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Wave packets in discrete quantum phase space

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The properties of quantum mechanics with a discrete phase space are studied. The minimum uncertainty states are found and these states become the Gaussian wave packets in the continuum limit. With a suitably chosen Hamiltonian that gives free particle motion in the continuum limit, it is found that full or approximate periodic time evolution can result. This represents an example of revivals of wave packets that in the continuum limit is the familiar free particle motion on a line. Finally we examine the uncertainty principle for discrete phase space and obtain the correction terms to the continuum case.

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I. INTRODUCTION

There are fundamental differences between quantum mechanics on a continuous configuration space with that on a discrete one. In recent years, it has become fashionable to consider the possibility that space time is discrete and the continuous space time we experience is an emergent property that is evident only at long distances. In particular, the discreteness is suggested by quantum mechanics, while classical gravity and continuous space time are described by general relativity. Therefore a discrete space time is natural concept that arises naturally in attempts to combine quantum mechanics and general relativity.

In Ref. [1], the authors presented a simple model in which momentum was compactified and the corresponding position operator has discrete eigenvalues. The continuum limit corresponds to the familiar quantum mechanics on a line. In the context of the model, it was shown that the uncertainty principle obtains corrections of the form of a generalized uncertainty principle (GUP) [2]. The model provided a way to see explicitly corrections that are naturally present in a quantum theory with a discrete configuration space. It must be considered a toy model as it makes no attempt to incorporate gravity. Further, since the continuum theory can be the limit of different discrete theories, the results presented there should be only considered as representative of the type of corrections that can be obtained.

In addition to the possible applications in describing a quantum space time, discrete quantum systems are interesting in their own right. The concept of considering compactified momentum has been considered briefly before in the literature. Schwinger [3] introduced a complete basis of unitary operators that form a realization of the Heisenberg group. The discrete quantum phase space was examined in a series of subsequent papers by Santhanam and co-workers [4] and Galetti and co-workers [5]. In these later papers, the connection with the continuum limit was investigated and it was shown how the usual case of quantum mechanics on a line can be obtained by taking the lattice spacing to zero (in both position and momentum space).

In this paper, we consider dynamics in a discrete quantum phase space. Our interest is twofold. First, we show how one can define a Hamiltonian on the discrete quantum phase space which has the appropriate classical limit of a free particle. A Hamiltonian allows one to study time evolution and therefore possible dynamics. Here, this is naturally achieved in the free particle motion where phase space is a discrete torus. We are able to show that a suitably defined Hamiltonian gives time evolution that is in some cases exactly periodic. This behavior is reminiscent of the revival of Rydberg states [6–8]. In the discrete quantum phase space considered here, there is no potential and the revivals occur for a freely moving wave packet. For small times and sufficiently fine discretization of the phase space, the motion of the wave packet will appear to spread like the Gaussian wave packet on the line. For some cases, the revivals are approximate and occur only for states that are minimum uncertainty states. Our second result continues the direction of Ref. [1], where a full quantum-mechanical model with a discrete configuration was considered. In that paper, the momentum was compactified on a circle leaving a discrete spectrum for the position (phase) operator. It was shown that the uncertainty principle involving the position and momentum operators receives the expected corrections that have been postulated in quantum gravity theories including string theory. The modified uncertainty principle has been called the GUP in this context.

II. MINIMUM UNCERTAINTY STATE

Consider a quantum system for which both the position operator $\hat{U}$ and the momentum operator $\hat{V}$ have discrete eigenvalues. The most straightforward example case occurs when both are compactified on circles and are related by a discrete Fourier transform. This may be called a discrete torus and the model was first considered by Schwinger [3]. It differs from the model(s) in Ref. [1] where only one variable (space or momentum) was discretized at a time.

