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Photon distribution in the dynamical Casimir effect with an account of dissipation

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A theory of quantum damped oscillator with arbitrary time dependence of the frequency and damping coefficient, based on the Heisenberg-Langevin equations with delta-correlated stochastic force operators, is applied to the case of the dynamical Casimir effect in a cavity with a periodically photo-excited semiconductor boundary. Accompanying results for the mean number of created photons, its variance, and the photon distribution function are given. In the asymptotical regime, the field mode goes to the so-called superchaotic quantum state.

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I. INTRODUCTION

Almost 40 years ago, Moore [1] showed that the motion of neutral boundaries could result in a creation of quanta of the electromagnetic field from the initial vacuum state. Since then, this phenomenon, which is frequently called nowadays as the *dynamical Casimir effect* (DCE), was a subject of numerous theoretical studies. However, the effect is extremely small, if the velocities of boundaries are much less than the velocity of light. A possibility of its enhancement inside cavities with *oscillating* boundaries under the conditions of parametric resonance was pointed out in [2], and calculations performed in the frameworks of different approaches [3–6] confirmed this idea (extensive lists of publications related to the DCE can be found in [7–9]). Namely, it was shown that a significant amount of photons could be created from vacuum, if boundaries of a high-Q cavity perform small oscillations at a frequency which is multiple of some cavity eigenfrequency. In particular, if a plane boundary of a *three-dimensional* cavity performs *harmonical* oscillations with an amplitude a at the frequency $\omega_w = 2\omega_0$, where ω_0 is the eigenfrequency of the lowest electromagnetic mode in the cavity with fixed dimensions, then the mean number of photons created from vacuum in this mode is given by the formula [3]

$$\langle n\rangle(t) = \sinh^2(\varepsilon\omega_0 t\eta^3), \tag{1}$$

where $\varepsilon = a/\lambda$ is the maximal relative displacement with respect to the wavelength $\lambda = 2\pi c/\omega_0$ and $\eta = \lambda/(2L_0) < 1$ is a numerical coefficient, which depends on the cavity geometry (L_0 is the average distance between vibrating walls). Under realistic conditions, the value of ε cannot exceed [3] $\varepsilon_{\max} \sim 10^{-8}$ for the lowest cavity mode with the frequency $\omega_0 \sim c\pi/L_0$, which makes an experimental observation of DCE with *really moving* boundaries an extremely difficult task.

On the other hand, what one really needs to create photons from a vacuum is the possibility to change the resonance frequency in a periodical way. This can be achieved not only by changing the geometry, but by changing the electric properties of the walls or some medium inside the cavity. Hence the idea of simulating "nonadiabatic Casimir effect" and other quantum phenomena using a medium with a rapidly decreasing in time refractive index ("plasma window") was formulated by Yablonovitch [10], who also pointed out that fast changes of electric properties can be achieved in semiconductors illuminated by laser pulses. Similar ideas were discussed in [11]. Only recently was the possibility of creating an effective "plasma mirror" in a semiconductor slab confirmed experimentally [12]. This confirmation supported a proposal [called the motion-induced radiation (MIR) experiment] [13,14] to simulate a motion of a boundary, using an electron-hole "plasma mirror," which can be created periodically on the surface of a semiconductor slab (attached to some part of the superconducting wall of a high-Q cavity) by illuminating it with a sequence of short laser pulses. The amplitude of effective displacements is determined in such a case by the thickness of the semiconductor slab, which can be made up to a few millimeters, thus giving an effective parameter $\varepsilon \sim 10^{-2}$, instead of 10^{-8} for mechanically driven mirrors.

The aim of this paper is to calculate *statistical properties* of the quantum state of a selected mode of electromagnetic field, created inside a cavity with a time-dependent semiconductor mirror after many periodical pulses. Namely, the main goal is to calculate the photon distribution function, the mean number of photons and its variance in the selected mode. The paper is organized as follows. In Sec. II, a model of a quantum damped oscillator with arbitrary time-dependent parameters, based on the Heisenberg-Langevin approach, is considered. General formulas for the mean number and variance of created quanta are derived in Sec. III. They show that the field mode goes asymptotically to the so-called "superchaotic" quantum state. The photon distribution function is considered in Sec. IV. A simple universal formula, describing highly excited asymptotical states, is found. The special case of *periodical* changes of parameters is analyzed in Sec. V, where analytical formulas for the mean number of quanta are obtained as functions of the number of pulses and other parameters. The last Sec. VI contains a brief discussion of results.

II. QUANTUM DAMPED OSCILLATOR WITH TIME-DEPENDENT PARAMETERS IN THE HEISENBERG-LANGEVIN APPROACH

The model used in this paper is based on the idea of an *effective Hamiltonian*, proposed in [15] and developed by

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other authors, especially in the papers [16,17] (for other references see [7]). In brief, the scheme is as follows. Suppose that the set of Maxwell's equations in a medium with *time-independent* parameters and boundaries can be reduced to an equation of the form

$$\hat{\mathcal{K}}(\{L\})\mathbf{F}_\alpha(\mathbf{r};\{L\}) = \omega_\alpha^2(\{L\})\mathbf{F}_\alpha(\mathbf{r};\{L\}), \tag{2}$$

where $\{L\}$ means a set of parameters, including, e.g., the distance L between the walls, $\omega_\alpha(\{L\})$ is the eigenfrequency of the field mode, labeled by the number (or a set of numbers) α, and $\mathbf{F}_\alpha(\mathbf{r};\{L\})$ is some (in general, vector) function, whose knowledge enables one to calculate all components of the electromagnetic field (e.g., the vector potential, dual potential, Hertz vector, etc.). In the simplest cases, Eq. (2) is reduced to the Helmholtz equation, and operator $\hat{\mathcal{K}}(\{L\})$ is reduced to the Laplace operator. Usually, the operator $\hat{\mathcal{K}}(\{L\})$ is self-adjoint, and the set of functions $\{\mathbf{F}_\alpha(\mathbf{r};\{L\})\}$ is orthonormalized and complete in some sense.

Now suppose that parameters $\{L\}$ become time dependent (for example, a part of the boundary is a plane surface moving according to a prescribed law of motion $L(t)$ or the dielectric function depends on time in some region inside the cavity). If one can satisfy automatically the boundary conditions, expanding the field $\mathbf{F}(\mathbf{r},t)$ over "instantaneous" eigenfunctions,

$$\mathbf{F}(\mathbf{r},t) = \sum_\alpha q_\alpha(t)\mathbf{F}_\alpha(\mathbf{r};\{L(t)\}) \tag{3}$$

(this is true, e.g., for the fixed boundaries and time-dependent properties of a medium inside the cavity—the case considered in this paper), then the field dynamics is described completely by the generalized coordinates $q_\alpha(t)$, whose equations of motion can be derived from the *effective time-dependent Hamiltonian* [16]

$$H = \frac{1}{2}\sum_\alpha \{p_\alpha^2 + \omega_\alpha^2[L(t)]q_\alpha^2\} + \frac{\dot{L}(t)}{L(t)}\sum_{\alpha\neq\beta} p_\alpha m_{\alpha\beta} q_\beta, \tag{4}$$

$$m_{\alpha\beta} = -m_{\beta\alpha} = L\int dV \frac{\partial \mathbf{F}_\alpha(\mathbf{r};L)}{\partial L}\mathbf{F}_\beta(\mathbf{r};L). \tag{5}$$

In a generic case of arbitrary time-dependent frequencies ω_α and coefficients $m_{\alpha\beta}$, finding solutions of the Schrödinger or Heisenberg equations corresponding to Hamiltonian Eq. (4), which contains the generalized coordinate and momentum operators of an infinite number of coupled modes, is an extremely difficult problem. However, it can be simplified in some special cases of practical importance. Namely, having in mind that the eigenfrequency spectra of *realistic* three-dimensional cavities are *nonequidistant*, one may suppose that for *periodical* perturbations satisfying some *resonance conditions*, different modes practically do not interact *in the long-time limit*, and one can consider only some *single mode* (marked by the label "$\alpha 0$"), which is in resonance with the perturbation. Then the problem is reduced to a simple model of a *one-dimensional quantum oscillator with a time-dependent frequency* $\omega(t)$, which is determined by the *instantaneous* geometry of the cavity and the dielectric function inside it [3]. If this problem is solved, then the field distribution inside the cavity is given (in the long-time limit, when the contribution of nonresonant modes can be neglected) by Eq. (3), where one can consider the resonance mode function $\mathbf{F}_{\alpha 0}(\mathbf{r};\{L(t)\})$ only (or a few coupled modes in some special cases [17,18]).

