Entangled state for constructing a generalized phase-space representation and its statistical behavior

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We construct a generalized phase-space representation (GPSR) based on the idea of Einstein-Podolsky-Rosen quantum entanglement, i.e., we generalize the Torres-Vega-Frederick phase-space representation to the entangled case, which is characteristic of the features when two particles' relative coordinate, total momentum operators, and their conjugative variables, respectively, operate on the GPSR. This representation is complete and nonorthogonal. The Weyl-ordered form of the density operator of GPSR is derived, and its identification with the generalized Husimi operator is recognized, which clearly exhibit its statistical behavior. The minimum uncertainty relation obeyed by the GPSR is also demonstrated.

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I. INTRODUCTION

Phase-space formalism of quantum mechanics began with Wigner's celebrated paper [1] in 1932, since then the (generalized) phase-space techniques have found useful applications in various branches of physics [2–5]. The main idea of this formalism is to represent the density operator as a quasidistribution function over the classical phase-space (q,p). Phase-space formalism implies that there is no a unique way to represent a quantum state as a wave function [in the standard approach one usually uses the coordinate $\psi(q)$ or the momentum $\psi(p)$ representations]. By observing this, Torres-Vega and Frederick (TF) constructed a quantum mechanical phase-space representation in 1993 [6], $\psi=\psi(\Gamma)$, where Γ denotes a point in the (q,p) space characteristic of parameters $(\alpha,\beta,\gamma,\delta)$, using it the standard Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(q,t)=\left[-\frac{1}{2m}\frac{\partial^2}{\partial q^2}+V(q)\right]\psi(q,t),\hbar=1 \tag{1}$$

changes to the following quantum Liouville equation in phase-space

$$i\frac{\partial}{\partial t}\psi(\Gamma,t)=\left[-\frac{1}{2m}\frac{\partial^2}{\partial q^2}+V\left(q+i\frac{\partial}{\partial p}\right)\right]\psi(\Gamma,t), \tag{2}$$

where $\psi(\Gamma)$ obtained from $\psi(q)$ via $\psi(\Gamma)=\int_{-\infty}^{\infty}dq\langle\Gamma|q\rangle\psi(q)$, i.e., Torres-Vega and Frederick employed Dirac symbol to associate Eq. (2) with the state vector $|\Gamma\rangle$. The bra $\langle\Gamma|$ exhibits the remarkable features when coordinate operator Q and momentum operator P, respectively, operates on it, i.e.,

$$\langle\Gamma|Q=\left(\alpha q+i\beta\frac{\partial}{\partial p}\right)\langle\Gamma|, \tag{3}$$

$$\langle\Gamma|P=\left(\gamma p+i\delta\frac{\partial}{\partial q}\right)\langle\Gamma|, \tag{4}$$

so $\langle\Gamma|$ is a state of phase-space variables (q,p); α, β, γ, and δ are all real number parameters specifying a whole family of $\langle\Gamma|$ states, satisfying

$$\beta\gamma-\alpha\delta=1. \tag{5}$$

It has been demonstrated that this representation best fits the correspondence between the classical and quantum Liouville equations and have wide applications [6–12].

To make the TF theory complete, in Ref. [13], we have found the explicit form of $|\Gamma\rangle$ in Fock space,

$$|\Gamma\rangle\equiv(2\sqrt{-\alpha\beta\gamma\delta})^{1/2}\times\exp\left[\frac{\alpha q^2}{2\delta}-\frac{\gamma p^2}{2\beta}+\sqrt{2}(\alpha q+i\gamma p)a_1^\dagger+\frac{\beta\gamma+\alpha\delta}{2}a_1^{\dagger 2}\right]|0\rangle_1, \tag{6}$$

where $|0\rangle_1$ is the single-mode vacuum state, a_1 and $a_1^\dagger$ are the Bose annihilation and creation operators, respectively, obeying $[a_1,a_1^\dagger]=1$. By using the technique of integration within an ordered product (IWOP) of operators [14,15], we have shown that $|\Gamma\rangle$ is a special coherent squeezed state which makes up a quantum mechanical representation. $|\Gamma\rangle$ contains elements of both the momentum eigenstate and the coordinate eigenstate. Remarkably, $|\Gamma\rangle\langle\Gamma|$ reduces to the Husimi operator for some special values of $(\alpha,\beta,\gamma,\delta)$.

When we move to tackling quantum entanglement in two degree of freedom (DOF) case, enlightened by the paper of Einstein-Podolsky-Rosen (EPR) in 1935 [16], who noticed that two particles' relative coordinate Q_1-Q_2 and total momentum P_1+P_2 are commutative and can be simultaneously measured, we are naturally led to consider first the common eigenvector $\langle\eta|$ of the relative coordinate operator Q_1-Q_2 and the momentum sum operator P_1+P_2, as well as the common eigenvector $\langle\xi|$ of their conjugative variables P_1-P_2 and Q_1+Q_2, since $[Q_1+Q_2,P_1+P_2]=2i$ and $[Q_1-Q_2,P_1-P_2]=2i$. Correspondingly, because in the entangled case only those states simultaneously describing two entangled particles can be endowed with physical meaning, phase-space should be understood with regard to $\langle\eta|$ and $\langle\xi|$. Besides, $|\eta=re^{i\varphi}\rangle$ also obeys the eigenequations

$$\frac{Q_1-Q_2}{\sqrt{(Q_1-Q_2)^2+(P_1+P_2)^2}}|\eta=re^{i\varphi}\rangle=\cos\varphi|\eta=re^{i\varphi}\rangle,$$

and

$$\frac{P_1+P_2}{\sqrt{(Q_1-Q_2)^2+(P_1+P_2)^2}}|\eta=re^{i\varphi}\rangle=\sin\varphi|\eta=re^{i\varphi}\rangle,$$

by noticing $[(Q_1-Q_2)^2+(P_1+P_2)^2]|\eta=re^{i\varphi}\rangle=2r^2|\eta=re^{i\varphi}\rangle$, we can define a phase operator. Therefore we should construct generalized phase-space representation based on two mutually conjugative EPR variables (sum and difference variables). As one can see shortly later, this generalization would result in the development of both the Wigner function and the Husimi function theory for entangled states. It is expected that states that are entangled or partly entangled would form a compact "blob" in the sum and difference phase-space, as opposed to the strongly oscillating Wigner function that they would have in the "old" (Q,P) representation. This would have two useful consequences: (1) it would become much easier to "eyeball" what is going on physically when, for example, some sort of complicated dynamics is going on; and (2) compact distributions are much better for Monte Carlo sampling if one were to make calculations this way.