To make contact with the continuum limit, we relate these unitary operators to the usual Hermitian position $\hat{P}$ and momentum $\hat{Q}$ operators. Our phase space is a torus so we define

$$\hat{V} = \exp[i\beta\hat{P}], \tag{1}$$

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$$\hat{U} = \exp[i\alpha\hat{Q}], \tag{2}$$

where

$$\alpha\beta = \frac{2\pi}{\hbar N}. \tag{3}$$

It is well known that the appropriate operators to use for quantum mechanics on a circle should be unitary ones such as $\hat{U}$ and $\hat{V}$ and can be referred to as phase operators. In the limit $N \to \infty$, one imagines that the solutions approach the continuum limit, which is simply the quantum mechanics of a particle on the line. The usual position and momentum operators, $\hat{Q}$ and $\hat{P}$, recover their usual meaning in the continuum limit. For finite N, one can attempt to calculate the $1/N$ corrections that arise from the small (for that case) level of discreteness that persists. The commutation relations are

$$\hat{V}\hat{U} = \exp[2\pi i/N]\hat{U}\hat{V}. \tag{4}$$

This algebra arises in the study of confinement and is sometimes also called the 't Hooft algebra [9]. It also arises in the matrix theory approach to string theory (see, for example, Ref [10]). It is also of interest in studies of the noncommutative torus [11]. On the eigenstates $|U_j\rangle$ of the U operator

$$\hat{V}|u_j\rangle = |u_{j-1}\rangle, \tag{5}$$

$$\hat{U}|u_j\rangle = u_j|u_j\rangle, \tag{6}$$

where the eigenvalue is the phase

$$u_j = \exp[2\pi ij/N]. \tag{7}$$

So $\hat{V}$ and $\hat{V}^\dagger$ are the lowering and raising operators, respectively (this choice is conventional), when acting on the discrete eigenstates of position.

We now proceed to discuss the minimum uncertainty states (MUS) for this quantum system. By minimum uncertainty states, we mean states that saturate the uncertainty principle. In the continuum limit, these states approach the familiar Gaussian states with minimum uncertainty. These states can be derived by a straightforward generalization of the standard argument for Hermitian operators.

A minimum uncertainty state with respect to the operators $\hat{U}$ and $\hat{V}$ must satisfy the following [1,12]:

$$(\hat{V} - \langle\hat{V}\rangle)|\psi\rangle = -\lambda(\hat{U} - \langle\hat{U}\rangle)|\psi\rangle. \tag{8}$$

This can be expressed as follows:

$$\hat{M}|\psi\rangle = \mu|\psi\rangle, \tag{9}$$

where $\hat{M}=\hat{V}+\lambda\hat{U}$, $\mu=\langle\hat{V}\rangle+\lambda\langle\hat{U}\rangle$, and we define coefficients c_j so that $|\psi\rangle=\Sigma_{j=0}^{N-1}c_j|u_j\rangle$. Solving this yields

$$\frac{c_{j+1}}{c_j} = \Delta_j, \tag{10}$$

where

$$\Delta_j = \mu - \lambda \exp[2\pi ij/N]. \tag{11}$$

The periodic condition ($c_0=c_N$) requires

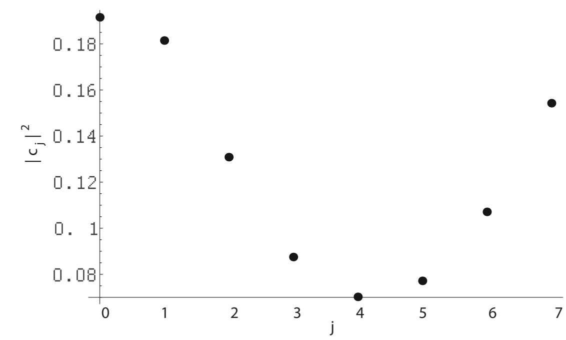

FIG. 1. The coefficients $|c_j|^2$ for $N=8$ and $\langle V\rangle=\langle U\rangle=1/2$. This wave packet has approximately a (discrete) Gaussian profile.

$$\prod_{k=0}^{N-1}\Delta_k = 1. \tag{12}$$

This condition determines λ in terms of μ or equivalently in terms of $\langle\hat{U}\rangle$ and $\langle\hat{V}\rangle$. It reads

$$\mu^N - \lambda^N = 1. \tag{13}$$

The coefficients c_j are then determined up to normalization (the unnormalized coefficients are simply $c_j=\Pi_{k=0}^{j-1}\Delta_k$). One can establish a momentum space wave function

$$|\psi\rangle = \sum_{k=0}^{N-1} d_k|v_k\rangle, \tag{14}$$

where the coefficients are determined from the c_j by a discrete Fourier transform. A simple case of interest is the state $|\psi\rangle=|u_j\rangle$ for some j. This state is an eigenstate of the position operator $\hat{U}$ and its possible momentum space wave functions are characterized by $|d_k|^2=1/N$. There is no dispersion in $\hat{U}$, while the dispersion in $\hat{V}$ is finite, since there are only a finite number N of sites $|v_k\rangle$.

