsymbol{n}}_\pm\rangle,\tag{9}$$

with a phase $\tilde{\gamma}_\pm$. Generally, $\tilde{\gamma}_\pm$ includes both the dynamical and the geometric components. We employ this noise model in order to ensure the existence of a definite AA phase, although we aware of its artificiality. An analysis based on a noncyclic geometric phase [12,16] may be needed for more comprehensive discussions.

We derive the Hamiltonian generating the time evolution corresponding to Eq. (7). By differentiating Eq. (7) with respect to $t\in(t_{i-1},t_i)(i=1,2,3)$, we obtain the Bloch equation. Then, we find the Hamiltonian in this time interval. Hence, the Hamiltonian at $t\in[0,1]$ is given by

$$\tilde{H}(t)=\frac{1}{2}\tilde{\omega}(t)\tilde{\boldsymbol{m}}(t)\cdot\boldsymbol{\sigma}+\frac{1}{2}\frac{dg(t)}{dt}\sigma_z,\tag{10}$$

where

$$\tilde{\omega}(t)=2\pi+\frac{df(t)}{dt},\quad \tilde{\boldsymbol{m}}(t)=\begin{pmatrix}\cos[\phi(t)+g(t)]\\ \sin[\phi(t)+g(t)]\\ 0\end{pmatrix}.$$

We find that

$$\tilde{\boldsymbol{m}}(t)\cdot\tilde{\boldsymbol{n}}(t)=0.\tag{11}$$

at any $t\in[0,1]$. The derivative of $f(t)$ is a fluctuation of the rf amplitude, while that of $g(t)$ is that of the resonance off-set. A fluctuation of the rf phase is described by $g(t)$. From Eq. (2), the dynamical component $\tilde{\gamma}_{\mathrm{d}\pm}$ of $\tilde{\gamma}_\pm$ is given by

$$\tilde{\gamma}_{\mathrm{d}\pm}=\mp\frac{1}{2}\int_{t_0}^{t_3}\frac{dg(t)}{dt}\cos[\theta(t)+f(t)]dt.\tag{12}$$

We show that the following two cases exactly lead to $\tilde{\gamma}_{\mathrm{d}\pm}=0$. Namely, (i) $g(t)=0$ and (ii) $f(t)$ and $g(t)$ have a certain symmetric property under time translation. The validity of the case (i) is obvious from Eq. (12). We focus on the case (ii). We note that $90_x180_y90_x$ has several interesting properties under time translation: $\theta(t+1/2)=\theta(t)+\pi$, for example. We divide the total time interval $I_{\mathrm{all}}=\{t\in[t_0,t_3]\}$ into the four intervals, $I_1=\{t\in[t_0,t_1]\}$, $I_2=\{t\in[t_1,1/2]\}$, $I_3=\{t\in[1/2,t_2]\}$, and $I_4=\{t\in[t_2,t_3]\}$. Let us consider a case when the conditions

$$f(t+1/2)=f(t),\quad \frac{dg}{dt}(t+1/2)=\frac{dg}{dt}(t),\tag{13}$$

are satisfied. The contribution from $I_1(I_2)$ to $\tilde{\gamma}_{\mathrm{d}\pm}$ is canceled out by that from $I_3(I_4)$. Thus, this case leads to $\tilde{\gamma}_{\mathrm{d}\pm}=0$. Let us consider another case, in which the conditions

$$f(1-t)=-f(t),\quad \frac{dg}{dt}(1-t)=\frac{dg}{dt}(t),\tag{14}$$

are satisfied. We note that $f(1/2)=0$ is imposed in Eq. (14). In this case, the contribution from $I_1(I_2)$ is canceled out by