Recalling the form of $\langle\eta|$ in two-mode Fock space [17]

$$|\eta\rangle=\exp\left\{-\frac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^* a_2^\dagger+a_1^\dagger a_2^\dagger\right\}|00\rangle,\quad \eta=\eta_1+i\eta_2, \tag{7}$$

where a_i, $a_i^\dagger$ $(i=1,2)$ are the two-mode Bose annihilation and creation operators obeying $[a_i,a_j^\dagger]=\delta_{ij}$, we see that $|\eta\rangle$ satisfies the eigenequations:

$$(Q_1-Q_2)|\eta\rangle=\sqrt{2}\eta_1|\eta\rangle,\ \ (P_1+P_2)|\eta\rangle=\sqrt{2}\eta_2|\eta\rangle, \tag{8}$$

and possesses the orthogonal-complete relation

$$\int\frac{d^2\eta}{\pi}|\eta\rangle\langle\eta|=1,\ \ \langle\eta'\,|\eta\rangle=\pi\delta(\eta'-\eta)\delta(\eta'^*-\eta^*). \tag{9}$$

EPR entanglement involved in $|\eta\rangle$ can be clearly seen from its Schmidt decompositions, i.e., $|\eta\rangle=e^{-i\eta_1\eta_2}\int_{-\infty}^{\infty}dq|q\rangle_1\otimes|q-\sqrt{2}\eta_1\rangle_2 e^{i\sqrt{2}\eta_2 q}$, where $|q\rangle_i$ $(i=1,2)$ is the eigenvector of coordinate Q_i. Based on $\langle\eta|$ and $\langle\xi|$ we generalize Eq. (6) to an enlarged phase-space representation ${}_e\langle\Gamma|$, where the subscript "e" implies entanglement. Then how to construct this representation? Similar in spirit to Eqs. (3) and (4) we start with considering what are the four self-consistent and reasonable equations (involving α, β, γ, and δ) in two DOF case. After doing tries, we find that $|\Gamma\rangle_e$ is characteristic of the features when Q_1-Q_2 and P_1+P_2, respectively, operates on it,

$${}_e\langle\Gamma|\frac{Q_1-Q_2}{\sqrt{2}}=\left(\alpha\sigma_1+i\beta\frac{\partial}{\partial\tau_2}\right){}_e\langle\Gamma|,$$

$${}_e\langle\Gamma|\frac{P_1+P_2}{\sqrt{2}}=\left(\alpha\sigma_2-i\beta\frac{\partial}{\partial\tau_1}\right){}_e\langle\Gamma|, \tag{10}$$

where $\sigma_1+i\sigma_2=\sigma$ and $\tau_1+i\tau_2=\tau$ are complex variables indicating the phase-space representation ${}_e\langle\Gamma|$. Simultaneously, under the action of the center-of-mass operator Q_1+Q_2 and the relative momentum operator P_1-P_2, the state ${}_e\langle\Gamma|$ should exhibit

$${}_e\langle\Gamma|\frac{Q_1+Q_2}{\sqrt{2}}=\left(\gamma\tau_1-i\delta\frac{\partial}{\partial\sigma_2}\right){}_e\langle\Gamma|,$$

$${}_e\langle\Gamma|\frac{P_1-P_2}{\sqrt{2}}=\left(\gamma\tau_2+i\delta\frac{\partial}{\partial\sigma_1}\right){}_e\langle\Gamma|. \tag{11}$$

Equations (10) and (11) are the nontrivial extension of Eqs. (3) and (4), so ${}_e\langle\Gamma|$ corresponds to a generalized phase-space representation (GPSR).

The work of extending $\langle\Gamma|$ to the entangled case ${}_e\langle\Gamma|$ is somehow like the extension from the single-mode squeezed state to the two-mode squeezed state (an entangled state too) [18]. In the following we shall derive the explicit form of $|\Gamma\rangle_e$ and further analyze its properties, in so doing, the Torres-Vega-Frederick theory can be developed and enriched.

Our paper is arranged as follows. In Sec. II using the $|\eta\rangle$ representation we shall derive the explicit form of $|\Gamma\rangle_e$ in two-mode Fock space and then analyze its properties. In Sec. III, the completeness relation and nonorthogonality of $|\Gamma\rangle_e$ are demonstrated. In Sec. IV the minimum uncertainty relation of two pairs of quadrature operators in $|\Gamma\rangle_e$ is shown. In Secs. V and VI we derive the Weyl-ordered form of $|\Gamma\rangle_{ee}\langle\Gamma|$, which yields its classical correspondence, and then examine its marginal distributions in "$|\eta\rangle$ direction" and "$|\xi\rangle$ direction." Sections VII and VIII are devoted to the identification of the density operator $|\Gamma\rangle_{ee}\langle\Gamma|$ with the generalized Husimi operator, and to the derivation of Wigner function of $|\Gamma\rangle_e$, respectively.

II. STATE $|\Gamma\rangle_e$ IN TWO-MODE FOCK SPACE

We find that the explicit form of the state $|\Gamma\rangle_e$ in two-mode Fock space is (see the Appendix),

$$|\Gamma\rangle_e\equiv 2\sqrt{-\alpha\beta\gamma\delta}\exp\left[\frac{\alpha|\sigma|^2}{2\delta}-\frac{\gamma|\tau|^2}{2\beta}+(\alpha\sigma+\gamma\tau)a_1^\dagger\right.$$
$$\left.+(\gamma\tau^*-\alpha\sigma^*)a_2^\dagger-(\beta\gamma+\alpha\delta)a_1^\dagger a_2^\dagger\right]|00\rangle, \tag{12}$$

where α, β, γ, and δ satisfy the relation Eq. (5); to satisfy the square integrable condition for wave function in the phase-space of $|\Gamma\rangle_e$, $\frac{\alpha}{\delta}<0$, and $\frac{\gamma}{\beta}>0$ are demanded. From Eq. (12) we see that the representation $|\Gamma\rangle_e$ involves elements of both $|\eta\rangle$ and $|\xi\rangle$ representations, which are both entangled states: i.e., the set of values $\alpha\delta=-1$, $\gamma=0$ gives the standard EPR entangled state $|\Gamma\rangle_e\to|\eta=\alpha\sigma\rangle$ [comparing with Eq. (7)]; while the set $\beta\gamma=1$, $\alpha=0$ yields $|\Gamma\rangle_e\to|\xi=\gamma\tau\rangle$ [comparing with Eq. (39) below]. To certify that Eq. (12) really obeys Eqs. (10) and (11) we operate a_i on $|\Gamma\rangle_e$,

$$a_1|\Gamma\rangle_e=[(\alpha\sigma+\gamma\tau)-(\beta\gamma+\alpha\delta)a_2^\dagger]|\Gamma\rangle_e,$$

$$a_2|\Gamma\rangle_e=[(\gamma\tau^*-\alpha\sigma^*)-(\beta\gamma+\alpha\delta)a_1^\dagger]|\Gamma\rangle_e. \qquad (13)$$