More generally, the minimum uncertainty states become approximately Gaussians in the continuum limit. An illustrative example is shown in Fig. 1 for the case of $N=8$ for $\langle\hat{V}\rangle=\langle\hat{U}\rangle=1/2$. An important point is that the expectation values of $\hat{U}$ and $\hat{V}$ for a wave packet do not need to lie on the points of either the position or momentum circles (but, as an average, must lie within each circle). The shape is similar to the Gaussian for the continuous case and exhibits the mod N behavior necessary for a properly defined wave packet. In fact, one can show that as the discretization becomes increasingly fine, the minimum uncertainty state approaches the familiar Gaussian minimum uncertainty wave packet for quantum mechanics on the line. The momentum space wave function also approaches the Gaussian shape in the continuum limit. Choosing a discretization for both the position and momentum spaces allows us to maintain a symmetry between the two.

III. TIME EVOLUTION

For quantum mechanics on a line, the minimum uncertainty states (Gaussians in both position and momentum spaces) do not continue to saturate the uncertainty relation under the time evolution dictated by the free particle Hamiltonian. The spreading of the wave packet in the position representation is easy to understand as a consequence of the nonzero dispersion of the wave in the momentum representation. The various momentum components of the position space wave packet move with different "velocities" resulting in the increased spread at later times.

However, placing the wave packet on the discrete toroidal lattice resulting from compactifying both coordinate and momentum spaces allows the minimum uncertainty wave packet to disperse and at some later time arrive again at its starting position with approximately its original shape. We now proceed to define a Hamiltonian on our discrete space that describes the time evolution.

Schwinger has shown that one can construct a complete orthogonal base system using the following set of operators [3]:

$$\hat{S}_{mn} = e^{(i\pi/N)mn}\hat{U}^m\hat{V}^n. \tag{15}$$

Some of their properties are:

(1) Any operator can be written in this base

$$\hat{O} = \sum_{m,n} O_{mn}\hat{S}_{mn},$$

where

$$O_{mn} = \mathrm{tr}[\hat{S}^\dagger_{mn}\hat{O}]. \tag{16}$$

(2) They have the following action on the ket:

$$\hat{S}_{mn}|u_j\rangle = e^{(i\pi/N)[2j-n]m}|u_{j-n}\rangle. \tag{17}$$

(3) Their (group) product becomes

$$\hat{S}_{rs}\hat{S}_{mn} = e^{(i\pi/N)[ms-nr]}\hat{S}_{(m+r)(n+s)}. \tag{18}$$

(4) There is the (group) identity

$$\hat{S}_{00} \equiv 1. \tag{19}$$

(5) The Hermitian conjugate and the inverse of a given element is

$$\hat{S}^\dagger_{mn} = \hat{S}^{-1}_{mn} = \hat{S}_{-m-n}. \tag{20}$$

(6) Under a similarity transformation, they become

$$\hat{S}_{mn}\hat{S}_{rs}\hat{S}^{-1}_{mn}|u_j\rangle = e^{(2\pi i/N)[2j-n]m}\hat{S}_{rs}|u_j\rangle. \tag{21}$$

(7) They satisfy associativity

$$(\hat{S}_{mn}\hat{S}_{rs})\hat{S}_{kl} = \hat{S}_{mn}(\hat{S}_{rs}\hat{S}_{kl}). \tag{22}$$

The labels (m,n) of the operator $\hat{S}_{mn}$ occupy discrete points on the lattice defined on the toroidal surface and one can use these labels to describe the quantum phase space. These operators are constructed in such a way to exhibit features resembling the symplectic structure of classical mechanics [13]. For our purposes, it is not crucial to understand these properties of the operators. However it is necessary to define a new representation of them for which the overall phase depends on the state on which the operator acts.

Define operators

$$\hat{T}^j_{mn} \equiv e^{-i\alpha_1(j;(m,n))}\hat{S}_{mn}, \tag{23}$$

where

$$\alpha_1(j;(m,n)) = (\pi/N)[2j-m]n. \tag{24}$$

This is something of an abuse of notation since, as stated above, the phase of the operator depends on the state $|u_j\rangle$ on which it is acting. The detailed description of the representation theory for the operators can be found in Ref. [13]. From our perspective, the purpose of the extra phase in the operator is to simply multiply the primitive $\hat{S}$ operators by the appropriate phase for each $|u_j\rangle$ as one moves around the discretized torus.

