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Bipartite concurrence and localized coherence

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Based on a proposed coherence measure, we show that the local coherence of a bipartite quantum pure state (coherence of its reduced density matrix) is exactly the same as the minimal average coherence with all potential pure-state realizations under consideration. In particular, it is shown that bipartite concurrence of pure states just captures the maximal difference between local coherence and the average coherence of one subsystem induced by local operations on the other subsystem with the assistance of classical communications, which provides an alternative operational meaning for bipartite concurrence of pure states. The relation between concurrence and the proposed coherence measure can also be extended to bipartite mixed states.

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I. INTRODUCTION

Coherence and entanglement arise from quantum superposition, the most distinctive and puzzling feature of quantum mechanics. Quantum coherence is an important subject in quantum mechanics, where decoherence due to the interaction with an environment is a crucial issue that is of fundamental interest. If there exists coherence among multiple quantum subsystems, a special nonlocal coherence-quantum entanglement may be generated besides the local coherence of each constituent subsystem. As an ingredient of quantum information, quantum entanglement has been recognized to be an important physical resource in quantum-information processing including quantum communication and quantum computation and plays a key role in quantum-information theory [1–4]. Recently, many works based on some special models have been written to show the relation between local decoherence and disentanglement of a composite quantum system by considering the interaction with environments [5–10]. In fact, so long as there exists interactions between two subsystems, the coherence of each subsystem might also be changed. For example, if a composite quantum system is maximally entangled, each subsystem is completely incoherent. It is natural to ask how entanglement is related to local coherence.

In fact, the previous question means finding some kind of operational meaning of the entanglement measure that we are going to employ. Even though quantification of entanglement has attracted many interests in recent years and a lot of entanglement measures have been proposed and explored from different viewpoints [11–17], one is usually concerned mainly about the monotonicity of entanglement measure under local operation and classical communication (LOCC), i.e., not increase under LOCC operations [18–21], hence only a few entanglement measures have been considered from the operational meanings point of view [22–26]. The most popular two examples are entanglement cost [22,23] and distillable entanglement [14,23,24] which show the conversion rate between the entangled state of interests and maximally entangled state. As a remarkable entanglement measure, concurrence [16] has been widely employed in lots of cases of quantum-information theory. However, to our knowledge, concurrence per se of pure states is related to the purity of one subsystem which only roughly or qualitatively shows the effect of the other subsystem [27].

In this paper, we focus on the relation between concurrence and localized coherence, which can provide an alternative operational meaning for concurrence. Suppose Alice and Bob share a composite bipartite state, Alice's local coherence is determined by her reduced density matrix but independent of its pure-state realization. However, if Bob performs some operations on his subsystem, with the assistance of classical communication Alice might owe her quantum ensemble with different average local coherence. For example, for a Bell state in σ_z representation, Alice's reduced density matrix is completely mixed. But if Bob performs a σ_x measurement on his subsystem and tell his outcome to Alice, Alice will obtain a pure state with maximal coherence. In this sense, we say that the coherence can be localized assisted by Bob (or LOCC). In this paper we propose a coherence measure with explicit geometric meaning by collecting the contribution of all off-diagonal elements of a density matrix. Based on this coherence measure, we show that local coherence of a bipartite pure state is just the same as the minimal average coherence with all potential pure-state realizations taken into consideration. In particular, it is shown that with this coherence measure, concurrence can be regarded as the difference between the maximal and the minimal localized (local) coherence. Thus it provides an operational meaning for concurrence. This is much like what we have found for $(2\otimes 2\otimes n)$-dimensional three-tangles which can be considered as the difference between concurrence of assistance and concurrence of $(2\otimes 2)$-dimensional subsystem [28]. This paper is organized as follows. In Sec. II, we consider coherence measure of quantum systems of a qubit and show the relation between the coherence measure and the concurrence of $(2\otimes n)$-dimensional quantum systems; in Sec. III, we focus on coherence measure of general high-dimensional quantum system and consider the relation between coherence measure and concurrence of a general $(n_1\otimes n_2)$-dimensional quantum systems; in Sec. IV, we extend both the relations given in Secs. II and III to concurrence of bipartite mixed states. The conclusion is drawn in Sec. V.