Then noting the relation $Q_i=(a_i+a_i^\dagger)/\sqrt{2}$, $P_i=(a_i-a_i^\dagger)/(\sqrt{2}i)$, and Eq. (5) as well as

$$\frac{\partial}{\partial\sigma}{}_e\langle\Gamma|={}_e\langle\Gamma|\left(\frac{\alpha\sigma^*}{2\delta}-\alpha a_2\right),\frac{\partial}{\partial\sigma^*}{}_e\langle\Gamma|={}_e\langle\Gamma|\left(\frac{\alpha\sigma}{2\delta}+\alpha a_1\right),$$

$$\frac{\partial}{\partial\tau}{}_e\langle\Gamma|={}_e\langle\Gamma|\left(-\frac{\gamma\tau^*}{2\beta}+\gamma a_2\right),\frac{\partial}{\partial\tau^*}{}_e\langle\Gamma|={}_e\langle\Gamma|\left(-\frac{\gamma\tau}{2\beta}+\gamma a_1\right), \qquad (14)$$

we see, for example,

$$\begin{aligned}{}_e\langle\Gamma|\frac{Q_1+Q_2}{\sqrt{2}}&={}_e\langle\Gamma|[-\delta(\alpha a_1+\alpha a_2)-i\alpha\sigma_2+\gamma\tau_1]\\&=\left[\gamma\tau_1+\delta\left(\frac{\partial}{\partial\sigma}-\frac{\partial}{\partial\sigma^*}\right)\right]{}_e\langle\Gamma|\\&=\left(\gamma\tau_1-i\delta\frac{\partial}{\partial\sigma_2}\right){}_e\langle\Gamma|, \qquad (15)\end{aligned}$$

which is the first equation in Eq. (11). In a similar way, we can check that ${}_e\langle\Gamma|$ satisfies the other relations in Eqs. (10) and (11). Using Eqs. (10) and (11) and noticing the commutator $[\frac{Q_1\pm Q_2}{\sqrt{2}},\frac{P_1\pm P_2}{\sqrt{2}}]=i$, we have

$${}_e\langle\Gamma|\left[\frac{Q_1\pm Q_2}{\sqrt{2}},\frac{P_1\pm P_2}{\sqrt{2}}\right]=i(\beta\gamma-\alpha\delta){}_e\langle\Gamma|, \qquad (16)$$

which results in the condition $\beta\gamma-\alpha\delta=1$ as shown in Eq. (5).

III. PROPERTIES OF $|\Gamma\rangle_e$

A. Completeness relation of $|\Gamma\rangle_e$

Next we prove the completeness relation of $|\Gamma\rangle_e$ in Eq. (12). Using the normally ordered form of the vacuum projector

$$|00\rangle\langle 00|=:\exp(-a_1^\dagger a_1-a_2^\dagger a_2):, \qquad (17)$$

where the symbol : : denotes the normal product, which means all the bosonic creation operators are standing on the left of annihilation operators in a monomial of $a^\dagger$ and a [19], and remembering that a normally ordered product of operators can be integrated with respect to c numbers provided the integration is convergent, we can use Eq. (12) and the IWOP technique to perform the following integration

$$\begin{aligned}&\frac{1}{\beta^2\delta^2}\int\frac{d^2\sigma d^2\tau}{4\pi^2}|\Gamma\rangle_{ee}\langle\Gamma|\\&=-\frac{\alpha\gamma}{\beta\delta}\int\frac{d^2\sigma d^2\tau}{\pi^2}:\exp\left[\frac{\alpha|\sigma|^2}{\delta}+\sigma\alpha(a_1^\dagger-a_2)+\sigma^*\alpha(a_1-a_2^\dagger)-a_1^\dagger a_1\right.\\&\left.-\frac{\gamma|\tau|^2}{\beta}+\tau\gamma(a_1^\dagger+a_2)+\tau^*\gamma(a_2^\dagger+a_1)-(\beta\gamma+\alpha\delta)(a_1^\dagger a_2^\dagger+a_1a_2)-a_2^\dagger a_2\right]:=-\frac{\alpha\gamma}{\beta\delta}\int\frac{d^2\sigma d^2\tau}{\pi^2}:\exp\left\{\frac{\alpha}{\delta}[\sigma+\delta(a_1-a_2^\dagger)][\sigma^*\right.\\&\left.+\delta(a_1^\dagger-a_2)]-\frac{\gamma}{\beta}[\tau-\beta(a_2^\dagger+a_1)][\tau^*-\beta(a_1^\dagger+a_2)]\right\}:=:\exp[-(a_1^\dagger a_1+a_2^\dagger a_2)(\alpha\delta-\beta\gamma+1)]:=1, \qquad (18)\end{aligned}$$

where we have used the integral formula [20]

$$\int\frac{d^2\beta}{\pi}e^{\varsigma|\beta|^2+\xi\beta+\eta\beta^*}=-\frac{1}{\varsigma}e^{-\frac{\xi\eta}{\varsigma}},\ \ \mathrm{Re}\,\varsigma<0. \qquad (19)$$

Thus $|\Gamma\rangle_e$ is capable of making up a quantum mechanical representation.

B. Nonorthogonality of $|\Gamma\rangle_e$

Noticing the overlap

$$\begin{aligned}{}_e\langle\Gamma|z_1,z_2\rangle&=2\sqrt{-\alpha\beta\gamma\delta}\exp\left[-\frac{|z_1|^2}{2}+\frac{\alpha|\sigma|^2}{2\delta}-\frac{\gamma|\tau|^2}{2\beta}\right.\\&\left.+(\alpha\sigma^*+\gamma\tau^*)z_1\right]\exp\left[-\frac{|z_2|^2}{2}+(\gamma\tau-\alpha\sigma)z_2\right.\\&\left.-(\beta\gamma+\alpha\delta)z_1z_2\right], \qquad (20)\end{aligned}$$

where $|z\rangle=\exp(-|z|^2/2+za^\dagger)|0\rangle$ is the coherent state [21,22] and using the overcompleteness relation of coherent states $\int\frac{d^2z_1d^2z_2}{\pi^2}|z_1,z_2\rangle\langle z_1,z_2|=1$, we can derive the inner product ${}_e\langle\Gamma|\Gamma'\rangle_e$, ($|\Gamma'\rangle_e$ has the same β, γ, α, and δ as in $|\Gamma\rangle_e$),