For the purpose of proposing a time evolution for the wave packet, we assume here that time can be taken to be a continuous variable and propose a Hermitian Hamiltonian so that the evolution is unitary. (One can alternatively assume that time evolution is given by a discrete shift operator, say the operator $\hat{V}$, that trivially gives a free particle motion for any wave function. A similar approach was considered in Ref. [14]. This does not approach the usual continuum limit where a wave packet is expected to spread because of its distribution in momenta.) To maintain the symmetry between configuration space and momentum space, the Hamiltonian should involve those operators $\hat{T}^j_{mn}$ with $m+n=N$ or, equivalently, $m=k$, $n=-k$, where $1\leq k\leq N-1$. The actions on $|u_j\rangle$ and $|v_j\rangle$ are $\hat{T}^j_{k,-k}|u_j\rangle=|u_{j-k}\rangle$ and $\hat{T}^j_{k,-k}|v_j\rangle=|v_{j-k}\rangle$. From the (unitary) operator $\hat{T}^j_{k,-k}$ and its conjugate, one can construct a Hermitian Hamiltonian. For $k\ll N$, it will also yield time evolution that gives the usual free particle motion in the continuum limit.

Consider then the following (dimensionless) Hamiltonian:

$$\hat{H} = 2 - \hat{T}^j_{k,-k} - \hat{T}^{j\dagger}_{k,-k}. \tag{25}$$

Our convention will be to take time to be dimensionless, but one can put in a scale t_0 that is otherwise undetermined. Notice also that the operator $\hat{T}^j_{k,-k}$ will generate the same amount of translation in both configuration and momentum spaces.

Consider a wave packet localized near $j=0$. In the continuum limit, $\hat{H}$ takes the following form:

$$\begin{aligned}\hat{H} &\approx k^2\beta^2\Bigg[\hat{P}^2 + \frac{\alpha^2}{\beta^2}\hat{Q}^2 - \frac{\alpha}{\beta}\{\hat{Q},\hat{P}\} - \left(\alpha k\hbar - \frac{4\pi j}{\beta N}\right)\hat{P} \\ &\quad + \left(\frac{k\hbar\alpha^2}{\beta} - \frac{4\pi j\alpha}{\beta^2 N}\right)\hat{Q} + \cdots\Bigg] \\ &= k^2\beta^2\Bigg[\hat{P}^2 + \frac{\alpha^2}{\beta^2}\hat{Q}^2 - \frac{\alpha}{\beta}\{\hat{Q},\hat{P}\} - \frac{1}{\beta}\hat{P}\left(\frac{2\pi}{N}[k-2j]\right) \\ &\quad + \frac{\alpha}{\beta^2}\hat{Q}\left(\frac{2\pi}{N}[k-2j]\right) + \cdots\Bigg].\end{aligned} \tag{26}$$

The parameter k^2 in the definition can be interpreted as proportional to the inverse mass of the wave packet.

