Nonadiabatic fluctuation in the measured geometric phase

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We study how the nonadiabatic effect causes the observable fluctuation in the "geometric phase" for a two-level system, which is defined as the experimentally measurable quantity in the adiabatic limit. From the Rabi exact solution to this model, we give a reasonable explanation to the experimental discovery of phase fluctuation in the superconducting circuit system [P. J. Leek, J. M. Fink, A. Blais, R. Bianchetti, M. Göppl, J. M. Gambetta, D. I. Schuster, L. Frunzio, R. J. Schoelkopf, and A. Wallraf, Science **318**, 1889 (2007)], which seemed to be regarded as the conventional experimental error.

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I. INTRODUCTION

It was discovered by Rabi [1] that the nonadiabatic transition of a quantum system in a time-dependent magnetic field was subject to the sign of its magnetic momentum. The exact solution was first given in 1937, but its physical significance for the relative phase acquired under adiabatic evolution was not clarified until five decades later [2]. It was Berry who found that this phase might contain a geometric part, now called Berry's phase. Then the quantum adiabatic approximation theorem (QAAT) [3] was reproved to naturally include Berry's phase [4] and generalized to deal with the nonadiabatic effects for many cases [5–9]. On the other hand, because of its geometric dependence, conditional geometric phase was proposed as an intrinsically fault-tolerant way of performing quantum computation [10].

In this Brief Report, associated with a recent experiment about Berry's phase in the superconducting circuit system [11], the above Rabi solution is used to study in details the nonadiabatic effects for a two-level system (TLS) in a harmonically rotated field (see Fig. 1). This field can be realized with a microwave field perpendicular to the static magnetic field both applied to the system. With the phase of the microwave linearly varying with time, the Hamiltonian harmonically rotates in the parametric space. Our theoretical analysis offers an explanation to part of the error in the experimental result [11].

Generally speaking, Berry's phase is always accompanied with the dynamical phase, and thus its pure effect cannot be observed directly. However, we apply a π pulse to the TLS, so that the evolution is divided into two parts with both of them in the same path but in the opposite directions. In this case, the effect of the dynamical phase can be completely eliminated [10]. This is the technique referred to as spin-echo technique [12]. Thus, the pure geometric effect can be observed and that may result in an observable fluctuation of the measured geometric phase due to the Rabi nonadiabatic transitions [13].

II. NONADIABATIC EFFECT WITH BERRY'S PHASE

The evolution of the system can be well described with the Hamiltonian

$$H(t) = \tfrac{1}{2}(\Delta\sigma_z + \Omega_R\sigma_x \cos \omega_R t + \Omega_R\sigma_y \sin \omega_R t), \quad (1)$$

where Δ is the energy splitting without the microwave field, Ω_R is the Rabi frequency of the microwave, ω_R is the oscillating frequency of the microwave phase, and $\sigma_{x,y,z}$ are the Pauli matrices. Note that the Hamiltonian (1) is exactly the effective Hamiltonian realized in Ref. [11], where the rotating wave approximation [14] was applied. Straightforwardly, its instantaneous eigenstates are obtained as

$$|e(t)\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\omega_R t}|1\rangle, \quad (2)$$

$$|g(t)\rangle = \sin\frac{\theta}{2}e^{-i\omega_R t}|0\rangle - \cos\frac{\theta}{2}|1\rangle, \quad (3)$$

with corresponding eigenenergies $\pm\omega/2$. Here, the energy splitting is $\omega=\sqrt{\Delta^2+\Omega_R^2}$ and the mixing angle is $\theta = \tan^{-1}(\Omega_R/\Delta)$. We also emphasize that, due to the requirement of the single-value-ness of the eigenfunctions for a given Hamiltonian without singularity, the phase factor $\exp(\pm i\omega_R t)$ in $|e(t)\rangle$ or $|g(t)\rangle$ is fixed once the factor in the other state is chosen [15].