$$
\begin{aligned}
{}_e\langle\Gamma|\Gamma'\rangle_e &= \int \frac{d^2z_1d^2z_2}{\pi^2}\,{}_e\langle\Gamma|z_1,z_2\rangle\langle z_1,z_2|\Gamma'\rangle_e \\
&= -4\alpha\beta\gamma\delta\int \frac{d^2z_1d^2z_2}{\pi^2}\exp[-|z_1|^2+(\alpha\sigma^*+\gamma\tau^*)z_1 \\
&\quad +(\alpha\sigma'+\gamma\tau')z_1^*]-|z_2|^2+(\gamma\tau-\alpha\sigma)z_2+(\gamma\tau'^* \\
&\quad -\alpha\sigma'^*)z_2^*-(\beta\gamma+\alpha\delta)z_1z_2+\frac{\alpha}{2\delta}(|\sigma|^2+|\sigma'|^2) \\
&\quad -\frac{\gamma}{2\beta}(|\tau|^2+|\tau'|^2)-(\beta\gamma+\alpha\delta)z_1^*z_2^*\Bigg].
\end{aligned}
\tag{21}
$$

With the aid of the integral formula in Eq. (19), we perform the integral over $d^2z_1d^2z_2$ in Eq. (21) and finally obtain

$$
\begin{aligned}
{}_e\langle\Gamma|\Gamma'\rangle_e = \exp\Bigg[&\frac{\alpha}{4\beta\gamma\delta}|\sigma-\sigma'|^2 \\
&-\frac{1}{4\beta\delta}(\tau'\sigma^*-\sigma\tau'^*+\sigma'\tau^*-\tau\sigma'^*)+\frac{\gamma}{4\alpha\beta\delta}|\tau \\
&-\tau'|^2-\frac{\beta\gamma+\alpha\delta}{4\beta\delta}(\tau'\sigma'^*-\sigma'\tau'^*+\sigma\tau^*-\tau\sigma^*)\Bigg].
\end{aligned}
\tag{22}
$$

From Eq. (22) one can see that ${}_e\langle\Gamma|\Gamma'\rangle_e$ is nonorthogonal, only when $\sigma=\sigma'$ and $\tau=\tau'$ Eq. (22) reduces to ${}_e\langle\Gamma|\Gamma\rangle_e=1$.

IV. MINIMUM UNCERTAINTY RELATION FOR $|\Gamma\rangle_e$

Due to Heisenberg's uncertainty principle, it is impossible that $|\Gamma\rangle_e$ is the simultaneous eigenvectors of both (Q_1-Q_2,P_1+P_2) and (Q_1+Q_2,P_1-P_2). Note that $|\eta\rangle$ and $|\xi\rangle$ are related to each other by

$$
\langle\xi|\eta\rangle=\frac{1}{2}\exp\left(\frac{\xi^*\eta-\xi\eta^*}{2}\right),
\tag{23}
$$

we see

$$
\begin{aligned}
\langle\eta|\Gamma\rangle_e &= \int\frac{d^2\xi}{\pi}\langle\eta|\xi\rangle\langle\xi|\Gamma\rangle_e=\int\frac{d^2\xi}{2\pi}e^{(\xi\eta^*-\xi^*\eta)/2}\langle\xi|\Gamma\rangle_e, \\
\langle\xi|\Gamma\rangle_e &= \int\frac{d^2\eta}{\pi}\langle\xi|\eta\rangle\langle\eta|\Gamma\rangle_e=\int\frac{d^2\eta}{2\pi}e^{(\xi^*\eta-\xi\eta^*)/2}\langle\eta|\Gamma\rangle_e,
\end{aligned}
\tag{24}
$$

which are conjugated each other. It then follows from Eq. (24) that once the value of (Q_1-Q_2,P_1+P_2) has been measured, one can find the system with any value for (Q_1+Q_2,P_1-P_2), and vice versa.

In order to see clearly how the state $|\Gamma\rangle_e$ obeys uncertainty relation, we introduce two pairs of quadrature phase amplitudes for two-mode field

$$
Q_\pm=\frac{Q_1\pm Q_2}{\sqrt{2}},\quad P_\pm=\frac{P_1\pm P_2}{\sqrt{2}},\quad [Q_\pm,P_\pm]=i.
\tag{25}
$$

In similar to deriving Eq. (22), using Eqs. (7), (12), and (39) (see below), we calculate the overlap between $\langle\eta|$ and $|\Gamma\rangle_e$,

$$
\begin{aligned}
\langle\eta|\Gamma\rangle_e = \sqrt{-\frac{\alpha\delta}{\beta\gamma}}\exp\Bigg\{&\frac{\alpha\delta}{2\beta\gamma}\left|\frac{\sigma}{\delta}+\eta\right|^2 \\
&+\frac{1}{2\beta}[\tau(\eta^*-\alpha\sigma^*)-\tau^*(\eta-\alpha\sigma)]\Bigg\},
\end{aligned}
\tag{26}
$$

and the overlap between $\langle\xi|$ and $|\Gamma\rangle_e$

$$
\begin{aligned}
\langle\xi|\Gamma\rangle_e = \sqrt{-\frac{\beta\gamma}{\alpha\delta}}\exp\Bigg\{&\frac{\beta\gamma}{2\alpha\delta}\left|\frac{\tau}{\beta}-\xi\right|^2 \\
&-\frac{1}{2\delta}[\sigma(\xi^*-\gamma\tau^*)-\sigma^*(\xi-\gamma\tau)]\Bigg\}.
\end{aligned}
\tag{27}
$$

Then employing the completeness relation of $|\eta\rangle$ and Eq. (26), we have

$$
\begin{aligned}
\langle Q_-\rangle &= \int\frac{d^2\eta}{\pi}\eta_1|\langle\eta|\Gamma\rangle_e|^2=-\frac{\sigma_1}{\delta}, \\
\langle Q_-^2\rangle &= \int\frac{d^2\eta}{\pi}\eta_1^2|\langle\eta|\Gamma\rangle_e|^2=\frac{\sigma_1^2}{\delta^2}-\frac{\beta\gamma}{2\alpha\delta},
\end{aligned}
\tag{28}
$$

which leads to

$$
\langle P_-\rangle=\frac{\tau_2}{\beta},\quad \langle P_-^2\rangle=\frac{\tau_2^2}{\beta^2}-\frac{\alpha\delta}{2\beta\gamma}.
\tag{29}
$$