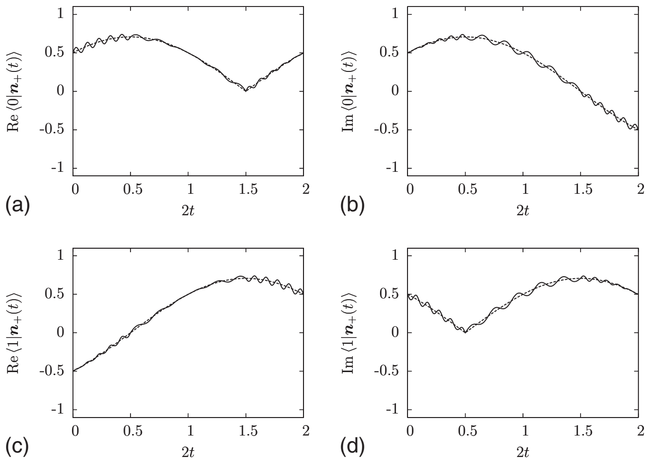

FIG. 2. Temporal behavior of the state vector corresponding to the basis Bloch vector ${}^t(0,1,0)$ during $90_x 180_y 90_x$. The initial state vectors are chosen as $|\boldsymbol{n}_+\rangle = e^{i\pi/4}(|0\rangle + i|1\rangle)/\sqrt{2}$. The solid line is the model with the fluctuations. The fluctuations are described by Eq. (15), where $f_0 = g_0 = 0.1$ and $\xi = \eta = 5$. The dashed line is the ideal case. (a) $\mathrm{Re}\langle 0|\boldsymbol{n}_+(t)\rangle$. (b) $\mathrm{Im}\langle 0|\boldsymbol{n}_+(t)\rangle$. (c) $\mathrm{Re}\langle 1|\boldsymbol{n}_+(t)\rangle$. (d) $\mathrm{Im}\langle 1|\boldsymbol{n}_+(t)\rangle$.

$I_4(I_3)$. This cancellation is related to the symmetry $\theta(1-t) = -\theta(t) + \pi$. When $f(t)$ and $g(t)$ have a certain symmetric property compatible with the pulse sequence, the dynamical phase is vanishing. In addition, a case (iii) $f(t)$ and $g(t)$ rapidly oscillate with no correlation, leads to $\tilde{\gamma}_{\mathrm{d}\pm} \approx 0$. We can confirm the validity of the case (iii) by numerically solving the Schrödinger equation with Eq. (10). The case (i) often happens in experiments. From Eq. (10), one can find $f(t)$ is associated only with the amplitude of an external controlled field. This quantity often shows an overshoot or an undershoot before settling a desired strength. One can also encounter the case (ii) in experiments. A typical example for Eq. (13) may be an oscillating function, as shown in Eq. (16). A linear combination of such oscillating functions leads to $\tilde{\gamma}_{\mathrm{d}\pm} = 0$. Thus, we expect that a lot of rapid oscillating fluctuations approximately satisfy Eqs. (13) and (14), and then $\tilde{\gamma}_{\mathrm{d}\pm} \approx 0$. The case (iii) is natural when the origins of $f(t)$ and $g(t)$ are independent. These three conditions lead to $\tilde{\gamma}_{\mathrm{d}\pm} = 0$. Thus, the quantum gate under them is still regarded as a GQG. It is necessary to examine about more realistic control processes [21,22]. Nevertheless, the present discussion is meaningful to understand nature of robustness of a geometric phase.

We directly solve the Schrödinger equation with Eq. (10) in order to calculate the geometric component of $\tilde{\gamma}_\pm$. First, we choose

$$f(t) = f_0 \sin[2\pi\xi u_i(t)], \quad g(t) = g_0 \sin[2\pi\eta u_i(t)], \quad (15)$$

at $t \in [t_{i-1}, t_i]$, where $u_i(t) = (t - t_{i-1})/(t_i - t_{i-1})$ and $\xi, \eta \in \mathbb{N}$. The above functions are piecewise smooth in $[t_0, t_3]$ [20]. We show that the temporal evolution of the basis Bloch vector ${}^t(0,1,0)$ during the composite pulse $90_x 180_y 90_x$ with the fluctuations in Fig. 1(b). This example corresponds to the case (ii), since Eq. (14) is satisfied. We display the temporal behaviors of $|\boldsymbol{n}_+(t)\rangle$ and $|\tilde{\boldsymbol{n}}_+(t)\rangle$ in Fig. 2. The state vector $|\tilde{\boldsymbol{n}}_+(t)\rangle$ is fluctuated around $|\boldsymbol{n}_+(t)\rangle$, but $|\tilde{\boldsymbol{n}}_+(t_3)\rangle = |\boldsymbol{n}_+(t_3)\rangle$. We find that $\tilde{\gamma}_\pm = \mp\pi/2$. Thus, $\tilde{\gamma}_{\mathrm{g}\pm} = \mp\pi/2$ is confirmed. Let us discuss another example,