It is worth emphasizing an important point, that the "uncoupling" of different modes was demonstrated in [3] for the *harmonical* motion of the boundary. It is *assumed* in this paper that the interaction between different modes can be neglected for more or less arbitrary periodical motions of boundary. Although such an assumption seems reasonable from the point of view of physical intuition, a strict proof is absent, and this problem needs and deserves a further investigation (actually, it is supposed that all higher harmonics of different modes are out of resonance). After these remarks, let us assume that the one-dimensional quantum harmonic oscillator with some time-dependent effective frequency can serve as a reasonable model for the description of the process of photon creation from vacuum in a three-dimensional nondegenerate cavity with time-dependent parameters.

As was told in the introduction, one of the most simple and practical ways to change the cavity eigenfrequency is to put some thin dielectric slab inside it. Quantum effects caused by a time dependence of properties of thin slabs inside resonance cavities were studied by several authors [19–21]. However, only very simple models of the media were considered: lossless homogeneous dielectrics with time-dependent permeability [21], ideal dielectrics or ideal conductors, suddenly removed from the cavity [19], or infinitely thin conducting slabs, modeled by δ potentials with time-dependent strength (a "plasma sheet") [20].

Unfortunately, these models, as well as estimations of the photon generations rate based on the simple formula (1), cannot be applied to the MIR experiment, because they are based on the assumption that the time-dependent dielectric function of the slab (or an equivalent delta potential in the plasma sheet model) is *real*. To see the realm of validity of this assumption, remember that a simple Drude model gives the following dependence of the dielectric function on the circular frequency in the semiconductor material (in the Gauss system of units)

$$\epsilon(\omega) = \epsilon_a + \frac{4\pi i\sigma_0}{\omega(1-i\omega\tau)}, \quad \sigma_0 = ne^2\tau/m, \tag{6}$$

where a real constant ϵ_a describes the contribution of bounded electrons and ions, n is the concentration of free carriers (created by laser pulses in the case involved) with charge e, m is their effective mass, and τ the relaxation time (time between collisions). The imaginary part of Eq. (6) can be neglected under the condition $\omega\tau \gg 1$, which means that the low-frequency mobility $b=|e|\tau/m$ (related to the low-frequency conductivity σ_0 as $\sigma_0=n|e|b$) must be much bigger than $b_*(\omega)=|e|/(m\omega)$. For the optical frequencies, $\omega\sim 3\times 10^{15}$ s^{-1}, and for $m\sim m_e$ (the mass of free electron) one has $b_*(\omega)\sim 5\times 10^{-5}$ m^2 V^{-1} s^{-1}, so that the condition $b\gg b_*$ can be easily fulfilled. However, to perform an experiment on the DCE at optical frequencies is an extremely difficult task, and the available schemes are designed for the microwave frequencies. For example, the resonance frequency of the cavity used in the MIR experiment is about 2.3 GHz, which is equivalent to $\omega \approx 1.4\times 10^{10}$ s^{-1}. For this frequency one obtains $b_*(\omega)\sim 10$ m^2 V^{-1} s^{-1}, whereas the reported values of the mobility in the highly doped GaAs samples used in this experiment do not exceed 0.5 m^2 V^{-1} s^{-1} [14], and hardly the mobility can be increased by two orders of magnitude (maintaining the necessary very small recombination time) to satisfy the condition $b \gg b_*$. Consequently, the dielectric function which should be used in the analysis of realistic DCE experiments with semiconductor time-dependent mirrors has the form $\epsilon(\omega)=\epsilon_a + 4\pi i\sigma_0/\omega = \epsilon_1 + i\epsilon_2$, where the imaginary part ϵ_2 is responsible for losses inside the semiconductor slab. Although these losses can be neglected if $\epsilon_2 \ll 1$ (an almost ideal dielectric) or $\epsilon_2 \gg 1$ (an almost ideal conductor), they become very important in the intermediate regime, when the high concentration of carriers in the excited semiconductor returns continuously to the initial (almost zero) value during the recombination process [22]. These observations show that without taking into account inevitable losses inside the semiconductor slab during the excitation-recombination process one cannot predict the results of the realistic DCE experiments even qualitatively.

This paper is based on the assumption, that even in the presence of dissipation and nonmonochromatic periodical variations, the field problem still can be reduced approximately to the dynamics of a *single selected mode*, described in the classical limit as a harmonic oscillator with time-dependent *complex* frequency $\omega_c(t)=\omega(t)-i\gamma(t)$, which can be found from the solution of the classical electrodynamical problem by taking the instantaneous geometry and material properties (as was done in the nondissipative case in [20,21]). Our calculations are based on the *quantum noise operator* approach, first proposed in [23,24] for systems with time-independent parameters and generalized to the case of arbitrary time dependence of the frequency and damping coefficient in [8,22]. The field noise operators have been widely used in studying different problems of the cavity QED: see, e.g., Refs. [25–32]. Following this approach, we assume that effects of dissipation can be described by means of the Heisenberg-Langevin operator equations. For the model considered in this paper, these equations can be written as

$$d\hat{x}/dt = \hat{p} - \gamma_x(t)\hat{x} + \hat{F}_x(t), \tag{7}$$

$$d\hat{p}/dt = - \gamma_p(t)\hat{p} - \omega^2(t)\hat{x} + \hat{F}_p(t). \tag{8}$$

Here $\hat{x}$ and $\hat{p}$ are the dimensionless quadrature operators of the selected mode, normalized in such a way that the mean number of photons equals ($\hbar=1$)

$$\mathcal{N} = \frac{1}{2}\langle \hat{p}^2 + \hat{x}^2 - 1\rangle. \tag{9}$$

In other words, $\omega(t)$ and $\gamma_j(t)$ are the frequency and damping coefficients, normalized by the initial frequency ω_i. It is worth emphasizing that $\hat{x}$ here is a dimensionless canonical "coordinate" operator, which has nothing in common with the real coordinates inside the cavity: $\hat{x}(t)$ corresponds to the coefficient $q_{\alpha 0}(t)$ at the resonant field mode in the field operator decomposition Eq. (3) over instantaneous modes of the cavity. Two noncommuting noise operators $\hat{F}_x(t)$ and $\hat{F}_p(t)$ (with zero mean values) are necessary to preserve the canonical commutator $[\hat{x}(t),\hat{p}(t)]=i$ between the Heisenberg operators describing the quantum dynamics of the selected resonant mode [24,30,33] (it is supposed that $\hat{F}_x(t)$ and $\hat{F}_p(t)$ commute with $\hat{x}$ and $\hat{p}$). Physically, the meaning of operators $\hat{F}_x(t)$ and $\hat{F}_p(t)$ is the replacement of complicated dissipative processes inside a thin dielectric (semiconductor) slab attached to one of the cavity walls. In the phenomenological model used in this paper, the net result of all those processes is encoded in the correlators of the noise operators [see Eqs. (25) and (27) below]. Again, these operators act in the corresponding Hilbert spaces, which have no relations to the coordinates of any point inside the cavity.

At first glance, the presence of two extra terms, $-\gamma_x(t)\hat{x} + \hat{F}_x(t)$, in Eq. (7) could seem unusual (from the point of view of the classical theory of Brownian motion). However, these terms arise quite naturally in quantum optics, for example, if one rewrites the standard Heisenberg-Langevin equation of motion for the annihilation operator [24], $d\hat{a}/dt = (-i\omega - \gamma)\hat{a} + \hat{F}_a$, in terms of quadrature components.