It then follows

$$
\begin{aligned}
\langle\Delta Q_-^2\rangle &= \langle Q_-^2\rangle-\langle Q_-\rangle^2=-\frac{\beta\gamma}{2\alpha\delta}, \\
\langle\Delta P_-^2\rangle &= \langle P_-^2\rangle-\langle P_-\rangle^2=-\frac{\alpha\delta}{2\beta\gamma},
\end{aligned}
\tag{30}
$$

which yields

$$
\sqrt{\langle\Delta Q_-^2\rangle\langle\Delta P_-^2\rangle}=\frac{1}{2}.
\tag{31}
$$

Similarly, using Eq. (27) we can derive

$$
\begin{aligned}
&\langle Q_+\rangle=\frac{\sigma_2}{\delta},\quad \langle Q_+^2\rangle=\frac{\sigma_2^2}{\delta^2}-\frac{\beta\gamma}{2\alpha\delta}, \\
&\langle P_+\rangle=\frac{\tau_1}{\beta},\quad \langle P_+^2\rangle=\frac{\tau_1^2}{\beta^2}-\frac{\alpha\delta}{2\beta\gamma},
\end{aligned}
\tag{32}
$$

and

$$
\sqrt{\langle\Delta Q_+^2\rangle\langle\Delta P_+^2\rangle}=\frac{1}{2}.
\tag{33}
$$

Equations (30)–(33) show that $|\Gamma\rangle_e$ is a minimum uncertainty state for the two pairs of quadrature operators.

V. WEYL-ORDERED FORM OF $|\Gamma\rangle_{ee}\langle\Gamma|$

For a bipartite operator $\hat{A}$, we can convert it into its Weyl ordering form by using the formula [23–25]

$$\hat{A}=4\int\frac{d^2z_1d^2z_2}{\pi^2}\,\vdots\,\langle-z_1,-z_2|\hat{A}|z_1,z_2\rangle\exp[2(a_1^\dagger a_1+a_1z_1^*-z_1a_1^\dagger)+2(a_2^\dagger a_2+a_2z_2^*-z_2a_2^\dagger)]\,\vdots, \quad (34)$$

where the symbol $\vdots\;\vdots$ denotes the Weyl ordering, $|z_i\rangle$ is the coherent state, $\langle -z_i|z_i\rangle=\exp\{-2|z_i|^2\}$. Note that the order of Bose operators a_i and $a_i^\dagger$ within a Weyl-ordered product can be permuted. That is to say, even though $[a,a^\dagger]=1$, we can have $\vdots aa^\dagger\vdots=\vdots a^\dagger a\vdots$. Substituting Eq. (12) into Eq. (34) and performing the integration by virtue of the technique of integration within a Weyl ordered product (IWWOP) of operators [26], we finally obtain

$$\begin{aligned}|\Gamma\rangle_{ee}\langle\Gamma|&=-16\alpha\beta\gamma\delta\int\frac{d^2z_1d^2z_2}{\pi^2}\,\vdots\exp\Big[-|z_1|^2+(\sigma^*\alpha+\tau^*\gamma-2a_1^\dagger)z_1+(2a_1-\sigma\alpha-\tau\gamma)z_1^*-|z_2|^2+(\tau\gamma-\sigma\alpha-2a_2^\dagger)z_2\\&\quad+(2a_2-\tau^*\gamma+\sigma^*\alpha)z_2^*-(\beta\gamma+\alpha\delta)(z_1^*z_2^*+z_1z_2)+\frac{\alpha|\sigma|^2}{\delta}-\frac{\gamma|\tau|^2}{\beta}+2a_1^\dagger a_1+2a_2^\dagger a_2\Big]\vdots\\&=4\,\vdots\exp\left\{\frac{\alpha\delta}{\beta\gamma}\left[\frac{\sigma}{\delta}+(a_1-a_2^\dagger)\right]\left[\frac{\sigma^*}{\delta}+(a_1^\dagger-a_2)\right]+\frac{\gamma\beta}{\alpha\delta}\left[\frac{\tau}{\beta}-(a_1+a_2^\dagger)\right]\left[\frac{\tau^*}{\beta}-(a_2+a_1^\dagger)\right]\right\}\vdots,\end{aligned} \quad (35)$$

or

$$|\Gamma\rangle_{ee}\langle\Gamma|=4\,\vdots\exp\left\{\frac{\alpha\delta}{\beta\gamma}\left[\left(\frac{\sigma_1}{\delta}+\frac{Q_1-Q_2}{\sqrt{2}}\right)^2+\left(\frac{\sigma_2}{\delta}+\frac{P_1+P_2}{\sqrt{2}}\right)^2\right]+\frac{\beta\gamma}{\alpha\delta}\left[\left(\frac{\tau_1}{\beta}-\frac{Q_1+Q_2}{\sqrt{2}}\right)^2+\left(\frac{\tau_2}{\beta}-\frac{P_1-P_2}{\sqrt{2}}\right)^2\right]\right\}\vdots, \quad (36)$$

which is the Weyl ordering form of $|\Gamma\rangle_{ee}\langle\Gamma|$. We should notice the difference between Eq. (35) and Eq. (18), since they are in different operator ordering.

VI. MARGINAL DISTRIBUTIONS OF $|\Gamma\rangle_{ee}\langle\Gamma|$

The merit of Weyl ordering lies in the Weyl ordering invariance under similar transformations [27], which brings convenience for us to obtain the marginal distributions of $|\Gamma\rangle_{ee}\langle\Gamma|$. From the Weyl-ordered form Eq. (36) we obtain the marginal distributions of $|\Gamma\rangle_{ee}\langle\Gamma|$,

$$\int\frac{d^2\sigma}{\pi}|\Gamma\rangle_{ee}\langle\Gamma|=-\frac{4\beta\gamma\delta}{\alpha}\,\vdots\exp\left\{\frac{\beta\gamma}{\alpha\delta}\left[\left(\frac{\tau_1}{\beta}-\frac{Q_1+Q_2}{\sqrt{2}}\right)^2+\left(\frac{\tau_2}{\beta}-\frac{P_1-P_2}{\sqrt{2}}\right)^2\right]\right\}\vdots. \quad (37)$$

Noting $[Q_1+Q_2,P_1-P_2]=0$, there is no operator ordering problem involved in Eq. (37), so the symbol $\vdots\;\vdots$ in Eq. (37) can be neglected, i.e.,

$$\int\frac{d^2\sigma}{\pi}|\Gamma\rangle_{ee}\langle\Gamma|=-\frac{4\beta\gamma\delta}{\alpha}\exp\left\{\frac{\beta\gamma}{\alpha\delta}\left[\left(\frac{\tau_1}{\beta}-\frac{Q_1+Q_2}{\sqrt{2}}\right)^2+\left(\frac{\tau_2}{\beta}-\frac{P_1-P_2}{\sqrt{2}}\right)^2\right]\right\}. \quad (38)$$