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II. QUANTUM COHERENCE OF QUBIT AND CONCURRENCE OF $(2\otimes n)$-DIMENSIONAL PURE STATES

A. Quantification of coherence

It has been shown that a good definition of coherence depends on not only the state of the system ρ but also the alternatives under consideration which are usually attached to different eigenvalues of an observable A. Since the off-diagonal elements of ρ characterize interference, they are usually called *coherences* with respect to the basis in which ρ is written [29–31]. The measurements on the observables that do not commute with A can reveal the interference. It is obvious that if ρ is diagonalized, there is not any relevant coherences with respect to that basis. Thus one can straightforwardly quantify the coherence in given basis by measuring the distance between the quantum state ρ and the nearest incoherent state.

Definition 1. If ρ is written in some basis, the coherence with respect to the same basis can be measured by

$$D(\rho) = \|\rho - \sigma^*\|_1 = \sum_{i\neq j} |\rho_{ij}|, \quad (1)$$

where σ^* is the diagonal matrix with $\sigma^*_{ii}=\rho_{ii}$ and $\|\cdot\|_1$ is the "entrywise" norm. In fact, $\|\cdot\|_1$ can also be replaced by Frobenius norm $\|\cdot\|_F$ for some convenient applications.

It is easily to find that $D(\rho)=\min_{\sigma\in\mathcal{I}}\|\rho-\sigma\|_1=\|\rho-\sigma^*\|_1$, where $\mathcal{I}$ is the set of incoherent states with the same basis to ρ. This shows the direct geometric meaning of the coherence measure. In addition, the measure collects the contribution of all off-diagonal elements of ρ which is consistent with what we have stated previously.

B. Localizable coherence

There exists infinitely many pure-state realizations of a given mixed state. Unlike quantum entanglement of a bipartite quantum state ρ which is defined as the minimal average entanglement with all pure-state realizations of ρ taken into account, in usual it seems not to be meaningful to define the average coherence of a mixed state by considering the different pure-state realizations. However, it is not the case if we have known that ρ_A owned by Alice was reduced from a bipartite state ϱ_{AB} shared with Bob, i.e., $\rho_A=\mathrm{Tr}_B\,\varrho_{AB}$. Based on Gisin—Hughston—Jozsa—Wootters theorem [32], any pure-state realization of ρ_A can be obtained by appropriate positive operator valued measure performed on subsystem B [33]. Therefore, if Bob informs Alice of the measurement outcomes via classical communication, Alice can obtain the corresponding pure state $|\phi_i\rangle$ with probability p_i. In other words, Alice will obtain the corresponding coherence $D(|\phi_i\rangle)$ with probability p_i. Averagely, the coherence that Alice can obtain should be given by

$$\bar{D}(\rho_A) = \sum_i p_i D(|\phi_i\rangle). \quad (2)$$

In this case, $D(\rho_A)$ defined in Eq. (1) is called local coherence because it describes the coherence of the local subsystem A in contrast to the whole composite system ϱ_{AB}, and the average coherence given in Eq. (2) can also be called localized coherence because the average coherence is generated based on Bob's assistance.

Definition 2. The localizable coherence of ρ_A is defined as the maximal average coherence with all possible pure-state realizations taken into account, i.e.,

$$D_L(\rho_A) = \max \bar{D}(\rho_A). \quad (3)$$

It is implied in the definition that one can distinguish the different pure-state realizations with the help of LOCC between the two components A and B of the composite quantum system ϱ_{AB}.