The system of linear Eqs. (7) and (8) can be solved explicitly for arbitrary time-dependent functions $\gamma_{x,p}(t)$, $\omega(t)$, and $\hat{F}_{x,p}(t)$. It is convenient to represent the solutions as sums of two commuting operators,

$$\hat{x}(t) = \hat{x}_s(t) + \hat{X}(t), \quad \hat{p}(t) = \hat{p}_s(t) + \hat{P}(t), \tag{10}$$

where the first terms represent solutions of homogeneous parts of Eqs. (7) and (8):

$$\hat{x}_s(t) = e^{-\Gamma(t)}\{\hat{x}_0 \,\mathrm{Re}[\xi(t)] - \hat{p}_0 \,\mathrm{Im}[\xi(t)]\}, \tag{11}$$

$$\hat{p}_s(t) = e^{-\Gamma(t)}\{\hat{x}_0 \,\mathrm{Re}[\eta(t)] - \hat{p}_0 \,\mathrm{Im}[\eta(t)]\}. \tag{12}$$

Here $\hat{x}_0$ and $\hat{p}_0$ are the initial values of operators at $t=0$ (taken as the initial instant), and

$$\Gamma(t) = \int_0^t \gamma(\tau)d\tau, \quad \gamma(t) = \frac{1}{2}[\gamma_x(t) + \gamma_p(t)]. \tag{13}$$

Function $\xi(t)$ is a special solution to the classical oscillator equation

$$\ddot{\xi} + \omega_{ef}^2(t)\xi = 0, \tag{14}$$

where

$$\omega_{ef}^2(t) = \omega^2(t) + \dot{\delta}(t) - \delta^2(t), \tag{15}$$

$$\delta(t) = \frac{1}{2}[\gamma_x(t) - \gamma_p(t)]. \tag{16}$$

This special solution is selected by the initial condition $\xi(t) = \exp(-it)$ for $t\to -\infty$, which is equivalent to fixing the value of the Wronskian

$$\xi\dot{\xi}^* - \dot{\xi}\xi^* = 2i. \tag{17}$$

The function $\eta(t)$ is defined as

$$\eta(t) = \dot{\xi}(t) + \delta(t)\xi(t). \tag{18}$$

It satisfies the identity following from Eq. (17),

$$\mathrm{Im}[\xi(t)\eta^*(t)] \equiv 1. \tag{19}$$

The operators $\hat{X}(t)$ and $\hat{P}(t)$ have the form

$$\begin{pmatrix} \hat{X}(t) \\ \hat{P}(t) \end{pmatrix} = e^{-\Gamma(t)}\int_0^t d\tau e^{\Gamma(\tau)}\mathcal{A}(t;\tau)\begin{pmatrix} \hat{F}_x(\tau) \\ \hat{F}_p(\tau) \end{pmatrix}, \tag{20}$$

where the 2×2 matrix

$$\mathcal{A}(t;\tau) = \begin{pmatrix} a_x^x(t;\tau) & a_x^p(t;\tau) \\ a_p^x(t;\tau) & a_p^p(t;\tau) \end{pmatrix} \tag{21}$$

consists of the following elements:

$$a_x^x = \mathrm{Im}[\xi(t)\eta^*(\tau)], \quad a_x^p = \mathrm{Im}[\xi^*(t)\xi(\tau)], \tag{22}$$

$$a_p^x = \mathrm{Im}[\eta(t)\eta^*(\tau)], \quad a_p^p = \mathrm{Im}[\eta^*(t)\xi(\tau)]. \tag{23}$$

It seems natural to identify the functions $\omega(t)$ and $\gamma(t)$ in Eqs. (8), (13), and (15) with the real and imaginary parts of the instantaneous complex cavity eigenfrequency, $\omega_c(t) = \omega(t) - i\gamma(t)$. An immediate consequence of Eqs. (10)–(12) and (19) is the formula

$$[\hat{x}(t),\hat{p}(t)] = ie^{-2\Gamma(t)} + [\hat{X}(t),\hat{P}(t)]. \tag{24}$$

Using Eqs. (19) and (20), one can verify that the commutator $[\hat{x}(t),\hat{p}(t)]=i$ is preserved exactly for arbitrary functions $\omega(t)$ and $\gamma(t)$, if one assumes that the noise operators are delta correlated (the Markov approximation) with the following commutation relations:

$$[\hat{F}_x(t),\hat{F}_p(t')] = 2i\gamma(t)\delta(t-t'), \tag{25}$$

$$[\hat{F}_x(t),\hat{F}_x(t')] = [\hat{F}_p(t),\hat{F}_p(t')] = 0. \tag{26}$$

Indeed, under these conditions one obtains

$$[\hat{X}(t),\hat{P}(t)] = e^{-2\Gamma(t)}\int_0^t 2i\gamma(t)e^{2\Gamma(\tau)}d\tau = i[1 - e^{-2\Gamma(t)}].$$

In contrast to the classical Langevin equations, which contain a single stochastic force, in the quantum case one must use *two* noise operators, otherwise the canonical commutation relations cannot be saved [24,30,33]. The Markov approximation implies the relations

$$\langle \hat{F}_j(t)\hat{F}_k(t')\rangle = \delta(t-t')\chi_{jk}(t), \quad j,k = x,p. \tag{27}$$

Strictly speaking, the noise coefficients $\chi_{jk}(t)$ must be derived from some "microscopical" model, which takes into account explicitly (i) the coupling of the field mode with electron-hole pairs inside the semiconductor slab and (ii) the coupling of electrons and holes with phonons or other quasiparticles, responsible for the damping mechanisms. Unfortunately, it seems that no model of this kind was studied up to now. Nonetheless, some conclusions on the relations between the noise coefficients can be made, if one calculates the second-order moments of the quadrature operators

$$\langle \hat{f}\hat{g}\rangle = \langle \hat{f}_s\hat{g}_s\rangle + \sum_{\nu,\rho=x,p} I_{fg}^{\nu\rho}, \quad (f,g = x,p), \tag{28}$$

$$I_{fg}^{\nu\rho}(t) = e^{-2\Gamma(t)}\int_0^t d\tau e^{2\Gamma(\tau)}a_f^\nu(t;\tau)a_g^\rho(t;\tau)\chi_{\nu\rho}(\tau). \tag{29}$$

Let us consider the case of time-independent frequency, $\omega = \omega_i = 1$, and time-independent damping and noise coefficients. Assuming that $\gamma \ll 1$ (small damping) one can neglect the correction $\delta^2 \sim \gamma^2$ in function $\omega_{ef}(t)$ Eq. (15) and use the solution $\xi(t) = \exp(-it)$. Then all integrals in Eq. (29) can be calculated exactly. Supposing that $\gamma_{x,p} \sim \gamma$ and $\chi_{jk} \sim \gamma$ (in accordance with the fluctuation–dissipation theorem), one can obtain the following mean values at $t \to \infty$ (when contributions from $\hat{x}_s$ and $\hat{p}_s$ disappear)

$$\langle \hat{x}^2\rangle_\infty = \frac{1}{4\gamma}[\chi_{xx} + \chi_{pp} + 2\gamma_p\chi_s] + O(\gamma^2), \tag{30}$$

$$\langle \hat{p}^2\rangle_\infty = \frac{1}{4\gamma}[\chi_{xx} + \chi_{pp} - 2\gamma_x\chi_s] + \mathcal{O}(\gamma^2), \tag{31}$$

$$\langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle_\infty = \frac{1}{2\gamma}[\gamma_x\chi_{pp} - \gamma_p\chi_{xx}] + \mathcal{O}(\gamma^2), \tag{32}$$

where $\chi_{xp} + \chi_{px} = 2\chi_s(t)$. Expressions (30)–(32) coincide with the thermodynamical equilibrium values,

$$\langle \hat{x}^2\rangle_{eq} = \langle \hat{p}^2\rangle_{eq} = 1/2 + \langle n\rangle_{th}, \quad \langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle_{eq} = 0 \tag{33}$$

(where $\langle n\rangle_{th}$ is the mean number of quanta in the thermal state), with the accuracy of the order of γ^2, provided the noise coefficients are chosen as follows,

$$\chi_s = 0, \quad \chi_{xx} = \gamma_x G, \quad \chi_{pp} = \gamma_p G, \tag{34}$$

$$G = 1 + 2\langle n\rangle_{th} = \coth[\hbar\omega_i/(2k_B\Theta)], \tag{35}$$

where Θ is the temperature of the reservoir.