By using the simultaneous eigenstate $|\xi\rangle$ of the commutative operators (Q_1+Q_2,P_1-P_2) in two-mode Fock space [17]

$$|\xi\rangle=\exp\left\{-\frac{1}{2}|\xi|^2+\xi a_1^\dagger+\xi^*a_2^\dagger-a_1^\dagger a_2^\dagger\right\}|00\rangle,\quad \xi=\xi_1+i\xi_2, \quad (39)$$

which satisfies the eigenequations $(Q_1+Q_2)|\xi\rangle=\sqrt{2}\xi_1|\xi\rangle$, $(P_1-P_2)|\xi\rangle=\sqrt{2}\xi_2|\xi\rangle$, and the complete-orthogonal relation,

$$\int\frac{d^2\xi}{\pi}|\xi\rangle\langle\xi|=1,\quad d^2\xi=d\xi_1d\xi_2, \quad (40)$$

$$\langle\xi'|\xi\rangle=\pi\delta(\xi'-\xi)\delta(\xi'^*-\xi^*), \quad (41)$$

we see that the marginal distribution of function $|_e\langle\Gamma|\Psi\rangle|^2$ in "ξ direction" is given by

$$\begin{aligned}\langle\Psi|\int\frac{d^2\sigma}{\pi}|\Gamma\rangle_{ee}\langle\Gamma|\Psi\rangle&=\langle\Psi|\int\frac{d^2\xi}{\pi}|\xi\rangle\\&\quad\times\langle\xi|\int\frac{d^2\sigma}{\pi}|\Gamma\rangle_{ee}\langle\Gamma\int\frac{d^2\xi'}{\pi}|\xi'\rangle\langle\xi'|\Psi\rangle\\&=-\frac{4\beta\gamma\delta}{\alpha}\int\frac{d^2\xi}{\pi}|\Psi(\xi)|^2\exp\left[\frac{\beta\gamma}{\alpha\delta}\left|\frac{\tau}{\beta}-\xi\right|^2\right],\end{aligned} \quad (42)$$

which is a GAUSSIAN-broadened version of quantal distribution $|\Psi(\xi)|^2$ (measuring two particles' relative momentum and center-of-mass coordinate). Similarly, we can obtain another marginal distribution by performing the integration $d^2\tau$ over $|\Gamma\rangle_{ee}\langle\Gamma|$,

$$\int \frac{d^2\tau}{\pi}|\Gamma\rangle_{ee}\langle\Gamma| = -\frac{4\alpha\beta\delta}{\gamma}\exp\left\{\frac{\alpha\delta}{\beta\gamma}\left[\left(\frac{\sigma_1}{\delta}+\frac{Q_1-Q_2}{\sqrt{2}}\right)^2 + \left(\frac{\sigma_2}{\delta}+\frac{P_1+P_2}{\sqrt{2}}\right)^2\right]\right\}. \quad (43)$$

By using Eqs. (7), (8), and (43) we see that the other marginal distribution of $|_e\langle\Gamma|\Psi\rangle|^2$ in "η direction" is

$$\langle\Psi|\int \frac{d^2\tau}{\pi}|\Gamma\rangle_{ee}\langle\Gamma|\Psi\rangle = -\frac{4\alpha\beta\delta}{\gamma}\int \frac{d^2\eta}{\pi}|\Psi(\eta)|^2\exp\left\{\frac{\alpha\delta}{\beta\gamma}\left|\frac{\sigma}{\delta}+\eta\right|^2\right\}, \quad (44)$$

a GAUSSIAN-broadened version of the distribution $|\Psi(\eta)|^2$ (measuring two particles' relative coordinate and total momentum). Equations (42) and (44) describe the relationship between wave functions in the $_e\langle\Gamma|$ representation and those in EPR entangled state $|\xi\rangle$ ($|\eta\rangle$) representation, respectively.

VII. $|\Gamma\rangle_{ee}\langle\Gamma|$ AS A GENERALIZED HUSIMI OPERATOR

In Ref. [28], we have derived the Weyl-ordered form of the two-mode Wigner operator $\Delta_w(\rho,\varsigma)$ (in its entangled form)

$$\Delta_w(\rho,\varsigma) = \vdots\,\delta(a_1-a_2^\dagger-\rho)\delta(a_1^\dagger-a_2-\rho^*)\delta(a_1+a_2^\dagger-\varsigma)\times\delta(a_1^\dagger+a_2-\varsigma^*)\,\vdots, \quad (45)$$

where $\delta(\ldots)$ denotes delta-function, $\Delta_w(\rho,\varsigma)=\Delta_1(q_1,p_1)\otimes\Delta_2(q_2,p_2)$, $\rho=\bar{\alpha}-\bar{\beta}^*$, $\varsigma=\bar{\alpha}+\bar{\beta}^*$, $\bar{\alpha}=(q_1+ip_1)/\sqrt{2}$, and $\bar{\beta}=(q_2+ip_2)/\sqrt{2}$. Equation (45) indicates that the Weyl quantization scheme, for bipartite entangled system, is to take the following correspondence,

$$\rho \to (a_1-a_2^\dagger),\quad \varsigma \to (a_1+a_2^\dagger), \quad (46)$$

then from the form of Eq. (35) we see that the classical Weyl function corresponding to $|\Gamma\rangle_{ee}\langle\Gamma|$ is

$$4\exp\left[\frac{\alpha\delta}{\beta\gamma}\left|\frac{\sigma}{\delta}+\rho\right|^2+\frac{\gamma\beta}{\alpha\delta}\left|\frac{\tau}{\beta}-\varsigma\right|^2\right] \equiv h(\rho,\varsigma). \quad (47)$$

Thus the Weyl quantization in this case is expressed as

$$\begin{aligned}|\Gamma\rangle_{ee}\langle\Gamma| &= 4\int d^2\rho d^2\varsigma\,\vdots\,\delta(a_1-a_2^\dagger-\rho)\delta(a_1^\dagger-a_2-\rho^*)\times\delta(a_1+a_2^\dagger-\varsigma)\delta(a_1^\dagger+a_2-\varsigma^*)\\&\quad\vdots\exp\left[\frac{\alpha\delta}{\beta\gamma}\left|\frac{\sigma}{\delta}+\rho\right|^2+\frac{\gamma\beta}{\alpha\delta}\left|\frac{\tau}{\beta}-\varsigma\right|^2\right]\\&= 4\int d^2\rho d^2\varsigma\Delta_w(\rho,\varsigma)\exp\left[\frac{\alpha\delta}{\beta\gamma}\left|\frac{\sigma}{\delta}+\rho\right|^2+\frac{\gamma\beta}{\alpha\delta}\left|\frac{\tau}{\beta}-\varsigma\right|^2\right].\end{aligned} \quad (48)$$