At time t, the evolution state is assumed to be a superposition $|\psi(t)\rangle=\alpha(t)|e(t)\rangle+\beta(t)|g(t)\rangle$ of two instantaneous eigenstates. The time-dependent Schrödinger equation

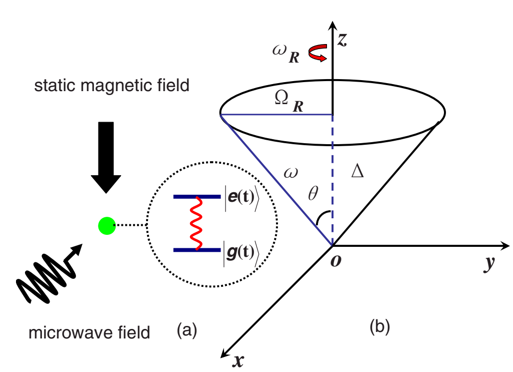

FIG. 1. (Color online) (a) Schematic of a TLS in a static magnetic field along with a microwave field. (b) Realized effective Hamiltonian rotating in the parametric space.

$H|\Psi(t)\rangle = i\partial_t|\Psi(t)\rangle$ leads to the following equations of coefficients:

$$\dot{\alpha} = -i\left(\frac{\omega}{2} + \omega_R \sin^2\frac{\theta}{2}\right)\alpha + i\beta' \frac{\omega_R}{2}\sin\theta, \tag{4}$$

$$\dot{\beta}' = i\left(\frac{\omega}{2} - \omega_R \cos^2\frac{\theta}{2}\right)\beta' + i\alpha \frac{\omega_R}{2}\sin\theta, \tag{5}$$

where $\beta'(t) = \beta(t)\exp(-i\omega_R t)$.

Under the adiabatic conditions

$$\left|\frac{\omega}{2} + \omega_R \sin^2\left(\frac{\theta}{2}\right)\right| \gg \left|\frac{\omega_R}{2}\sin\theta\right|, \tag{6}$$

$$\left|\frac{\omega}{2} - \omega_R \cos^2\left(\frac{\theta}{2}\right)\right| \gg \left|\frac{\omega_R}{2}\sin\theta\right|, \tag{7}$$

the adiabatic approximate solutions to Eqs. (4) and (5) is obtained by ignoring the terms with $\omega_R \sin\theta/2$. They show that both norms of the amplitudes remain the same as their initial values, while they acquire Berry's geometric phases $\pm\omega_R t(1-\cos\theta)/2$ in addition to the dynamical phases $\pm\omega t/2$, respectively.

On the other hand, the above equations (4) and (5) can be solved exactly and it should be done so when the adiabatic condition is broken under certain circumstances. Thus, we have

$$\alpha = A_1 e^{i\omega_+ t} + A_2 e^{i\omega_- t}, \tag{8}$$

$$\beta' = B_1 e^{i\omega_+ t} + B_2 e^{i\omega_- t}, \tag{9}$$

where $\omega_\pm = (-\omega_R \pm \sqrt{\omega^2 - 2\omega\omega_R \cos\theta + \omega_R^2})/2$, and the coefficients are determined by the initial values $\alpha(0)$ and $\beta(0)$ as follows:

$$A_1 = \frac{\beta(0)\omega_R \sin\theta + \alpha(0)(-\omega + \Sigma_+)}{2\Omega},$$

$$A_2 = \frac{-\beta(0)\omega_R \sin\theta + \alpha(0)(\omega + \Sigma_-)}{2\Omega},$$

$$B_1 = \frac{\beta(0)(\omega + \Sigma_-) + \alpha(0)\omega_R \sin\theta}{2\Omega},$$

$$B_2 = \frac{\beta(0)(-\omega + \Sigma_+) - \alpha(0)\omega_R \sin\theta}{2\Omega},$$

with $\Sigma_\pm = \omega_R(1 \pm \cos\theta) + 2\omega_+$.

Generally speaking, Berry's phase cannot be observed directly from the experiment since the dynamical phase always occurs along with Berry's phase. Here, we consider a concrete case where the total evolution is divided into two rounds, both of which are in the same path but with opposite directions. Provided that both the excited and the ground states acquire the same dynamical phase but with opposite signs in each round, the dynamical phase can be canceled when the amplitudes are exchanged after the first round of evolution. Thereafter, by experiencing the inverse rotation in the parametric space, the dynamical phase is canceled while Berry's phase is doubled since the latter depends on the sign of the angle velocity ω_R while the former does not. An excellent agreement may be expected with Berry's prediction

$$\phi_B = 2\omega_R T(1-\cos\theta) \tag{10}$$

provided that the adiabatic condition, i.e., Eq. (6), is satisfied. Here, T is the evolution time for each round.