The main assumption made in this paper is that relations Eq. (34) hold for time-dependent functions, $\gamma_{x,p}$, χ_{xx}, and χ_{pp}. Perhaps, it is not very important, but it greatly simplifies calculations and the analysis of results, reducing the number of independent functions and parameters. Namely, the condition $\chi_s = 0$ simplifies calculations due to the identities

$$I_{ab}^{px} + I_{ba}^{xp} \equiv 0, \quad a,b = x,p, \tag{36}$$

which result, in particular, in the simple formulas

$$\langle \hat{P}^2\rangle = I_{pp}^{xx} + I_{pp}^{pp}, \quad \langle \hat{X}^2\rangle = I_{xx}^{xx} + I_{xx}^{pp}. \tag{37}$$

Identity Eq. (19) leads to the relations

$$I_{xp}^{px} + I_{xp}^{xp} = -I_{px}^{xp} - I_{px}^{px} = \frac{i}{2}\theta(t), \tag{38}$$

$$\theta(t) \equiv 1 - \exp[-2\Gamma(t)], \tag{39}$$

so that

$$\langle \hat{X}\hat{P}\rangle = \langle \hat{P}\hat{X}\rangle^* = I_{xp}^{xx} + I_{xp}^{pp} + \frac{i}{2}\theta. \tag{40}$$

Let us introduce the "asymmetry parameter" y according to the relations

$$y = \frac{\gamma_p - \gamma_x}{\gamma_p + \gamma_x}, \quad \gamma_p = \gamma(1+y), \quad \gamma_x = \gamma(1-y). \tag{41}$$

Then [using the notation $f_t \equiv f(t)$]

$$\langle \hat{P}^2(t)\rangle = |\eta_t|^2 J_t - \mathrm{Re}(\eta_t^{*2}\tilde{J}_t), \tag{42}$$

$$\langle \hat{X}^2(t)\rangle = |\xi_t|^2 J_t - \mathrm{Re}(\xi_t^{*2}\tilde{J}_t), \tag{43}$$

$$\langle \hat{X}\hat{P}\rangle_t = \frac{i\theta}{2} + \mathrm{Re}(\xi_t\eta_t^* J_t - \xi_t^*\eta_t^*\tilde{J}_t), \tag{44}$$

$$J_t = \frac{G}{2}e^{-2\Gamma(t)}\int_0^t d\tau e^{2\Gamma(\tau)}\gamma(\tau)[|\xi_\tau|^2 + |\eta_\tau|^2 + y(|\xi_\tau|^2 - |\eta_\tau|^2)], \tag{45}$$

$$\tilde{J}_t = \frac{G}{2}e^{-2\Gamma(t)}\int_0^t d\tau e^{2\Gamma(\tau)}\gamma(\tau)[\xi_\tau^2 + \eta_\tau^2 + y(\xi_\tau^2 - \eta_\tau^2)]. \tag{46}$$

III. MEAN NUMBER AND NUMBER VARIANCE OF QUANTA

Formula (9) for the mean number of quanta can be splitted in two parts,

$$\mathcal{N}(t) = \mathcal{N}_s(t) + \mathcal{N}_r(t), \tag{47}$$

where the first term depends on the initial state ("signal"), and the second term is determined by the interaction with the reservoir. From Eqs. (42) and (43) one obtains

$$\mathcal{N}_r(t) = E_t J_t - \mathrm{Re}(\tilde{E}_t^*\tilde{J}_t), \tag{48}$$

$$E_t = \frac{1}{2}(|\xi_t|^2 + |\eta_t|^2), \quad \tilde{E}_t = \frac{1}{2}(\xi_t^2 + \eta_t^2). \tag{49}$$

In the special case of the initial *coherent* state $|\alpha\rangle$ (which corresponds to the initial "classical" signal in the cavity), the "signal" contribution is given by the formula

$$\mathcal{N}_s^{(coh)}(t) = e^{-2\Gamma(t)}\left(\{\mathrm{Re}[\alpha\xi(t)]\}^2 + \{\mathrm{Re}[\alpha\dot{\xi}(t)]\}^2 + \frac{1}{2}E(t)\right) - \frac{1}{2}. \tag{50}$$

For the initial thermal state one has

$$\mathcal{N}_s^{(th)}(t) = \frac{1}{2}\{G_0 e^{-2\Gamma(t)}E(t) - 1\}. \tag{51}$$

Note that coefficients G and G_0 can be different. One should remember that formulas (48), (50), and (51) have sense for sufficiently big values of time t, when the normalized frequency $\omega(t)$ returns to its initial unit value.

Fluctuations of the number of created quanta are characterized by the variance

$$\sigma_N = \langle \hat{\mathcal{N}}^2\rangle - \langle \hat{\mathcal{N}}\rangle^2, \quad \hat{\mathcal{N}} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2 - 1). \tag{52}$$

The quantity $\langle \hat{\mathcal{N}}^2\rangle$ contains combinations of various products of *four* operators $\hat{p}$ and $\hat{x}$. Calculations can be significantly simplified in the case of initial GAUSSIAN states of the field (which include, as special cases, thermal, coherent, and squeezed states). Indeed, it is well known [24,34] that the description of open quantum systems by means of the Heisenberg-Langevin equations with delta-correlated stochastic force operators is equivalent to the description in the Schrödinger picture by means of the master equation for the statistical operator. In the case of *linear* operator equations of motion, such as Eqs. (7) and (8), the corresponding master equations contain only *quadratic* terms (various products of *two* operators $\hat{p}$ and $\hat{x}$) [24,34–36]). Consequently, any initial GAUSSIAN state *remains* GAUSSIAN in the process of evolution. But it is well known that for GAUSSIAN states, all higher-order statistical moments of operators $\hat{p}$ and $\hat{x}$ can be expressed in terms of products of moments of the second order. I consider here only the simplest case of zero mean values of the first-order moments, $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$. Then the following formula is valid:

$$\langle \hat{a}\hat{b}\hat{c}\hat{d}\rangle = \langle \hat{a}\hat{b}\rangle\langle \hat{c}\hat{d}\rangle + \langle \hat{a}\hat{c}\rangle\langle \hat{b}\hat{d}\rangle + \langle \hat{a}\hat{d}\rangle\langle \hat{b}\hat{c}\rangle, \tag{53}$$

where $a,b,c,d = x,p$ (and the order of operators should be maintained). Using Eqs. (10) and (53), one can write

$$\begin{aligned}2\sigma_N = {} & \langle \hat{P}^2\rangle^2 + \langle \hat{X}^2\rangle^2 + \langle \hat{P}\hat{X}\rangle^2 + \langle \hat{X}\hat{P}\rangle^2 + \langle \hat{p}_s^2\rangle^2 + \langle \hat{x}_s^2\rangle^2 + \langle \hat{p}_s\hat{x}_s\rangle^2 \\ & + \langle \hat{x}_s\hat{p}_s\rangle^2 + 2\langle \hat{p}_s^2\rangle\langle \hat{P}^2\rangle + 2\langle \hat{x}_s^2\rangle\langle \hat{X}^2\rangle + 2\langle \hat{p}_s\hat{x}_s\rangle\langle \hat{P}\hat{X}\rangle \\ & + 2\langle \hat{x}_s\hat{p}_s\rangle\langle \hat{X}\hat{P}\rangle.\end{aligned} \tag{54}$$