In particular, when $\beta=-\delta=1$, and $\alpha=\frac{\kappa}{1+\kappa}$, $\gamma=\frac{1}{1+\kappa}$, Eq. (48) becomes

$$|\Gamma\rangle_{ee}\langle\Gamma| \to 4\int d^2\rho d^2\varsigma\Delta_w(\rho,\varsigma)\exp\left[-\frac{1}{\kappa}|\rho-\sigma|^2-\kappa|\varsigma-\tau|^2\right], \quad (49)$$

which is the generalization of single-mode Husimi operator [29,30], so Eq. (48) is a generalized two-mode Husimi operator, the result after averaging over a "coarse graining" function $\exp[\frac{\alpha\delta}{\beta\gamma}|\frac{\sigma}{\delta}+\rho|^2+\frac{\gamma\beta}{\alpha\delta}|\frac{\tau}{\beta}-\varsigma|^2]$ for the Wigner operator $\Delta_w(\rho,\varsigma)$. On the other hand, Eq. (48) can be considered as a Weyl correspondence formula, in this sense the course graining function can be considered as the Weyl classical correspondence of the density operator $|\Gamma\rangle_{ee}\langle\Gamma|$.

We can further check the validity of Eq. (48), recalling the normally ordered form of $\Delta_w(\rho,\varsigma)$ [31],

$$\Delta_w(\rho,\varsigma) = \frac{1}{\pi^2}:\exp[-(a_1-a_2^\dagger-\rho)(a_1^\dagger-a_2-\rho^*) -(a_1+a_2^\dagger-\varsigma)(a_1^\dagger+a_2-\varsigma^*)]:, \quad (50)$$

then substituting Eq. (50) into Eq. (48) to perform the integration yields the normal ordered form of $|\Gamma\rangle_{ee}\langle\Gamma|$,

$$\begin{aligned}|\Gamma\rangle_{ee}\langle\Gamma| &= 4\int\frac{d^2\rho d^2\varsigma}{\pi^2}:\exp\left\{-(a_1-a_2^\dagger-\rho)(a_1^\dagger-a_2-\rho^*)+\frac{\alpha\delta}{\beta\gamma}\left|\frac{\sigma}{\delta}+\rho\right|^2-(a_1+a_2^\dagger-\varsigma)(a_1^\dagger+a_2-\varsigma^*)+\frac{\gamma\beta}{\alpha\delta}\left|\frac{\tau}{\beta}-\varsigma\right|^2\right\}:\\&= -4\alpha\beta\gamma\delta:\exp\left\{\frac{\alpha}{\delta}[\sigma+\delta(a_1-a_2^\dagger)][\sigma^*+\delta(a_1^\dagger-a_2)]-\frac{\gamma}{\beta}[\tau-\beta(a_2^\dagger+a_1)][\tau^*-\beta(a_1^\dagger+a_2)]\right\}:,\end{aligned} \quad (51)$$

which confirms Eq. (18).

VIII. WIGNER FUNCTION OF $|\Gamma\rangle_e$

The Wigner function [32–35] plays an important role in studying quantum optics [18,36,37] and quantum statistics [38]. It gives the most analogous description of quantum mechanics in the phase-space to classical statistical mechanics of Hamilton systems and is also a useful measure for studying the nonclassical features of quantum states. For a bipartite system, the two-mode Wigner operator in the entangled state $|\eta\rangle$ representation is expressed as [31]

$$\Delta_w(\rho,\varsigma) = \int\frac{d^2\eta}{\pi^3}|\rho-\eta\rangle\langle\rho+\eta|e^{\eta\varsigma^*-\varsigma\eta^*}. \quad (52)$$

Then the Wigner function $W(\rho,\varsigma)$ of $|\Gamma\rangle_e$ is

$$W(\rho,\mathrm{s}) = \mathrm{Tr}[|\Gamma\rangle_{ee}\langle\Gamma|\Delta_w(\rho,\mathrm{s})] = \int \frac{d^2\eta}{\pi^3}\,{}_e\langle\Gamma\,|\rho-\eta\rangle\langle\rho+\eta|\Gamma\rangle_e e^{\eta \mathrm{s}^*-\mathrm{s}\eta^*}. \quad (53)$$

Substituting Eq. (26) into Eq. (53) and using the formula in Eq. (19), we obtain

$$\begin{aligned} W(\rho,\mathrm{s}) &= \int \frac{d^2\eta}{\pi^3}\,{}_e\langle\Gamma|\rho-\eta\rangle\langle\rho+\eta|\Gamma\rangle_e e^{\eta \mathrm{s}^*-\mathrm{s}\eta^*} \\ &= -\frac{\alpha\delta}{\beta\gamma}\int \frac{d^2\eta}{\pi^3}\exp\left\{\frac{\alpha\delta}{\beta\gamma}|\eta|^2 + \left(\mathrm{s}^* - \frac{\tau^*}{\beta}\right)\eta \right. \\ &\quad + \left(\frac{\tau}{\beta}-\mathrm{s}\right)\eta^* + \frac{\alpha}{2\beta\gamma\delta}[2|\sigma|^2 \\ &\quad \left. + 2\delta^2|\rho|^2 + 2\delta(\sigma\rho^* + \rho\sigma^*)]\right\} \\ &= \frac{1}{\pi^2}\exp\left[\frac{\alpha\delta}{\beta\gamma}\left|\frac{\sigma}{\delta}+\rho\right|^2 + \frac{\gamma\beta}{\alpha\delta}\left|\frac{\tau}{\beta}-\mathrm{s}\right|^2\right]. \quad (54)\end{aligned}$$

Comparing Eq. (54) with Eq. (48), we see that Wigner function of $|\Gamma\rangle_e$ is just the coarse graining function [up to a constant $(2\pi)^2$], this is another understanding of $|\Gamma\rangle_{ee}\langle\Gamma|$. Thus the state $|\Gamma\rangle_e$ can be such introduced as its Wigner function is just $\frac{1}{\pi^2}\exp[\frac{\alpha\delta}{\beta\gamma}|\frac{\sigma}{\delta}+\rho|^2+\frac{\gamma\beta}{\alpha\delta}|\frac{\tau}{\beta}-\mathrm{s}|^2]$. From Eq. (54) we see that the Wigner function's marginal distribution in "σ direction" is a general GAUSSIAN form $\exp\{\frac{\alpha\delta}{\beta\gamma}|\frac{\sigma}{\delta}+\rho|^2\}$, while its marginal distribution in "τ-direction" is $\exp\{\frac{\beta\gamma}{\alpha\delta}|\frac{\tau}{\beta}-\mathrm{s}|^2\}$. When $\beta\gamma+\alpha\delta=0$ and $\beta=-\delta=1$, Eq. (54) reduces to $W(\rho,\mathrm{s}) = \frac{1}{\pi^2}e^{-|\sigma+\rho|^2-|\tau-\mathrm{s}|^2}$, which is just the Wigner function of two-mode canonical coherent state.