C. Relation between coherence and concurrence

Theorem 1. Suppose $\mathcal{E}=\{p_i,|\psi_i\rangle\}$ is a potential pure-state realization of a quantum state of qubit ρ, then the coherence measure

$$D(\rho) = \sum_{i\neq j} |\rho_{ij}| = \min_{\mathcal{E}} \bar{D}(\rho_A) = \min_{\mathcal{E}} \sum_i p_i D(|\psi_i\rangle) = \lambda_1 - \lambda_2 \quad (4)$$

and the localizable coherence

$$D_L(\rho) = \max_{\mathcal{E}} \sum_i p_i D(|\psi_i\rangle) = \lambda_1 + \lambda_2, \quad (5)$$

where λ_i is the square root of the eigenvalues of $\rho\sigma_x\rho^*\sigma_x$ and $\sigma_x=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

Proof. At first, by a simple algebra, one can easily find that Eq. (1) can be rewritten as $D(|\psi\rangle)=|\langle\psi^*|\sigma_x|\psi\rangle|$ for a pure state of qubit. Thus for a mixed state $\rho=\Sigma p_i|\psi_i\rangle\langle\psi_i|$, the average coherence can be given by

$$\bar{D}(\rho) = \sum_i p_i |\langle\psi_i^*|\sigma_x|\psi_i\rangle|. \quad (6)$$

Considering the matrix notation $\rho=\Psi W\Psi^\dagger$, where the columns of Ψ correspond to $|\psi_i\rangle$ and W is a diagonal matrix with diagonal entries corresponding to p_i, one can find that

$$\bar{D}(\rho) = \sum_i |W^{1/2}\Psi^T\sigma_x\Psi W^{1/2}|_{ii}, \quad (7)$$

with superscript T denoting transpose operation. Based on the eigenvalue decomposition: $\rho=\Phi M\Phi^\dagger$, where the columns of Φ correspond to the eigenvectors and M is a diagonal matrix with diagonal entries corresponding to the eigenvalues, it is easily find that $W^{1/2}\Psi=U^T\Phi^T M^{1/2}$, with $UU^\dagger=\mathbf{1}$ and $\mathbf{1}$ as the identity. Thus Eq. (7) can be rewritten as

$$\bar{D}(\rho) = \sum_i |U^T M^{1/2}\Phi^T\sigma_x\Phi M^{1/2}U|_{ii}. \quad (8)$$

The minimal and maximal (localizable coherence) average coherence can be directly calculated from Eq. (8) based on Thompson theorem [34,35] and Ref. [36]. In this way, we have

$$\bar{D}^{\min}(\rho) = \min_U \sum_i |U^T M^{1/2}\Phi^T\sigma_x\Phi M^{1/2}U|_{ii} = \lambda_1 - \lambda_2 \quad (9)$$

and

$$D_L(\rho) = \max_U \sum_i |U^T M^{1/2}\Phi^T \sigma_x \Phi M^{1/2} U|_{ii} = \lambda_1 + \lambda_2, \quad (10)$$

where λ_i is the singular values of matrix $M^{1/2}\Phi^T\sigma_x\Phi M^{1/2}$ in decreasing order or the square roots of the eigenvalues of $\rho\sigma_x\rho^*\sigma_x$.

In order to explicitly show the $\bar{D}^{\min}(\rho)$ and $D_L(\rho)$, we suppose $\rho=\left(\begin{smallmatrix} a & b^* \\ b & c\end{smallmatrix}\right)$, where a and $c=1-a$ are real and $ac-|b|^2\geq 0$ due to the positive ρ. From Eq. (1), it is obvious that the coherence of ρ is $D(\rho)=2|b|$. Substitute ρ into Eq. (9), we can obtain that

$$\rho\sigma_x\rho^*\sigma_x = \begin{pmatrix} ac+|b|^2 & 2ab^* \\ 2bc & ac+|b|^2 \end{pmatrix}. \quad (11)$$

The eigenvalue equation of $\rho\sigma_x\rho^*\sigma_x$ can be given by

$$\Lambda^2 - 2(ac+|b|^2)\Lambda + (ac-|b|^2)^2 = 0. \quad (12)$$

Thus based on Vieta's theorem [37], one can easily find that

$$\bar{D}^{\min}(\rho) = \lambda_1 - \lambda_2 = 2|b| = D(\rho) \quad (13)$$

and

$$D_L(\rho) = \lambda_1 + \lambda_2 = 2\sqrt{ac}. \quad (14)$$

In particular, Eq. (13) shows that $\bar{D}^{\min}(\rho)$ is exactly the same as the coherence of ρ. In this sense, we can redescribe the coherence of ρ as the minimal average coherence. ■