Using Eqs. (42)–(44) and the relation $[\mathrm{Re}(\xi\eta^*)]^2 = |\xi\eta|^2 - 1$, following from Eq. (19), one can obtain the following formula for the initial *thermal* state, characterized by parameter G_0 (which can be different from the reservoir factor G)

$$\begin{aligned}\sigma_N = {} & (2E^2 - 1)\left(J + \frac{G_0}{2}e^{-2\Gamma}\right)^2 + E^2|\tilde{J}|^2 - \frac{1}{4} \\ & + \mathrm{Re}[\tilde{E}^2\tilde{J}^{*2} - 4E\tilde{E}J\tilde{J}^* - 2G_0 E\tilde{E}^*\tilde{J}e^{-2\Gamma}].\end{aligned} \tag{55}$$

The correctness of formula (55) can be verified in the special case of relaxation from one temperature (characterized by parameter G_0) to another (with parameter G) without changing the frequency $\omega = const = 1$. If $y=0$, then $E \equiv 1$, $\tilde{E} = \tilde{J} \equiv 0$, $J(t) = G\theta(t)/2$, so that one obtains, for an arbitrary function $\gamma(t)$,

$$\sigma_N(t) = \frac{1}{4}[G_{ef}^2(t) - 1] \equiv \mathcal{N}(t)[\mathcal{N}(t) + 1], \tag{56}$$

where

$$G_{ef}(t) = 1 + 2\mathcal{N}(t) = G\theta(t) + G_0[1 - \theta(t)], \tag{57}$$

as it must be for thermal states.

Comparing Eq. (55) with Eqs. (48) and (51), one can obtain the formula (using the identity $E^2 - |\tilde{E}|^2 = [\mathrm{Im}(\xi\eta^*)]^2 \equiv 1$)

$$\sigma_N - 2\mathcal{N}^2 = 2\mathcal{N} + |\tilde{J}|^2 - \left(J + \frac{G_0}{2}e^{-2\Gamma}\right)^2 + \frac{1}{4}. \tag{58}$$

I am interested here in the case, when the number of quanta exponentially grows with time due to the effect of parametric amplification, so that $\mathcal{N} \gg 1$. This happens if four functions, $E(t)$, $\tilde{E}(t)$, $J(t)$, and $\tilde{J}(t)$, assume big values. In such a case, the right-hand side of Eq. (58) has the same order of magnitude as $\mathcal{N}$, i.e.,

$$\sigma_N = 2\mathcal{N}^2 + \mathcal{O}(\mathcal{N}) \quad \text{if} \quad \mathcal{N} \gg 1. \tag{59}$$

This means that the field mode goes asymptotically to the so-called superchaotic [37] quantum state, whose statistics is essentially different from the statistics of the initial thermal state, characterized by formula (56).

IV. ASYMPTOTICAL PHOTON STATISTICS

The photon statistics in the most general mixed GAUSSIAN states was studied in [38–41]. If the mean values of the quadrature operators (or electric and magnetic fields in the cavity mode) are equal to zero, then this statistics is determined completely by two parameters

$$\tau = \sigma_{xx} + \sigma_{pp} \equiv 1 + 2\mathcal{N}, \tag{60}$$

$$\Delta = \sigma_{xx}\sigma_{pp} - \sigma_{px}^2 \equiv 1/(4\mu^2), \tag{61}$$

where σ_{xx} and σ_{pp} are the variances of the quadrature operators, $\sigma_{xp} = \sigma_{px}$ is their covariance, and $\mu \equiv \mathrm{Tr}(\hat{\rho}^2)$ is the *purity* of the GAUSSIAN quantum state of the field mode described by the statistical operator $\hat{\rho}$. The photon distribution function $f(m) \equiv \langle m|\hat{\rho}|m\rangle$ (i.e., the probability to detect m quanta in the state $\hat{\rho}$) can be expressed in terms of the Legendre polynomials as follows [38–41]:

$$f(m) = \frac{2}{\sqrt{1 + 2\tau + 4\Delta}}\left(\frac{1 + 4\Delta - 2\tau}{1 + 4\Delta + 2\tau}\right)^{m/2} \times P_m\left(\frac{4\Delta - 1}{\sqrt{(4\Delta + 1)^2 - 4\tau^2}}\right). \tag{62}$$

Formula (62) is exact. However, if the number of photons is big (say, $m \sim \mathcal{N} > 1000$), then it is more convenient to find an asymptotical form of Eq. (62) for $m \gg 1$. Note that the argument of the Legendre polynomial in Eq. (62) is always outside the interval $(-1, 1)$, being equal to unity only for thermal states with $\tau = 2\sqrt{\Delta}$. Therefore it is convenient to use the following asymptotical formula [42]:

$$P_m(\cosh\xi) \approx \left(\frac{\xi}{\sinh\xi}\right)^{1/2} I_0([m + 1/2]\xi) \tag{63}$$

(where $I_0(z)$ is the modified Bessel function), because it holds even for complex values of variable ξ, provided $\mathrm{Re}\,\xi \geq 0$ and $|\mathrm{Im}\,\xi| < \pi$. The parameter ξ equals

$$\xi = \ln\left(\frac{4\Delta - 1 + 2\sqrt{\tau^2 - 4\Delta}}{\sqrt{(4\Delta + 1)^2 - 4\tau^2}}\right). \tag{64}$$

The variance of the photon number distribution for an arbitrary GAUSSIAN state (with zero quadrature mean values) is given by the formula [40,41]

$$\sigma_N = \frac{1}{2}\tau^2 - \Delta - \frac{1}{4}. \tag{65}$$

Comparing this formula with Eqs. (48), (51), and (58) one gets

$$\Delta = \left(J + \frac{G_0}{2}e^{-2\Gamma}\right)^2 - |\tilde{J}|^2. \tag{66}$$

Consequently, $\Delta/\tau \sim \Delta/\mathcal{N} = \mathcal{O}(1)$ if $\tau \sim \mathcal{N} \gg 1$. This means that parameter ξ is of the order of unity in the case concerned. Then function $I_0(x)$ in Eq. (63) can be replaced by its asymptotical form $I_0(x) \approx (2\pi x)^{-1/2}\exp(x)$, because its argument $x = (m + 1/2)\xi$ is much bigger than unity. Using Eq. (64) to calculate $\sinh(\xi)$, one can obtain an approximate formula

$$f(m) \approx [\pi(m + 1/2)\sqrt{\tau^2 - 4\Delta}]^{-1/2} \times \left(\frac{4\Delta - 1 + 2\sqrt{\tau^2 - 4\Delta}}{4\Delta + 1 + 2\tau}\right)^{m+1/2}. \tag{67}$$

Obviously, one can neglect the term $4\Delta \ll \tau^2$ in the first factor. Using the Taylor expansion $\sqrt{\tau^2 - 4\Delta} = \tau - 2\Delta/\tau + \mathcal{O}(1/\tau)$, one can simplify the fraction in the second factor as follows:

$$\frac{4\Delta - 1 + 2\sqrt{\tau^2 - 4\Delta}}{4\Delta + 1 + 2\tau} = 1 - \frac{1}{\tau} + \mathcal{O}(1/\tau^2).$$

Replacing $(1 - x)^m \approx \exp(-mx)$ for $x \ll 1$, one can transform Eq. (67) to the following simple form:

$$f(m) \approx \frac{\exp[-(m + 1/2)/\tau]}{\sqrt{\pi\tau(m + 1/2)}} \approx \frac{\exp[-(m + 1/2)/(2\mathcal{N})]}{\sqrt{2\pi\mathcal{N}(m + 1/2)}}. \tag{68}$$

Equation (68) holds under the conditions $\tau \approx 2\mathcal{N} \gg 1$ and $m \gg 1$. Using the Euler-MacLaurin summation formula, one can verify that the distribution function (68) has the correct normalization with an accuracy $\mathcal{O}(\tau^{-1/2})$

$$\sum_{m=0}^{\infty} f(m) \approx \int_0^{\infty} f(m)dm + \mathcal{O}(f(0)) \approx \int_0^{\infty} \frac{\exp(-x/\tau)}{\sqrt{\pi\tau x}}dx + \mathcal{O}(\tau^{-1/2}) = 1 + \mathcal{O}(\tau^{-1/2}).$$

With the same accuracy, the moments of the distribution function are given by the formula

$$\langle m^k\rangle \equiv \sum_{m=0}^{\infty} m^k f(m) \approx \int_0^\infty x^k \frac{\exp(-x/\tau)}{\sqrt{\pi\tau x}}dx = \tau^k \frac{(2k-1)!!}{2^k} \approx \mathcal{N}^k (2k-1)!!\,. \tag{69}$$

For $k=2$, Eq. (69) reproduces the result Eq. (59): $\sigma_N = \langle m^2\rangle - \langle m\rangle^2 \approx 2\mathcal{N}^2$.