In summary, based on the concept of quantum entanglement of Einstein-Podolsky-Rosen, we have introduced the entangled state $|\Gamma\rangle_e$ for constructing generalized phase-space representation, which possesses well-behaved properties. The set of $|\Gamma\rangle_e$ make up a complete and nonorthogonal representation, so it may have some applications, for examples: (1) It can be chosen as a good representation for solving dynamic problems for some Hamiltonians which include explicitly the function of quadrature operators $Q_\pm$, and/or $P_\pm$; (2) $|\Gamma\rangle_{ee}\langle\Gamma|$ may be considered as a generalized Husimi operator, since from Eq. (48) we see that it is expressed as smoothing out the usual Wigner operator by averaging over a coarse graining function $\exp[\frac{\alpha\delta}{\beta\gamma}|\frac{\sigma}{\delta}+\rho|^2+\frac{\gamma\beta}{\alpha\delta}|\frac{\tau}{\beta}-\mathrm{s}|^2]$, and the corresponding generalized distribution function is positive definite. (3) The representation $|\Gamma\rangle_e$ may be used to analyze entanglement degree for some entangled states. (4) The $|\Gamma\rangle_e$ state may be taken as a quantum channel for quantum teleportation, such channel may make the teleportation fidelity flexible, since it involves adjustable parameters α, β, γ, and δ. We expect these applications would work in the near future.

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APPENDIX: DERIVATION OF EQ. (12)

In this Appendix, we show how to derive Eq. (12). In $|\eta\rangle$ representation we have

$$\frac{Q_1+Q_2}{\sqrt{2}}|\eta\rangle = -i\frac{\partial}{\partial\eta_2}|\eta\rangle = \left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\eta^*}\right)|\eta\rangle, \quad (A1)$$

$$\frac{P_1-P_2}{\sqrt{2}}|\eta\rangle = i\frac{\partial}{\partial\eta_1}|\eta\rangle = i\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\eta^*}\right)|\eta\rangle. \quad (A2)$$

Therefore, as required by Eqs. (10) and (11), we see

$$\left[\frac{\gamma}{2}(\tau+\tau^*) + \delta\left(\frac{\partial}{\partial\sigma}-\frac{\partial}{\partial\sigma^*}\right)\right]{}_e\langle\Gamma|\eta\rangle = \left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\eta^*}\right){}_e\langle\Gamma|\eta\rangle, \quad (A3)$$

$$\left[\frac{\alpha}{2i}(\sigma-\sigma^*) - i\beta\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\tau^*}\right)\right]{}_e\langle\Gamma|\eta\rangle = \frac{\eta-\eta^*}{2i}{}_e\langle\Gamma|\eta\rangle, \quad (A4)$$

and

$$\left[\frac{\alpha}{2}(\sigma+\sigma^*) - \beta\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial\tau^*}\right)\right]{}_e\langle\Gamma|\eta\rangle = \frac{\eta+\eta^*}{2}{}_e\langle\Gamma|\eta\rangle, \quad (A5)$$

$$\left[\frac{\gamma}{2i}(\tau-\tau^*) + i\delta\left(\frac{\partial}{\partial\sigma}+\frac{\partial}{\partial\sigma^*}\right)\right]{}_e\langle\Gamma|\eta\rangle = i\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\eta^*}\right){}_e\langle\Gamma|\eta\rangle, \quad (A6)$$

Combining Eqs. (A3)–(A6) yields

$$\left(\alpha\sigma + 2\beta\frac{\partial}{\partial\tau^*}\right){}_e\langle\Gamma|\eta\rangle = \eta\,{}_e\langle\Gamma|\eta\rangle, \quad (A7)$$

$$\left(\alpha\sigma^* - 2\beta\frac{\partial}{\partial\tau}\right){}_e\langle\Gamma|\eta\rangle = \eta^*\,{}_e\langle\Gamma|\eta\rangle, \quad (A8)$$

$$\left(\gamma\tau^* + 2\delta\frac{\partial}{\partial\sigma}\right){}_e\langle\Gamma|\eta\rangle = 2\frac{\partial}{\partial\eta}{}_e\langle\Gamma|\eta\rangle \quad (A9)$$

$$\left(\gamma\tau - 2\delta\frac{\partial}{\partial\sigma^*}\right){}_e\langle\Gamma|\eta\rangle = -2\frac{\partial}{\partial\eta^*}{}_e\langle\Gamma|\eta\rangle. \quad (A10)$$

The solution to Eqs. (A7)–(A10) is

$$\begin{aligned}{}_e\langle\Gamma|\eta\rangle = C\exp&\left\{\frac{\alpha\delta}{2\beta\gamma}\left|\frac{\sigma}{\delta}+\eta\right|^2\right. \\ &\left. + \frac{1}{2\beta}[\tau^*(\eta-\alpha\sigma) - \tau(\eta^*-\alpha\sigma^*)]\right\}, \quad (A11)\end{aligned}$$

where C is the normalization constant determined by ${}_e\langle\Gamma|\Gamma\rangle_e=1$.

Using the completeness relation of EPR entangled state Eq. (9) and the integral formula in Eq. (19), we obtain

$$
\begin{aligned}
{}_e\langle\Gamma| &= \int \frac{d^2\eta}{\pi}\, {}_e\langle\Gamma|\eta\rangle\langle\eta| \\
&= C\langle 00| \int \frac{d^2\eta}{\pi} \exp\left[-\frac{1}{2}|\eta|^2 + \eta^* a_1 - \eta a_2 + a_1 a_2\right] \\
&\quad \times \exp\left\{\frac{\alpha}{2\beta\gamma\delta}|\sigma+\delta\eta|^2 + \frac{1}{2\beta}[\tau^*(\eta-\alpha\sigma) - \tau(\eta^*-\alpha\sigma^*)]\right\} \\
&= \langle 00| C \exp\left[\frac{\alpha|\sigma|^2}{2\delta} - \frac{\gamma|\tau|^2}{2\beta} + (\alpha\sigma^* + \gamma\tau^*)a_1 + (\gamma\tau - \alpha\sigma)a_2 - (\beta\gamma + \alpha\delta)a_1 a_2\right],
\end{aligned}
\tag{A12}
$$

which is Eq. (12) while $C = 2\sqrt{-\alpha\beta\gamma\delta}$.

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