Theorem 2. For a bipartite $(2\otimes n)$-dimensional quantum pure state $|\varphi\rangle_{AB}$ with $\rho_A=Tr_B|\varphi\rangle_{AB}\langle\varphi|$ defined in two dimension, the concurrence $C(|\varphi\rangle_{AB})$ of $|\varphi\rangle_{AB}$ satisfies

$$C^2(|\varphi\rangle_{AB}) = D_L^2(\rho_A) - D^2(\rho_A). \quad (15)$$

Proof. Suppose the reduced density matrix of the bipartite pure state $|\varphi\rangle_{AB}$ is given by

$$\rho_A = \mathrm{Tr}_B|\varphi\rangle_{AB}\langle\varphi| = \begin{pmatrix} a & b^* \\ b & c \end{pmatrix}, \quad (16)$$

then the concurrence of $|\varphi\rangle_{AB}$ is defined [38] as

$$C(|\varphi\rangle_{AB}) = \sqrt{2(1-\mathrm{Tr}\,\rho_A^2)}. \quad (17)$$

Substitute Eq. (16) into Eq. (17), one can have

$$C(|\varphi\rangle_{AB}) = \sqrt{4(ac-|b|^2)}. \quad (18)$$

Based on Eqs. (13) and (14), it is obvious that

$$D_L^2(\rho_A) - D^2(\rho_A) = 4(ac-|b|^2). \quad (19)$$

Therefore, Eq. (15) holds. ■

III. QUANTUM COHERENCE OF QUDIT AND CONCURRENCE OF GENERAL BIPARTITE PURE STATES

A. Quantification of coherence and localizable coherence for a qudit

In order to study the previous question for high-dimensional quantum states, the key question is how to generalize the coherence measure and the average coherence of quantum qubit states. The discussion in Sec. II provides a direct understanding of average coherence, especially for qubit systems. In a different matter, we can give an alternative understanding to average coherence of high-dimensional quantum system. Since coherence is closely related to the nonzero off-diagonal elements, it requires at least two levels for a given quantum system (for example, the excited and ground states of an atom) in order to demonstrate the coherence. In other words, a two-level system can be considered as the minimal unit in researching coherence, which just corresponds to two off-diagonal elements of density matrix in terms of definition 1. In this sense, if ρ_{AB} is shared by Alice and Bob, Alice can first be concerned about the coherence with respect to the given basis in some 2×2 subspace and then collect all the contributions of different subspace.

For an n-dimensional density matrix ρ_A, there exist $N=\frac{n(n-1)}{2}$ alternative 2×2 subspace with all potential choices of two levels under consideration. It happened that each such choice for a density matrix just corresponds to one generator S_i of the group SO(n). Suppose L_i is a $2\times n$ matrix derived from $|S_i|$ ($|\cdot|$ denotes the absolute value of the matrix elements) by deleting the row where all the elements are zero, then the quantum state in each 2×2 subspace (or corresponding to a generator S_i) can be achieved by

$$\rho_i = \frac{L_i\rho_A L_i^\dagger}{\mathrm{Tr}\,L_i\rho_A L_i^\dagger}, \quad (20)$$

where $\mathrm{Tr}(L_i\rho_A L_i^\dagger)$ is normalization factor. The average coherence in the ith subspace can be given by $\bar{D}(\rho_i)$ defined as Eq. (2). Defining an N-dimensional average coherence vector as

$$\mathcal{D}(\rho_A) = [\bar{D}(\rho_1), \bar{D}(\rho_2), \ldots, \bar{D}(\rho_N)] \quad (21)$$

and the corresponding weightlike vector as

$$\mathcal{P}(\rho_A) = [\mathrm{Tr}\,L_1\rho_A L_1^\dagger, \mathrm{Tr}\,L_2\rho_A L_2^\dagger, \ldots, \mathrm{Tr}\,L_N\rho_A L_N^\dagger], \quad (22)$$

then the total average coherence of all subspace can be defined as the length of the weighted vector, i.e.,

$$\bar{D}_F(\rho_A) = \|\mathcal{P}\circ D\|, \quad (23)$$

where $\circ$ denotes the Hadamard product, $\|\cdot\|$ denotes the L_2 norm of a vector, and the subscript F will be explained later. It is obvious that $\bar{D}_F(\rho_A)$ and $\mathcal{D}(\rho_A)$ depend on Bob's operations. In this sense, we can define a vector of maximal average coherence as