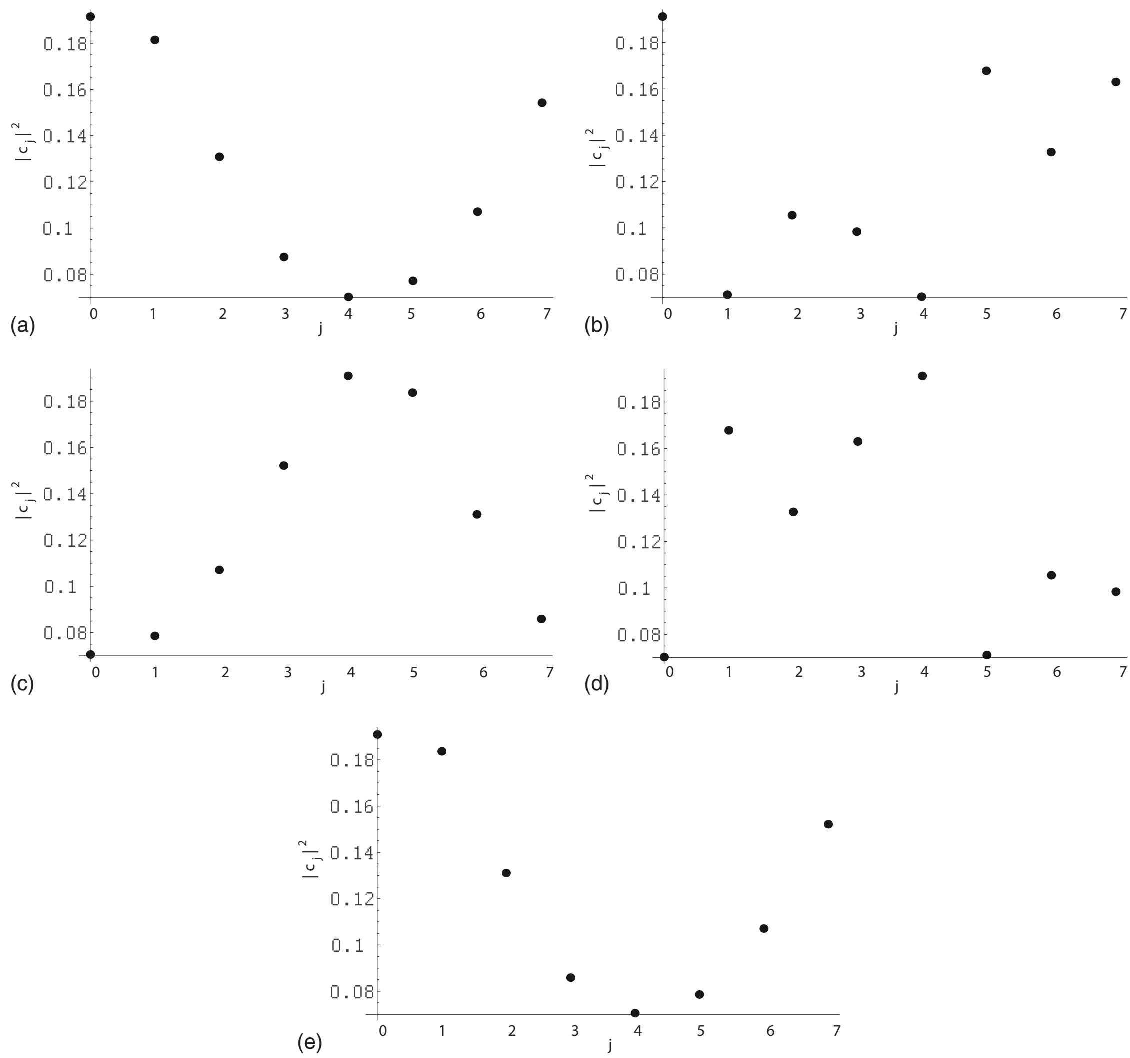

FIG. 2. Evolution of a MUS wave packet for $N=8$, $k=2$ with $\langle V\rangle=\langle U\rangle=1/2$. The plots correspond to times (a) $t=0$ (b) $t=\pi/4$, (c) $t=\pi/2$, (d) $t=3\pi/4$, and (e) $t=\pi$. The motion is periodic with period π and at half periods, the wave packet is located halfway around the circle.

Motivated by a potential application to quantum gravity, one can take the discretization scale to be the Planck length ℓ_p. The compactification radius R of the configuration space is

$$N\ell_p = 2\pi R. \tag{27}$$

A specification of α and β consistent with Eq. (3) is

$$\alpha = \sqrt{\frac{2\pi}{N^{3/2}\ell_p^2}}; \quad \beta = \sqrt{\frac{2\pi\ell_p^2}{N^{1/2}\hbar^2}}, \tag{28}$$

which then for large N, Eq. (26) has the interpretation of a free particle Hamiltonian.

Examples of time evolution are shown in Fig. 2 where $N=8$, $k=2$, in Fig. 3 where $N=8$, $k=4$, in Figs. 4 and 5 where $N=100$, $k=25$, and in Fig. 6 where $N=100$, $k=2$. The points correspond to the probability distribution $|c_j|^2$ and the wave packet at $t=0$ in Figs. 2 and 3 is taken to be the minimum uncertainty state shown in Fig. 1. The demonstration that the motion is periodic, and thus that a revival occurs, proceeds by brute force examination of all the terms in the time evolution operator. It would be desirable to obtain a more elegant proof of the periodicity. For the special cases $N/k=2,4,6$, one can sum the time evolution into trigonometric functions, in which case, the periodicity becomes obvious. We find for an arbitrary ket $|u_j\rangle$ that

$$N/k=2: \quad \exp[-i\hat{H}t]|u_j\rangle = \cos(2t)|u_j\rangle,$$

$$N/k=4: \quad \exp[-i\hat{H}t]|u_j\rangle = \cos^2(t)|u_j\rangle,$$

$$N/k=6: \quad \exp[-i\hat{H}t]|u_j\rangle = \left(\frac{2}{3}\cos^2(t) + \frac{2}{3}\cos(t) - \frac{1}{3}\right)|u_j\rangle. \tag{29}$$