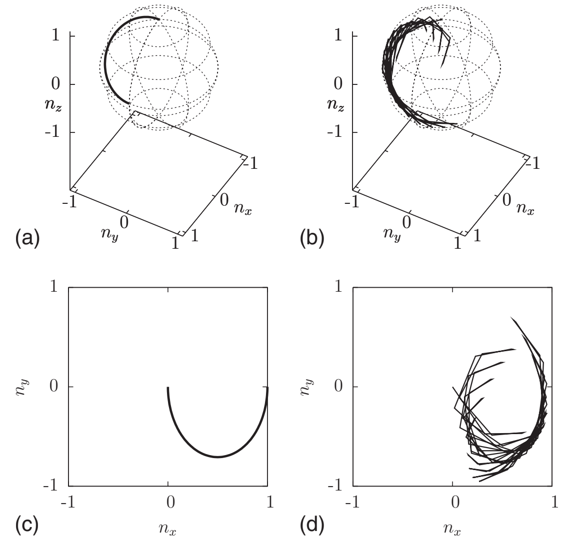

FIG. 3. Temporal behavior of the Bloch vector starting from ${}^t(0,0,1)$ under the Hamiltonian H_{B} is shown in (a) and its trajectory projected on $n_x n_y$-plane is shown in (c). The final point is ${}^t(1,0,0)$. Temporal behavior of the Bloch vector starting from ${}^t(0,0,1)$ under the fluctuating Hamiltonian $\tilde{H}_{\mathrm{B}}$ ($f_0 = g_0 = 1.0$ and $\xi = \eta = 10$ in Eq. (16)) is show in (b) and its trajectory projected on $n_x n_y$ plane is shown in (d). The final point is ${}^t(0.95, -0.26, -0.16)$.

$$f(t) = f_0 \sin(8\pi\xi t), \quad g(t) = g_0 \sin(8\pi\eta t), \quad (16)$$

where $f_0(g_0)$ is a positive real number and $\xi(\eta)$ is an integer $(t_0 \le t \le t_3)$. The above functions also satisfy Eq. (8). Solving the Schrödinger equation numerically leads to $\tilde{\gamma}_\pm = \tilde{\gamma}_{\mathrm{g}\pm} = \mp\pi/2$. The above results mean that the solid angle surrounded by $\tilde{\boldsymbol{n}}_\pm(t)$ is always π. We conjecture that, as long as the fluctuations are introduced by Eqs. (7) and (8), no dynamical phase should exactly lead to $\tilde{\gamma}_{\mathrm{g}\pm} = \gamma_{\mathrm{g}\pm}$.

It is interesting to study the case in which $\boldsymbol{m}(t) \cdot \boldsymbol{n}(t) \neq 0$. Let us consider a simple operation on the Bloch sphere: ${}^t(0,0,1) \to {}^t(1,0,0)$. This process is realized by using either $e^{-iH_{\mathrm{A}}t}$ or $e^{-iH_{\mathrm{B}}t}$ $(0 \le t \le 1)$, where $H_{\mathrm{A}} = \pi\sigma_y/4$ and $H_{\mathrm{B}} = \pi(\sigma_x + \sigma_z)/2\sqrt{2}$. The former satisfies the condition $\mathbf{m}(t) \cdot \boldsymbol{n}(t) = 0$, but the latter does not. We describe fluctuations in the two models such as Eq. (10),

$$\tilde{H}_{\mathrm{A}}(t) = \left(\frac{\pi}{2} + \frac{df}{dt}\right)\frac{\tilde{\boldsymbol{m}}_{\mathrm{A}}(t) \cdot \boldsymbol{\sigma}}{2} + \frac{dg}{dt}\frac{\sigma_z}{2},$$

$$\tilde{H}_{\mathrm{B}}(t) = \left(\frac{\pi}{\sqrt{2}} + \frac{df}{dt}\right)\frac{\tilde{\boldsymbol{m}}_{\mathrm{B}}(t) \cdot \boldsymbol{\sigma}}{2} + \left(\frac{\pi}{\sqrt{2}} + \frac{dg}{dt}\right)\frac{\sigma_z}{2},$$

where $\tilde{\boldsymbol{m}}_{\mathrm{A}}(t) = {}^t(\cos[\pi/2 + g(t)], \sin[\pi/2 + g(t)], 0)$ and $\tilde{\boldsymbol{m}}_{\mathrm{B}}(t) = {}^t(\cos g(t), \sin g(t), 0)$. Since $f(0) = f(1) = g(0) = g(1) = 0$, which corresponds to Eq. (8), the unitary operator generated by $\tilde{H}_{\mathrm{A}}(t)$ maps ${}^t(0,0,1) \to {}^t(1,0,0)$ even in the presence of $f(t)$ and $g(t)$. On the other hand, the numerical calculation reveals that the one generated by $\tilde{H}_{\mathrm{B}}(t)$ maps

${}^{t}(0,0,1)\rightarrow{}^{t}(0.95,-0.26,-0.16)$ [Fig. 3]. The results mean that Eq. (8) does not always ensure robustness in the present-model. We can find an additional term appears in Eq. (6) when $\boldsymbol{m}(t)\cdot\boldsymbol{n}(t)\neq 0$. Thus, it may cause a large error in operation. We guess that $\boldsymbol{m}(t)\cdot\boldsymbol{n}(t)=0$ plays an important role for stable time evolution in the present model.