V. MEAN NUMBER OF QUANTA FOR PERIODICAL VARIATIONS OF PARAMETERS

Having in mind applications to the dynamical Casimir effect, let us suppose that functions $\omega(t)$ and $\gamma(t)$ have the form of *periodical* pulses with the periodicity T (so that the kth pulse begins at $t_k=(k-1)T$, $t_1=0$), separated by intervals of time with $\omega=1$ and $\gamma=0$ (I neglect the damping of the field between pulses, supposing that the quality factor of the cavity is big enough). Then integrals in Eqs. (45) and (46) are reduced to sums of n (the total number of pulses) integrals taken between the initial and final time moments of each pulse. In the interval between the kth and $(k+1)$th pulses the solution to Eq. (14) can be written as

$$\xi_k(t) = a_k e^{-it} + b_k e^{it}, \quad a_0 = 1, \quad b_0 = 0, \tag{70}$$

where a_k and b_k are constant coefficients. During these intervals, functions $E(t)$ and $\tilde{E}(t)$ assume constant values

$$E_k = |a_k|^2 + |b_k|^2, \quad \tilde{E}_k = 2a_k b_k. \tag{71}$$

The pairs of the nearest coefficients, (a_{k-1}, b_{k-1}) and (a_k, b_k), are related by means of a linear transformation

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = M_k \begin{pmatrix} a_{k-1} \\ b_{k-1} \end{pmatrix}, \quad M_k = \begin{Vmatrix} f & g^* e^{-2it_k} \\ g e^{2it_k} & f^* \end{Vmatrix},$$

where complex coefficients f and g can be expressed through two complex amplitude reflection coefficients and two complex amplitude transmission coefficients, which connect "plane waves" $\exp(\pm it)$ coming from the "left" and from the "right." It is important that matrix M_k is *unimodular*

$$\det M_k = |f|^2 - |g|^2 \equiv 1. \tag{72}$$

Actually, this condition is equivalent to the Wronskian identity Eq. (17). One can verify the relations

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \Phi^{\dagger n} (\Phi M_1)^n \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, \quad \Phi \equiv \begin{Vmatrix} e^{iT} & 0 \\ 0 & e^{-iT} \end{Vmatrix}.$$

Since $\det(\Phi M_1)=1$, one can use the formula for powers of a two-dimensional unimodular matrix S (see, e.g., [43])

$$S^n = U_{n-1}(z)S - U_{n-2}(z)E, \quad z \equiv \frac{1}{2}\mathrm{Tr}\, S, \tag{73}$$

where E means here the unit matrix and $U_n(z)$ is the Tchebyshev polynomial of the second kind. In the case involved one has $z=\frac{1}{2}\mathrm{Tr}(\Phi M_1)=\mathrm{Re}[f\exp(iT)]$. An amplification can happen if $|z|>1$. Thus it is convenient to use the parametrization

$$\frac{1}{2}\mathrm{Tr}(\Phi M_1) = \mathrm{Re}[f\exp(iT)] = \cosh(\nu). \tag{74}$$

If $\mathrm{Re}[f\exp(iT)]>1$, then ν is real positive parameter. If $\mathrm{Re}[f\exp(iT)]<-1$, then $\nu=\tilde{\nu}+i\pi$, where $\tilde{\nu}$ is real and positive. The maximal values of $|\mathrm{Re}[f\exp(iT)]|$ correspond to the cases of strict resonance with

$$T = T_{res} = \frac{1}{2}T_0(m - \varphi/\pi), \tag{75}$$

where $f=|f|\exp(i\varphi)$, T_0 is the period of oscillations in the selected field mode and $m=1,2,\ldots$ (even values of m correspond to $\mathrm{Re}[f\exp(iT)]>1$, whereas odd values of m correspond to $\mathrm{Re}[f\exp(iT)]<-1$). One can check the fulfillment of the identity $|a_n|^2-|b_n|^2\equiv 1$ as a consequence of the initial identity $|f|^2-|g|^2\equiv 1$.

I consider here only the simplest case of the *strict resonance*, $T=T_{res}$, when $\exp(-iT)=\exp(i\varphi+m\pi)$. Then $|f|^2=\cosh^2(\nu)$ and $|g|^2=\sinh^2(\nu)$, so that [8]

$$a_k = \cosh(k\nu)e^{-ikT}, \quad b_k = \sinh(k\nu)e^{iT(k-1)+i\phi} \tag{76}$$

(where ϕ is the phase of complex number g),

$$E_k = \cosh(2k\nu), \quad \tilde{E}_k = \sinh(2k\nu)e^{i(\phi-T)}. \tag{77}$$

To calculate the integrals Eqs. (45) and (46), one needs an explicit form of functions $\xi(t)$ and $\eta(t)$ during the pulse, i.e., when $\chi(t)\equiv\omega(t)-1\neq 0$ and $\gamma(t)\neq 0$. But $|\chi(t)|\ll 1$ and $\gamma(t)\ll 1$ (in the dimensionless variables) under realistic conditions of the DCE experiments. Consequently, as a matter of fact, the exact function $\xi(t)$ is very close to the form Eq. (70) even during the action of pulses, and not only in the intervals between them (see [9] for a detailed analysis). This means, in particular, that one can neglect the term $\delta(t)\xi$ in Eq. (18), writing simply $\eta(t)\approx\dot{\xi}(t)$ (this formula is exact if $y=0$). Therefore the functions $E(t)$ and $\tilde{E}(t)$ are close to constant values Eq. (71) for $t_{k-1}\le t\le t_k$, with small corrections of the order of $\chi_{\max}\sim\gamma_{\max}$. Neglecting these corrections, one can calculate the integrals over the duration of each pulse exactly, if parameter y does not depend on time [since $\gamma(t)=d\Gamma/dt$].

Let us do this first in the symmetrical case $y=0$. After n pulses one has

$$J_n \equiv J(nT) = \frac{G}{2}e^{-2\Lambda n}(1-e^{-2\Lambda})\sum_{k=1}^{n} e^{2\Lambda k}E_k \tag{78}$$

and a similar formula for $\tilde{J}_n$, where E_k is replaced by $\tilde{E}_k$. Here

$$\Lambda = \int_{t_i}^{t_f} \gamma(\tau)d\tau, \tag{79}$$

t_i and t_f being the initial and final moments of each pulse. Numerical evaluations confirmed a high accuracy of the approximate formula (78). Using Eq. (77) one can obtain the following explicit formulas:

$$J_n = A_n^{(+)} + A_n^{(-)}, \quad \tilde{J}_n = e^{i(\phi-T)}(A_n^{(+)} - A_n^{(-)}), \tag{80}$$

$$A_n^{(\pm)} = \frac{G\Lambda}{4(\Lambda \pm \nu)}(e^{\pm 2n\nu} - e^{-2n\Lambda}). \quad (81)$$

It is taken into account that $\Lambda, \nu \ll 1$. Besides, $1-\exp(-2\Lambda)$ is replaced by 2Λ. The coefficient ν in Eq. (81) is determined by the equivalent relations [taking into account Eq. (72) and the resonance conditions]

$$\cosh(\nu) = |f|, \quad \sinh(\nu) = |g|. \quad (82)$$

In view of Eqs. (48), (77), and (80), the mean number of "noise" quanta created after n pulses in the resonance case is given by a simple formula

$$\mathcal{N}_{rn} = A_n^{(+)}e^{-2n\nu} + A_n^{(-)}e^{2n\nu}, \quad (83)$$

where $\mathcal{N}_{rn}$ stands for the value $\mathcal{N}_r(t)$ of the function (48) at the moment $t=nT$. Adding to the quantity Eq. (83) the expression (51), one obtains the total number of quanta, created from the initial thermal state

$$\mathcal{N}_n = e^{2n\nu}\left(A_n^{(-)} + \frac{G_0}{4}e^{-2\Lambda n}\right) + e^{-2n\nu}\left(A_n^{(+)} + \frac{G_0}{4}e^{-2\Lambda n}\right) - \frac{1}{2}. \quad (84)$$

Only the first term in the right-hand side of Eq. (84) grows with the number of pulses n. So one obtains the following asymptotical formula for $2n\nu \gg 1$:

$$\mathcal{N}_n = \frac{1}{4}e^{2n(\nu-\Lambda)}\left(G_0 + \frac{G\Lambda}{\nu - \Lambda}\right) + \mathcal{O}(1). \quad (85)$$

It shows that the mean number of photons grows exponentially as function of the number of pulses, if $\nu > \Lambda$.