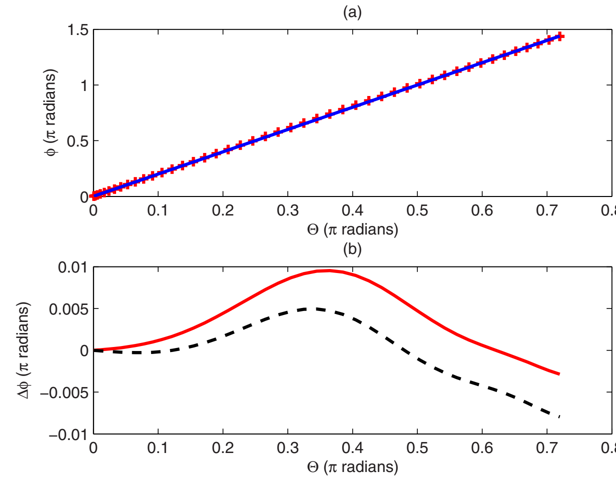

FIG. 2. (Color online) (a) Comparison between the Berry's phase (blue solid line) ϕ_B and nonadiabatic phase (red cross) ϕ_{na} with $n=1$ circular rotation in each round. (b) The discrepancy between them with red solid line for the numerical result and black dashed line for the second-order approximation.

However, when the nonadiabatic effect is considered, a small deviation is expected. In Fig. 2, we plot the exact phase calculated from Eqs. (8) and (9), which is defined as the phase of $\alpha(2T)\beta^*(2T)$, denoted as

$$\phi_{na} = \angle[\alpha(2T)\beta^*(2T)]. \tag{11}$$

This is an observable quantity in experiment, which can be determined by measuring the complex amplitudes $\alpha(2T)$ and $\beta(2T)$. Although it is not an Aharanov-Anandan phase [16], it just recovers Berry's phase in the adiabatic limit. In a sense, it can be considered as an additional phase associated with noncyclic time evolution [17].

It is predicted that Berry's phase is proportional to the solid angle $\Theta = 2\pi(1-\cos\theta)$ subtended by the path. Recently, a measurement of Berry's phase in the superconducting qubit was carried out [11]. In order to compare the theoretical analysis with the experimental result, we adopt the same parameters as those given in Ref. [11], i.e., $\Delta/2\pi = 50$ MHz, $\omega_R/2\pi = (4n+1)$ MHz with n being the number of loops. The tiny difference between those two can almost not be distinguished in Fig. 2(a). Moreover, as shown in Fig. 2(b), there are small oscillations in the deviation between them, $\Delta\phi = \phi_{na} - \phi_B$, with the root-mean-square deviation of 0.015 rad from the expected lines while the counterpart for $n=1.5$ is 0.043 rad. They are in reasonable agreement with the experimental result, i.e., 0.14 rad [11], considering that the rotating wave approximation [14] was applied to obtain

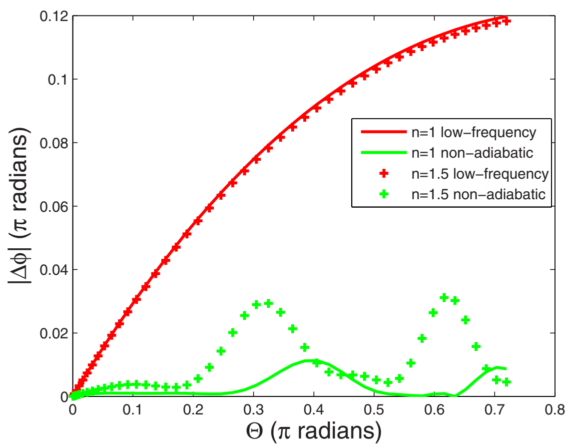

FIG. 3. (Color online) Comparison between deviations induced by a low-frequency noise (σ_ϕ, red lines) and the nonadiabatic effect ($|\Delta\phi|$, green lines), with solid lines for $n=1$ and cross lines for $n=1.5$.

the Hamiltonian (1) and the nonadiabatic effect in the process of applying the microwave field also accounted for part of the deviation.