Since the time evolution operator is unitary when the square of the coefficient of $|u_j\rangle$, namely, $|c_j|^2$, returns to 1, a revival must occur. This exactly periodic behavior will occur for any

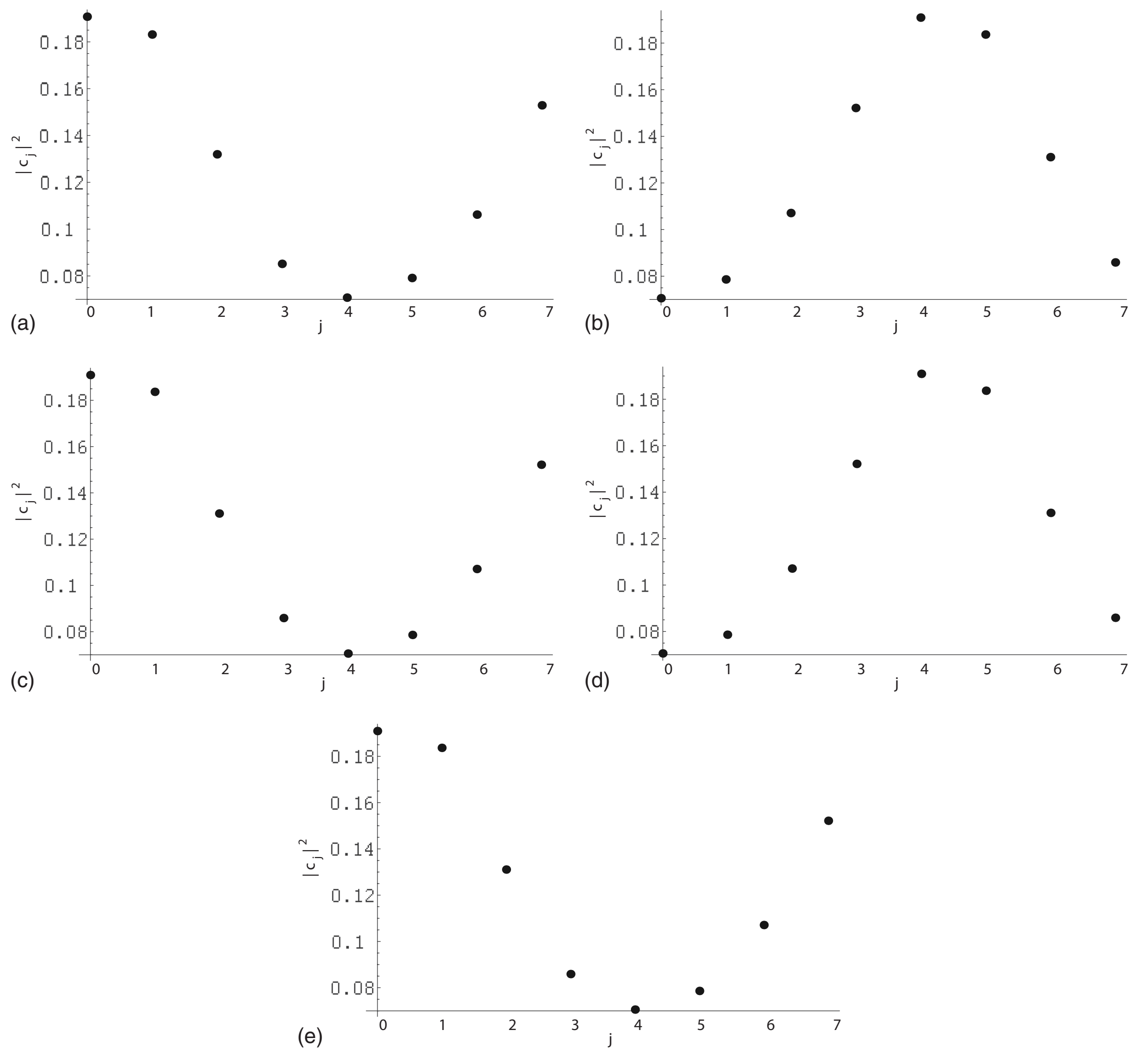

FIG. 3. Evolution of a MUS wave packet for $N=8$, $k=4$ with $\langle V\rangle=\langle U\rangle=1/2$. The plots correspond to times (a) $t=0$, (b) $t=\pi/4$, (c) $t=\pi/2$, (d) $t=3\pi/4$, and (e) $t=\pi$. The motion is periodic with period $\pi/2$ and at half periods, the wave packet is located halfway around the circle.

state, not necessarily the MUS. The periods are $\pi/2$, π, and 2π in our dimensionless time. We have also demonstrated numerically that there is a revival of the $t=0$ wave packet at later times for higher values of N/k for the MUS only. This behavior seems to be generic, while the periodic behaviors for the smaller values of N/k in Eq. (29) are special cases.

The cases of most interest to us are $N \gg 1$ and $N/k \gg 1$, which correspond to the case where the wave packet is close to the continuum Gaussian case and the motion for small times is the evolution of this Gaussian according to a free particle Hamiltonian. For this case, our numerical method is unable to follow the time evolution until a revival occurs. A rigorous demonstration that the revivals that occur for small N/k also occur for these very large N/k is still needed.

When one takes the limit in which the phase space radii to infinity so that discretized phase space approaches the continuum (and N goes to infinity), there is a residual part of the Hamiltonian that does not represent a free particle on the line (for which we know that there is a spreading of the wave packet). The spreading of the wave packet on the line does not represent a violation of Liouville's theorem which states the volume of phase space under unitary time evolution must be invariant. In fact, it can be easily shown that if one uses the quantum operators $\hat{p}$ and $\hat{x}'=\hat{x}-\hat{p}/mt$, the dispersions of a Gaussian wave packet saturate the uncertainty principle for all time. (In the standard variables p and x, the spreading of the wave packet causing the product of the dispersions, $(\Delta\hat{x})^2(\Delta\hat{p})^2$, to increase from the minimum uncertainty wave packet. Of course, under time reversal, the product of the dispersions can be made to decrease until it reaches a minimum of 1/4, after which it will subsequently increase. There is no contradiction with Liouville's theorem as a rectangle will evolve into a parallelogram under time evolution and preserve the area. It is easy to see that the operator x' represents the position coordinate with the average "velocity" of the wave packet taken into account.) The periodic behavior exhibited by the wave packets in the discrete quantum phase space can be seen to arise from the presence of these small correction factors which accumulate when times become