The mean number of quanta in the case when the periodicity of pulses T is different from the optimal value T_{res} given by Eq. (75) was calculated (for $y=0$ only) in [22,44]. The general expressions become rather cumbersome in this case, but the asymptotical behavior for $2n\nu \gg 1$ is similar to that given by Eq. (85). The main difference is that the coefficient ν in the argument of the exponential function in Eq. (85) should be replaced by the smaller coefficient $\nu_\delta = \sqrt{\nu^2 - \delta^2}$, where $\delta = \omega_0(T - T_{res})$. Besides, some changes in the pre-exponential coefficients should be made (but they are not very significant).

Additional terms in Eqs. (45) and (46), proportional to the asymmetry coefficient y, cannot be calculated explicitly in a general case, because they contain integrals of $\exp[2\Gamma(\tau)]\gamma(\tau)$ multiplied by $\text{Re}[a_k b_k^* \exp(-2i\tau)]$ or $a_k^2 \exp(-2i\tau) + b_k^2 \exp(2i\tau)$. But it was shown in [8,22,44], that the condition $\nu > \Lambda$ can be fulfilled only for very short pulses of functions $\chi(t)$ and $\gamma(t)$, whose duration T_d (determined mainly by the recombination time T_r in the semiconductor material) is much smaller than the period of field oscillations T_0 (which equals 2π in the dimensionless variables). Hence, a good approximation consists in replacing functions $\exp(\pm 2i\tau)$ in the integrals for the kth pulse by constant values $\exp(\pm 2it_k)$. Then the integrals can be calculated explicitly, and one can obtain the following corrections to the coefficients J_n and $\tilde{J}_n$, given by Eq. (80)

$$\delta J_n = y\ \text{Re}(e^{-i\psi}[C_n^{(+)} - C_n^{(-)}]), \quad (86)$$

$$\delta\tilde{J}_n = ye^{i\psi}\{\text{Re}(e^{-i\psi}[C_n^{(+)} + C_n^{(-)}]) + 2i\ \text{Im}(e^{-i\psi}D_n)\}, \quad (87)$$

where

$$C_n^{(\pm)} = \frac{G\Lambda}{4(\Lambda + 2i\varphi \pm \nu)}(e^{4in\varphi \pm 2n\nu} - e^{-2n\Lambda}), \quad (88)$$

$$D_n = \frac{G\Lambda}{4(\Lambda + 2i\varphi)}(e^{4in\varphi} - e^{-2n\Lambda}), \quad (89)$$

and the phases are defined as follows,

$$f = |f|e^{i\varphi}, \quad g = |g|e^{i\phi}, \quad \psi = \varphi + \phi. \quad (90)$$

The resonance relation $\exp(-2iT) = \exp(2i\varphi)$ was also taken into account. The correction to the mean number of photons is (the term with D_n drops out)

$$\delta\mathcal{N}_n = y\ \text{Re}(e^{-i\psi}[C_n^{(+)}e^{-2n\nu} - C_n^{(-)}e^{2n\nu}]), \quad (91)$$

so that for $n \gg 1$ one gets

$$\delta\mathcal{N}_n \approx \frac{y}{4}e^{2n(\nu-\Lambda)}\ \text{Re}\left(\frac{G\Lambda\exp(-i\psi)}{\Lambda - \nu + 2i\varphi}\right) + \mathcal{O}(1). \quad (92)$$

Simple approximate formulas for the coefficients g and f in the case of small variations of frequency and damping coefficients, $|\chi(t)|$, $\gamma(t) \ll 1$, were given in [8] (see also [9] for details). Neglecting the terms of the order of γ^2 in Eq. (15) and assuming that the asymmetry coefficient y Eq. (41) does not depend on time, one can write an effective relative frequency shift $\chi_{ef}(t)$ as $\chi_{ef} = \chi(t) - y\dot{\gamma}(t)/2$. Then (supposing that the pulse starts at $t_i = 0$)

$$g(y) \approx i\int_0^{t_f}[\chi(t) - y\dot{\gamma}(t)/2]e^{-2it}dt = i\int_0^{t_f}[\chi(t) - iy\gamma(t)]e^{-2it}dt \quad (93)$$

[the second equality is obtained from the first one by integrating by parts; remember that $\gamma(t_i) = \gamma(t_f) = 0$],

$$f = \left(1 + \frac{1}{2}|g|^2\right)e^{i\varphi}, \quad \varphi = -\int_{t_i}^{t_f}\chi(t)dt. \quad (94)$$

Note that neither φ nor Λ depend on the asymmetry coefficient y. On can check an accuracy of formulas (93) and (94), considering the case of "rectangular" pulse with $\chi = const$ and $\gamma = 0$. The difference between exact coefficients g or f and their approximate expressions (93) and (94) can be seen only in terms of the order of χ^2. Other consequencies of Eqs. (79), (93), and (94) are the inequality $|g(y=0)| \le |\varphi|$ and the relation

$$g(y) \approx y\Lambda - i\varphi \quad \text{if} \quad T_d \ll 1. \quad (95)$$

For cavities with photo-excited semiconductor slabs, the functions $\chi(t)$ and $\gamma(t)$ can be approximated by the following analytical expressions (in the simplest case of very short laser pulses and in the absence of surface recombination) [8,22,44]:

$$\chi(t) = \frac{\chi_m A_0^2 \exp(-2t/T_r)}{A_0^2 \exp(-2t/T_r) + 1}, \quad (96)$$

$$\gamma(t) = \frac{|\chi_m| A_0 \exp(-t/T_r)}{A_0^2 \exp(-2t/T_r) + 1}, \tag{97}$$

where $T_r \ll T_0$ is the recombination time and dimensionless parameter A_0 is proportional to the energy of a single laser pulse. Note that both functions, $\chi(t)$ and $\gamma(t)$, contain the same maximal amplitude factor χ_m (which can be positive or negative depending on the cavity shape). Both the functions equal zero for $t<0$ (they are continuous, as a matter of fact, but their exact dependence on time in the beginning of the process turns out to be insignificant, so that it can be approximated by an instantaneous "jump" at $t=0$). Using functions (96) and (97), one can obtain the following expressions for the parameters Λ and φ [8] (the upper limit of integration $t_f \gg T_r$ is replaced by ∞ due to the rapid convergence of integrals):

$$\Lambda = |\chi_m| T_r \tan^{-1}(A_0), \quad \varphi = -\frac{\chi_m}{2} T_r \ln(1 + A_0^2). \tag{98}$$

Consequently, $|\varphi|$ is several times bigger than Λ in the case of $A_0 \gg 1$ (when $\nu - \Lambda > 0$ [8,22]). Noticing that the phase φ is proportional to $\chi_m \ll 1$, whereas the phase ϕ does not depend on χ_m, one can write $\exp(-i\psi) \approx \exp(-i\phi) = g^*/|g|$. If $T_d \sim T_r \ll 1$, then $\nu \approx |g(y)| \approx |\varphi|$ for $A_0 \gg 1$. Using these relations, one can obtain a rough evaluation of the maximal possible ratio of the correction $\delta\mathcal{N}_n$ Eq. (92) to the term $\mathcal{N}_n$ Eq. (85) as $|\delta\mathcal{N}_n|/\mathcal{N}_n < |y|/2$ (if $G \gg G_0$), and concrete numerical calculations give even smaller values for this ratio. This means that the contribution of the correction Eq. (92) to the total mean number of created quanta can be not very essential, even if y is close to unity. Nonetheless, the influence of y on $\mathcal{N}_n$ can be significant, because y enters formula (93) for the coefficient g and $\nu \approx |g|$.