In Ref. [11], the dephasing effect was analyzed with a very special model where the noise was treated as a fluctuation of a classical field [18]. Now, the deviation due to the low-frequency noise is calculated in contrast with the one of the nonadiabatic effect. By abstracting the data from Fig. 4(B) of Ref. [11], we obtain that $\sigma_\omega/2\pi=19$ MHz, with σ_ω^2 being the variance of the fluctuation in ω. According to Ref. [18], the variance of the measured geometrical phase is Gaussian distributed around Berry's prediction with the variance

$$\sigma_\phi^2 = 2\frac{\sigma_\omega^2}{\omega^2}\left(\frac{\pi\sin^2\theta}{2T}\right)^2\frac{2\Gamma T-1+e^{-2\Gamma T}}{\Gamma^2}. \tag{12}$$

Here, the noise is modeled as a Gaussian stationary Markovian Ornstein-Uhlenbeck process with a Lorentzian bandwidth $\Gamma=1/T_2^{\text{echo}}=0.5$ MHz [11]. The deviations due to these two effects are plotted in Fig. 3. Besides the low-frequency deviation larger than the nonadiabatic one, two distinctions are observed obviously. For the former case, the deviations for different rotation loops almost stay the same as each other. That is because $\sigma_\phi \simeq \pi\sigma_\omega \sin^2\theta/\omega$ is independent of T and thus n for small T. The little discrepancy between different n's is attributed to the third-order term $O(\Gamma T)^3$. On the contrary, the fluctuation due to the nonadiabatic effect rises dramatically as n is enlarged. Moreover, the former increases monotonously with the solid angle as $\sigma_\phi \propto -(2-\Theta/2\pi)^2+4$ whereas there are sinusoidal oscillations in the latter. A detailed analysis will be given in the next section.

III. SECOND-ORDER FLUCTUATION

It was Yang who first pointed out that Berry's phase could be recovered from the original QAAT by retaining the first-order term $O(\omega_R/\omega)$ in the phase. And this point of view was shortly confirmed by one of the authors [9] using the Rabi exact solution. Here, with a careful calculation to the second-order term $O(\omega_R/\omega)^2$, the fluctuation in the phase is obtained. For an initial state with $\alpha(0)=\beta(0)=1/\sqrt{2}$, the measured deviation from the expected Berry's phase is given as

$$\begin{aligned}\Delta\phi = \lambda^2(&\sin\phi_1+4\sin\phi_2+2\sin\phi_3+2\sin\phi_4+2\sin\phi_5\\&+\sin\phi_6+2\sin\phi_7+\sin\phi_8+2\sin\phi_9+2\sin\phi_{10}\\&+4\sin\phi_{11}+4\sin\phi_{12}),\end{aligned} \tag{13}$$

where $\lambda=\omega_R\sin\theta/2\omega$, $\phi_j=\phi_j'-\phi_B$ $(j=1,2,\ldots)$ with

$$\phi_1' = \pi - 2T\omega_R\cos\theta,$$

$$\phi_2' = -T\omega_R,$$

$$\phi_3' = T\omega_R(-1+2\cos\theta),$$

$$\phi_4' = -\frac{T(\omega_R^2+8\omega_R\omega+4\omega^2-4\omega_R\omega\cos\theta-\omega_R^2\cos 2\theta)}{4\omega},$$

$$\phi_5' = \pi + \frac{T[-(\omega_R+2\omega)^2+4\omega_R\omega\cos\theta+\omega_R^2\cos 2\theta]}{4\omega},$$

$$\phi_6' = \pi + \frac{T[(\omega_R-2\omega)^2-\omega_R^2\cos 2\theta]}{2\omega},$$

$$\phi_7' = \frac{T(\omega_R^2-2\omega_R\omega+4\omega^2-\omega_R^2\cos 2\theta)}{2\omega},$$

$$\phi_8' = \pi + \frac{T(\omega_R^2+4\omega^2-\omega_R^2\cos 2\theta)}{2\omega},$$

$$\phi_9' = \pi + \frac{T[(\omega_R-2\omega)^2-4\omega_R\omega\cos\theta-\omega_R^2\cos 2\theta]}{4\omega},$$

$$\phi_{10}' = \frac{T(\omega_R^2+4\omega^2-4\omega_R\omega\cos\theta-\omega_R^2\cos 2\theta)}{4\omega},$$

$$\phi_{11}' = \frac{T(\omega_R^2-8\omega_R\omega+4\omega^2+4\omega_R\omega\cos\theta-\omega_R^2\cos 2\theta)}{4\omega},$$

$$\phi_{12}' = \pi + \frac{T[(\omega_R-2\omega)^2+4\omega_R\omega\cos\theta-\omega_R^2\cos 2\theta]}{4\omega}.$$