$$\mathcal{D}_L(\rho_A) = \|\mathcal{P}\circ\max\,\mathcal{D}\| = [\bar{D}_L(\rho_1), \bar{D}_L(\rho_2), \ldots, \bar{D}_L(\rho_N)], \quad (24)$$

with $\bar{D}_L(\cdot)$ is given by Eq. (3) and max (min) on a vector denotes the maximum (minimum) of every elements of the vector. In terms of Eq. (24) we can analogously define the localizable coherence as follows.

Definition 3. The localizable coherence of ρ_A is defined as the length of the weighted maximal average coherence vector $\mathcal{D}_L(\rho_A)$, i.e.,

$$D_{FL}(\rho_A) = \|\mathcal{P} \circ D_L\|. \tag{25}$$

At the end of this subsection, we would like to emphasize that the generalized coherence measures $\bar{D}_F(\rho_A)$ and $D_{FL}(\rho_A)$ can be reduced to $\bar{D}(\rho_A)$ and $D_L(\rho_A)$, respectively, when ρ_A is a density matrix of a qubit. We have shown that $D(\rho_A) = \min_{\mathcal{E}} \bar{D}(\rho_A)$ for a qubit density matrix ρ_A, the analogous relation with $\bar{D}_F(\rho_A)$ and $D_F(\rho_A)$ taken into account is also satisfied for a high-dimensional ρ_A, which will be proved in the next subsection. In addition, it should be noted that the subscripts F means that $D_F(\rho_A)=\sqrt{\Sigma_{i\neq j}|\rho_{Aij}|^2}$, namely, in definition 1 of coherence measure, we employ Frobenius norm.

B. Relation between coherence and concurrence

Theorem 3. For a quantum state of qudit σ, let $\mathcal{D}(\sigma)$ be the average coherence vector defined as Eq. (21) and $\mathcal{D}_L(\sigma)$ be the maximal average coherence with the corresponding weightlike vector $\mathcal{P}(\sigma)$ defined as Eq. (22). Then the coherence measure $D_F(\sigma)$ can be given by

$$D_F(\sigma) = \sqrt{\sum_{i\neq j} |\sigma_{ij}|^2} = \|\mathcal{P}(\sigma) \circ \min \mathcal{D}(\sigma)\| = \sqrt{\sum_j (\tilde{\lambda}_1^j - \tilde{\lambda}_2^j)^2} \tag{26}$$

and the localizable coherence

$$D_{FL}(\sigma) = \|\mathcal{P}(\sigma) \circ D_L(\sigma)\| = \sqrt{\sum_j (\tilde{\lambda}_1^j + \tilde{\lambda}_2^j)^2}, \tag{27}$$

where $\tilde{\lambda}_k^j$ is the square root of the eigenvalues of $\rho|S_j|\rho^*|S_j|$.

Proof. Let $\mathcal{E}=\{q_i,|\chi_i\rangle\}$ be a potential decomposition of n-dimensional density matrix σ. Substitute $\mathcal{E}$ into Eq. (26) [or Eq. (23)], one can find that

$$\begin{aligned}\mathrm{Tr}\, L_j\sigma L_j^\dagger \bar{D}\left(\frac{L_j\sigma L_j^\dagger}{\mathrm{Tr}\, L_j\sigma L_j^\dagger}\right) &= \sum_i q_i D(L_j|\chi_i\rangle\langle\chi_i|L_j^\dagger)\\ &= \sum_i q_i |\langle\chi_i^*|L_j^T\sigma_x L_j|\chi_i\rangle|\\ &= \sum_i |\tilde{U}^T\tilde{M}^{1/2}\tilde{\Phi}^T|S_j|\tilde{\Phi}\tilde{M}^{1/2}\tilde{U}|_{ii},\end{aligned} \tag{28}$$