In the case of functions (96) and (97), the integral in Eq. (93) can be expressed in terms of the Gauss hypergeometric function $F(a,b;c;z)$, if one makes the substitution $x = \exp(-2\tau/T_r)$ and uses an integral representation [45]

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{x^{b-1}(1-x)^{c-b-1}}{(1-xz)^a} dx.$$

The answer is

$$g/\chi_m = \frac{iT_r A_0^2}{2(1+iT_r)} F(1, 1+iT_r; 2+iT_r; -A_0^2) + \frac{yT_r A_0}{1+2iT_r} F\left(1, \frac{1}{2}+iT_r; \frac{3}{2}+iT_r; -A_0^2\right).$$

Numerical calculations show that the difference $\nu - \Lambda$ is positive and not too small, only if $A_0 \gg 1$. Therefore it is convenient to use one of Kummer's formulas, connecting Gauss hypergeometric functions with the arguments z and z^{-1}. In the special case concerned ($c=b+1$ and $a=1$) the required relation is [45]

$$F(1,b;b+1;z) = \frac{b}{z(1-b)} F(1, 1-b; 2-b; z^{-1}) + \frac{\pi b}{\sin(\pi b)} (-z)^{-b}.$$

Thus one can obtain (see [44] for a more general case)

$$g/\chi_m = \frac{1}{2} F(1, -iT_r; 1-iT_r; -A_0^{-2}) - \frac{yT_r}{A_0(1-2iT_r)} F\left(1, \frac{1}{2}-iT_r; \frac{3}{2}-iT_r; -A_0^{-2}\right) - \frac{\pi T_r \exp(-2iT_r \ln A_0)}{2\sinh(\pi T_r)} [1 - y\tanh(\pi T_r)]. \tag{99}$$

If $A_0 \gg 1$, then the hypergeometric functions in (99) can be replaced by unit values. For example, taking $T_r=0.2$ and $A_0=20$, which are reasonable values, according to numerical calculations done in [8] [in dimension variables, T_r in Eq. (99) should be replaced by $\omega_0 T_r$, where $\omega_0 = 2\pi/T_0$ is the frequency of the field mode in s^{-1}], one obtains from Eq. (99) the following numerical values of $\tilde{g} = g/\chi_m$ for $y = -1, 0, 1$ (in this case, $\tilde{\Lambda} = \Lambda/\chi_m \approx 0.31$ and $\tilde{\varphi} = \varphi/\chi_m \approx -0.60$): $\tilde{g}(-1) \approx 0.24 + 0.69i$, $\tilde{g}(0) \approx 0.33 + 0.44i$, and $\tilde{g}(1) \approx 0.42 + 0.19i$. The corresponding values of the amplification coefficient $\tilde{F} = \tilde{g} - \tilde{\Lambda}$ are

$$\tilde{F}(-1) \approx 0.42, \quad \tilde{F}(0) \approx 0.24, \quad \tilde{F}(1) \approx 0.15.$$

These numbers show that the influence of the parameter y on the coefficient $\tilde{F}$ can be significant. Achievable values of the coefficient χ_m in cavities with semiconductor layers are of the order of 10^{-2}. Therefore creating many thousands of quanta after a few thousand laser pulses seems to be quite realistic.

VI. SUMMARY AND DISCUSSION

One of main results of this study is a new asymptotical formula (68) for the distribution function of quanta, generated in the parametric amplification process in the presence of damping. This asymptotical distribution depends only on the mean number of quanta $\mathcal{N}$ (provided this number is big enough), being insensitive to concrete values of damping coefficients and correlation coefficients of the noise operators [at least under the conditions Eq. (34), which, perhaps, are not necessary]. But damping plays an important role, because in its absence the quantum purity is preserved, and the distribution function has well known oscillations (studied in many papers [38–41]), whereas formula (68) shows a smooth behavior. The asymptotical "superchaotic" statistics of quanta is highly superPoissonian, with $\sigma_N \approx 2\mathcal{N}^2$.

Another result consists in new analytical formulas for the mean number of quanta, showing explicitly its dependence on many parameters, especially on the coefficient y Eq. (41), which accounts for the asymmetry between damping coefficients. Knowledge of this coefficient is important for predicting the number of quanta, which could be created due to the

DCE in cavities with semiconductor slabs. Unfortunately, phenomenological models cannot deduce the value of y from some general principles.

On the one hand, the value $y=0$ seems to be distinguished (and the most attractive) for several reasons. For example, if $y=\delta\equiv 0$, then $\eta(t)\equiv\dot{\xi}(t)$ and the effective frequency in Eq. (15) coincides exactly with $\omega(t)$. For this special set of coefficients, the stationary asymptotical values of the second-order statistical moments coincide *exactly* with the equilibrium values Eq. (33) for an *arbitrary* (not necessarily small) function $\gamma(t)$ (if $\omega=const$). Moreover, all formulas are greatly simplified if $y=0$. Other reasonings are related to the fact that the noise and damping coefficients cannot be quite arbitrary and independent. Namely, they must obey the restriction $\chi_{pp}\chi_{xx}-\chi_s^2\geq\gamma^2$, which prevents from possible violations of the positivity of the density matrix during the evolution [35,41]. Consequently, the inequality

$$G^2\gamma_x\gamma_p \geq (\gamma_x+\gamma_p)^2/4 \tag{100}$$

must hold in the case of Eq. (34). At zero temperature of the reservoir ($G=1$), inequality Eq. (100) can be satisfied only for $\gamma_x=\gamma_p=\gamma$, i.e., for $y=0$. For reservoirs with nonzero temperature ($G>1$), the positivity of the density matrix can be preserved for unequal damping coefficients γ_x and γ_p. However, in this case a temperature-dependent restriction $y^2\leq 1-G^{-2}=\cosh^{-2}[\hbar\omega_i/(2k_B\Theta)]$ must be imposed on the asymmetry coefficient.

There are also some arguments in favor of the value $y=0$, based on the analysis of different "microscopical" models, describing an interaction of a selected harmonic oscillator (field mode) with an "environment" by means of multidimensional *quadratic* Hamiltonians of the most general form. Namely, it was shown [35,36] that time-independent damping and noise coefficients, satisfying all the requirements (i.e., not allowing violations of the positivity of the density matrix), arise in these models in the only case: when the coupling between the selected oscillator and the "bath" has the so-called "rotating wave approximation" form $\Sigma_j\hat{a}\hat{b}_j^\dagger+$H.c. (this special kind of coupling is considered in all textbooks on quantum optics, see, e.g., [46]). Under this restriction, the models of this kind result in the set of coefficients $\chi_{xx}=\chi_{pp}=\gamma G$ and $\chi_s=0$.

However, it is worth remembering that a usual justification for the exclusion of "antirotating" terms $\hat{a}\hat{b}_j+$H.c. from the interaction Hamiltonian is based on reasonings that such terms give rapidly oscillating corrections to the equations of motion, whose influence becomes small after averaging over many periods of the field mode oscillations. The problem is that the duration of laser pulses in the MIR experiment is much smaller than the period of the field oscillations. Thus one cannot exclude a possibility that antirotating terms can give an essential contribution under realistic experimental conditions. These observations show a need in constructing a rigorous microscopical theory of the dynamical Casimir effect in dissipative media with time-dependent parameters.

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