In conclusion, we showed that the composite pulse $90_x180_y90_x$ is regarded as a nonadiabatic GQG. In addition, we proposed a simple noise model based on a fluctuated curve on the Bloch sphere, and then classified fluctuations in terms of robustness of $90_x180_y90_x$. Although the present analysis is artificial, it is suitable for evaluating errors in nonadiabatic GQGs since a definite geometric phase exists even in the presence of fluctuations. It is important to improve the present method in order to examine a more realistic control process or a stochastic process. The fluctuations that we discussed should be called regular fluctuations, because the fluctuations are expressed by the two smooth functions $f(t)$ and $g(t)$. On the other hand, when fluctuations are given by uniform random variables, even a cyclic evolution may not be guaranteed [23] and thus the robustness is not expected as discussed in Ref. [7]. We emphasize that it is important to specify fluctuations in order to evaluate robustness of a gate.

The authors wish to acknowledge helpful discussion with M. Nakahara. This work was supported by "Open Research Center" Project for Private Universities: Matching fund subsidy from Ministry of Education, Culture, Sports, Science and Technology.

[1] M. Nakahara, *Geometry, Topology and Physics*, 2nd ed. (CRC Press, Boca Raton, London, 2003).
[2] M. Vojta, Rep. Prog. Phys. **66**, 2069 (2003).
[3] D. Chruściński and A. Jamiołkowski, *Geometric Phases in Classical and Quantum Mechanics* (Birkhäuser, Boston, 2004).
[4] I. Bengtsson and K. Zyczkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press, New York, 2006).
[5] J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Nature (London) **403**, 869 (2000); S.-L. Zhu and Z. D. Wang, Phys. Rev. A **67**, 022319 (2003).
[6] M. Tian, Z. W. Barber, J. A. Fischer, and Wm. Randall Babbitt, Phys. Rev. A **69**, 050301(R) (2004); R. Das, S. K. K. Kumar, and A. Kumar, J. Magn. Reson. **177**, 318 (2005); H. Imai and A. Morinaga, Phys. Rev. A **76**, 062111 (2007).
[7] A. Blais and A.-M. S. Tremblay, Phys. Rev. A **67**, 012308 (2003).
[8] S.-L. Zhu and P. Zanardi, Phys. Rev. A **72**, 020301(R) (2005).
[9] A. Nazir, T. P. Spiller, and W. J. Munro, Phys. Rev. A **65**, 042303 (2002).
[10] A. Carollo, I. Fuentes-Guridi, M. F. Santos, and V. Vedral, Phys. Rev. Lett. **90**, 160402 (2003); **92**, 020402 (2004).
[11] G. De Chiara and G. M. Palma, Phys. Rev. Lett. **91**, 090404 (2003); G. De Chiara, A. Łoziński, and G. M. Palma, Eur. Phys. J. D **41**, 179 (2007).
[12] J. Dajka, M. Mierzejewski, and J. Łuczka, J. Phys. A: Math. Theor. **41**, 012001 (2008).
[13] M. H. Levitt, Prog. Nucl. Magn. Reson. Spectrosc. **18**, 61 (1986).
[14] T. D. W. Claridge, *High-Resolution NMR Techniques in Organic Chemistry* (Pergamon, Amsterdam, 1999).
[15] Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987); D. N. Page, Phys. Rev. A **36**, 3479 (1987).
[16] N. Mukunda and R. Simon, Ann. Phys. **228**, 205 (1993); A. Friedenauer and E. Sjöqvist, Phys. Rev. A **67**, 024303 (2003).
[17] D. Suter, K. T. Mueller, and A. Pines, Phys. Rev. Lett. **60**, 1218 (1988).
[18] H. K. Cummins and J. A. Jones, New J. Phys. **2**, 6 (2000); M. Steffen, J. M. Martinis, and I. L. Chuang, Phys. Rev. B **68**, 224518 (2003); H. K. Cummins, G. Llewellyn, and J. A. Jones, Phys. Rev. A **67**, 042308 (2003).
[19] A part of pulse sequence commonly employed in NMR, called WALTZ [13], is also regarded as a GQG.
[20] When $f(t)$ and $g(t)$ are piecewise smooth, the current discussion is also possible. We should divide the time interval into smaller pieces, in which they are smooth.
[21] M. Mehring and J. S. Waugh, Rev. Sci. Instrum. **43**, 649 (1972).
[22] C. A. Ryan, M. Laforest, and R. Laflamme, New J. Phys. **11**, 013034 (2009).
[23] It is interesting to consider a supercycle [13,14] or a noncyclic geometric phase [16]. We can numerically find that a supercycle based on $90_x180_y90_x$ approximately corresponds to a cyclic evolution.