where $\tilde{U}\tilde{U}^\dagger=1$ by which any decomposition of $\sigma=\tilde{\Psi}\tilde{W}\tilde{\Psi}^\dagger$ is related to the eigenvalue decomposition $\sigma=\tilde{\Phi}\tilde{M}\tilde{\Phi}^\dagger$. Based on Thompson theorem and Ref. [36], one can find that

$$\mathrm{Tr}\, L_j\sigma L_j^\dagger \min_{\mathcal{E}} \bar{D}\left(\frac{L_j\sigma L_j^\dagger}{\mathrm{Tr}\, L_j\sigma L_j^\dagger}\right) = \tilde{\lambda}_1^j - \tilde{\lambda}_2^j, \tag{29}$$

$$\mathrm{Tr}\, L_j\sigma L_j^\dagger \max_{\mathcal{E}} \bar{D}\left(\frac{L_j\sigma L_j^\dagger}{\mathrm{Tr}\, L_j\sigma L_j^\dagger}\right) = \tilde{\lambda}_1^j + \tilde{\lambda}_2^j, \tag{30}$$

where $\tilde{\lambda}_k^j$ is the square root of the eigenvalues of $\sigma|S_j|\sigma^*|S_j|$ in decreasing order. In Eqs. (29) and (30), it should be emphasized that $\sigma|S_j|\sigma^*|S_j|$ has only two nonzero eigenvalues $(\tilde{\lambda}_1^j$ and $\tilde{\lambda}_2^j)$ since the nonzero block of $\sigma|S_j|\sigma^*|S_j|$ is completely the same as $L_j\sigma L_j^\dagger\sigma_x L_j\sigma^* L_j^T\sigma_x$. Thus

$$\|\mathcal{P} \circ \min \mathcal{D}\| = \sqrt{\sum_j (\tilde{\lambda}_1^j - \tilde{\lambda}_2^j)^2}, \tag{31}$$

$$\|\mathcal{P} \circ \max \mathcal{D}\| = \|\mathcal{P}(\sigma) \circ \mathcal{D}_L(\sigma)\| = \sqrt{\sum_j (\tilde{\lambda}_1^j + \tilde{\lambda}_2^j)^2}. \tag{32}$$

In fact, one can find that for each L_j, $L_j\sigma L_j^\dagger$ can be written by

$$L_j\sigma L_j^\dagger = \begin{pmatrix} \sigma_{kk} & \sigma_{kl} \\ \sigma_{kl}^* & \sigma_{ll} \end{pmatrix}, \tag{33}$$

where σ_{kk} and σ_{ll} are the kth and lth diagonal elements of σ and σ_{kl} is the off-diagonal element of σ subject to the two diagonal elements. Analogous to the proof of theorem 1, one can find that

$$\tilde{\lambda}_1^j - \tilde{\lambda}_2^j = 2|\sigma_{kl}|,$$

$$\tilde{\lambda}_1^j + \tilde{\lambda}_2^j = 2\sqrt{\sigma_{kk}\sigma_{ll}}. \tag{34}$$

Since each pair of off-diagonal elements of σ corresponds to a L_j, the contribution of all the off-diagonal elements can be described as

$$D_F(\sigma) = \sqrt{\sum_{i\neq j} |\sigma_{ij}|^2} = \sqrt{\sum_j (\tilde{\lambda}_1^j - \tilde{\lambda}_2^j)^2} = \|\mathcal{P} \circ \min \mathcal{D}\|. \tag{35}$$

Equations (31), (32), and (35) show that this theorem holds. ∎

Theorem 4. For a bipartite $(n_1\otimes n_2)$-dimensional quantum pure state $|\eta\rangle_{AB}$ with $\sigma_A=\mathrm{Tr}_B|\eta\rangle_{AB}\langle\eta|$ defined in n_1 dimension, the concurrence $C(|\eta\rangle_{AB})$ of $|\eta\rangle_{AB}$ satisfies

$$C^2(|\eta\rangle_{AB}) = D_{FL}^2(\sigma_A) - D_F^2(\sigma_A). \tag{36}$$

Proof. Since the concurrence of $|\eta\rangle_{AB}$ is defined as Eq. (17), based on σ_{ij} (the entries of σ_A), $C(|\eta\rangle_{AB})$ can be rewritten by

$$C(|\eta\rangle_{AB}) = \sqrt{4\sum_{ij} (\sigma_{ii}\sigma_{jj} - |\sigma_{ij}|^2)}. \tag{37}$$