It can be seen from Eq. (13) that the sinusoidal deviation from the expectation will be present in the measured phase. As shown in Fig. 2(b), the behavior is well described by the second-order approximation. Although there is a small difference between it and the exact result, it is believed that it is due to the higher-order terms. Based on the above theoretical analysis, we may safely arrive at the conclusion that, due to the nonadiabatic evolution of the TLS, not only do the relative phases of the amplitudes of the two states change with time, but also do their norms. In other words, the nonadia-

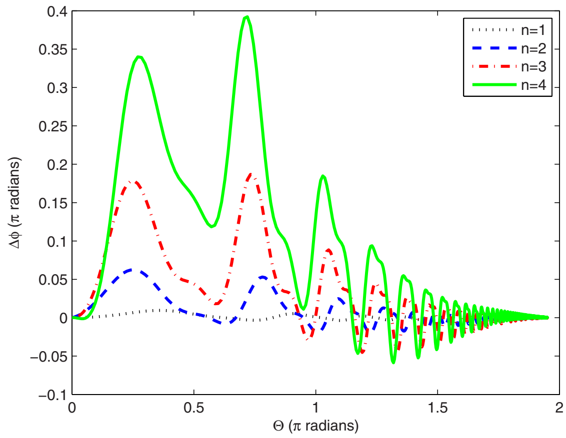

FIG. 4. (Color online) Comparison between different numbers of rotation loops n, with black dotted line for $n=1$, blue dashed line for $n=2$, red dashed-dotted line for $n=3$, and green solid line for $n=4$.

batic transition results in the small fluctuation of the measured phase.

To further explore the nonadiabatic effect, we investigate the phase fluctuation for different rotation velocities ω_R. In Fig. 4, the discrepancy between the measured phase (11) and the Berry's phase is plotted. Here, we extend the parameter n from the experimental case $n=1,1.5$ to $n=2,3,4$ since the nonadiabatic effect would be more remarkable as the rotation velocity is increased. In the future experiment, the nonadiabatic effect at large n can be demonstrated to test our theoretical prediction. We stress that the experimental parameters $n=1,1.5$ fulfill the adiabatic condition, Eqs. (6) and (7), while the ones for $n=3,4$ do not. As expected from Eq. (13), the sinusoidal oscillation is again witnessed. Additionally, the oscillating amplitudes rise as the rotating velocity is increased. Here, the total time for evolution is fixed [19]. This result is consistent with Eq. (13) as $\Delta\phi$ scales as ω_R^2. Notice that all of them nonexceptionally approach zero at the both ends. It is a reasonable result since $\Delta\phi$ vanishes as $\theta=0$. Actually, the Hamiltonian remains the same as its initial state in the parametric space. The only effect for time evolution is to acquire a dynamical phase. On the other hand, as θ approaches $\pi/2$,

$$\omega = \sqrt{\Omega_R^2 + \Delta^2} > \Omega_R \gg \Delta > \omega_R$$

since Δ is fixed. Due to vanishing of λ, we have a zero deviation from the predicted phase, i.e., $\Delta\phi=0$. In the limit $\theta\to\pi/2$, the initial state $|\psi(0)\rangle=(|0\rangle+|1\rangle)/\sqrt{2}$ is the eigenstate of the initial Hamiltonian $H(0)\simeq\Omega_R\sigma_x/2$. Intuitionally, the evolution of the system is similar to the situation that a classical magnetic moment initially parallel to the applied field closely follows the rotation of the field.

IV. CONCLUSION

In this Brief Report, we have investigated the fluctuation in the phase due to the nonadiabatic evolution for a general spin precession in an external field. In contrast to the adiabatic evolution, the sinusoidal deviation from the expected line drawn for Berry's phase is observed. Compared with the deviation induced by low-frequency noise, our theoretical analysis can explain part of the phase fluctuations discovered in the recent experiment. For a given time of evolution, it is predicted that the fluctuation from Berry's phase rises larger and larger as n, the number of the cycles of the field change, is increased whereas the deviation due to low-frequency noise is almost not dependent of n.

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[19] In the experiment, the total time consisted of the evolution of two rotations and the processes of applying and removing the microwave field. For the sake of excluding the phase decoherence, the total time for evolution $T'=500$ ns is fixed. In contrast, the time for each rotation $T=nT'/(2n+1/2)$ varies with n.