According to Eqs. (31) and (32), we have

$$D_{FL}^2(\sigma_A) - D_F^2(\sigma_A) = \sum_j (\tilde{\lambda}_1^j + \tilde{\lambda}_2^j)^2 - \sum_j (\tilde{\lambda}_1^j - \tilde{\lambda}_2^j)^2. \tag{38}$$

Substitute Eq. (34) into Eq. (36), one can find that

$$D_{FL}^2(\sigma_A) - D_F^2(\sigma_A) = 4\sum_{kl} (\sigma_{kk}\sigma_{ll} - |\sigma_{kl}|^2). \tag{39}$$

Comparing Eq. (37) with Eq. (39), one can conclude that Eq. (36) holds. ∎

IV. QUANTUM COHERENCE AND BIPARTITE CONCURRENCE OF MIXED STATES

In this section, we will show that theorems 2 and 4 can be extended to bipartite mixed states. For a bipartite mixed state σ_{AB}, one can always introduce an auxiliary system C such that $|\psi\rangle_{ABC}$ is a purification of σ_{AB}. If subsystem A is two dimensional based on theorem 2 one can obtain

$$C^2(|\psi\rangle_{A(BC)}) = D_L^2(\sigma_A) - D^2(\sigma_A). \tag{40}$$

If subsystem A is more than two dimensional based on theorem 4 one can obtain

$$C^2(|\psi\rangle_{A(BC)}) = D_{FL}^2(\sigma_A) - D_F^2(\sigma_A). \tag{41}$$

In Eqs. (40) and (41), $\sigma_A = \mathrm{Tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$. Since concurrence $C(|\psi\rangle_{A(BC)})$ is an entanglement monotone—it does not increase under LOCC [18,19] and σ_{AB} can always be obtained from $|\psi\rangle_{A(BC)}$ by local operations on subsystem C, one has

$$C(|\psi\rangle_{A(BC)}) \geq C(\sigma_{AB}). \tag{42}$$

Thus we can have the following theorem.

Theorem 5. For bipartite mixed state σ_{AB}, if subsystem A is two dimensional, the concurrence satisfies

$$C^2(\sigma_{AB}) \leq D_L^2(\sigma_A) - D^2(\sigma_A), \tag{43}$$

otherwise,

$$C^2(\sigma_{AB}) \leq D_{FL}^2(\sigma_A) - D_F^2(\sigma_A). \tag{44}$$

V. CONCLUSION AND DISCUSSION

In summary, we have shown that the local coherence based on a proposed coherence measure can be understood as the minimal average coherence with all potential pure-state realizations taken into account. In particular, we have revealed the relation between the local coherence including localizable coherence and bipartite concurrence of pure states which provides an alternative operational meaning for concurrence of pure states. In addition, it is also shown that the relation can also be extended to the case of bipartite mixed state.

Before the end, we would like to briefly discuss the potential applications of our relations. As mentioned in Sec. I, a lot of works have been done to study disentanglement and local decoherence by considering different $(2\otimes 2)$-dimensional physical models. However, for high-dimensional quantum systems, there does not generally exist an analytic entanglement measure which greatly limits the relevant researches. It can be easily found that the coherence measures presented in this paper can be analytically calculated, in particular the relations given in theorem 5 provide an upper bound of concurrence, therefore, one can find that a sufficient condition of disentanglement can be provided by local decoherence.

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$=\Sigma_i p_i \sigma_A^i$, $C(|\phi\rangle_{AB}) = \sqrt{2(1-\mathrm{Tr}[\Sigma_i p_i \sigma_A^i]^2)} \geq \Sigma_i p_i \sqrt{2(1-\mathrm{Tr}[\sigma_A^i]^2)}$ based on the convacity, where the equality holds if and only σ_A^i is the same for any i, but this condition is trivial. In particular, when σ_A^i is pure for all i, the above inequality only provides a trivial lower bound 0. In this sense, one cannot find the explicit role of subsystem B.

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