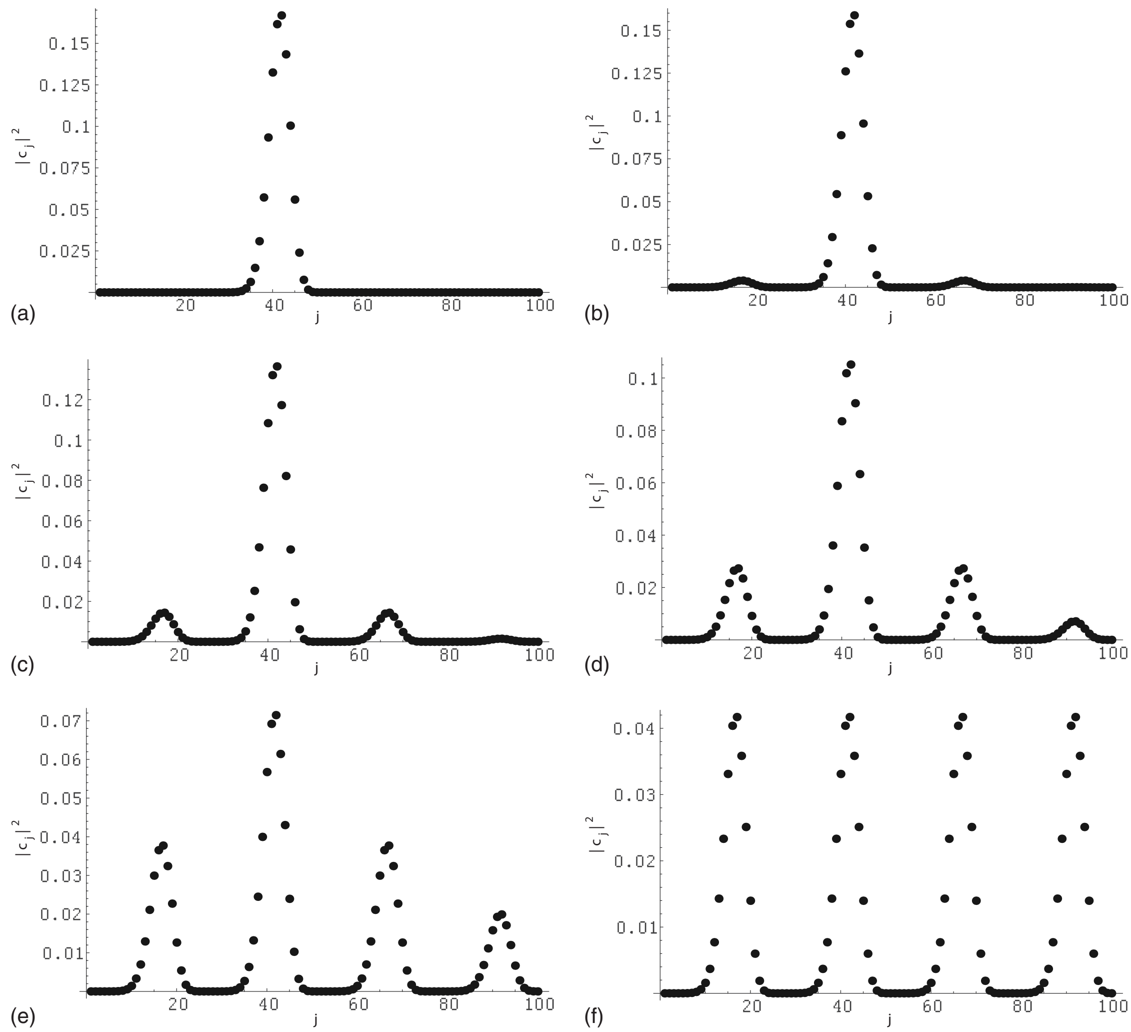

FIG. 4. Evolution of a MUS wave packet for $N=100$, $k=25$ with $\mu=1.5$, $\lambda=-1.497+0.094i$. The plots correspond to times (a) $t=0$, (b) $t=5\pi/100$, (c) $t=10\pi/100$, (d) $t=15\pi/100$, (e) $t=20\pi/100$, and (f) $t=25\pi/100$. The motion is periodic with period π and at half periods, the wave packet is located halfway around the circle.

large enough for the particle to make its way around the compactified direction.

IV. GENERALIZED UNCERTAINTY PRINCIPLE

For topologically nontrivial configuration space, it is known that there are consequences for the uncertainty principle. The simplest illustration is the motion of a particle on a circle. The appropriate position operator to employ is the phase operator rather than the angle operator which is not single valued. The (angular) momentum is then quantized unlike the usual quantum mechanics of a particle on a line. The wave function can be in a definite state of angular momentum $[\psi\sim\exp(i\langle L\rangle\phi)]$ in which case the probability distribution is constant on the circle. Clearly, if one insists on using the dispersions ΔL and $\Delta\phi$ for an uncertainty principle, then (since the state is an L eigenstate) the dispersion in ΔL is zero.

In this section, we will be interested in showing that there are small corrections to the usual uncertainty principle in position Q and momentum P when the continuum phase space is approximated by a discrete one. It is expected that a discretization of space or the existence of a minimum length will result in modifications to the uncertainty principle. In particular, one expects corrections of the form

$$\Delta x \geq \frac{1}{2\Delta p} + \alpha\ell_P^2\Delta p + \cdots . \tag{30}$$

One obtains this expression along with the coefficient α in the case momentum is compactified on a circle. Actually, one expects a whole series of terms on the right-hand side involving higher-order quantities such as $\langle p^4\rangle$. For the toy model of momentum compactified on a circle, the full expression can be worked out in detail [1] and a smooth extrapolation to the limit where the discretization of space becomes dominant can be performed. The uncertainty principle is easily generalized from the case involving Hermitian operators to the case of unitary operators $\hat{U}$ and $\hat{V}$ and has the following form:

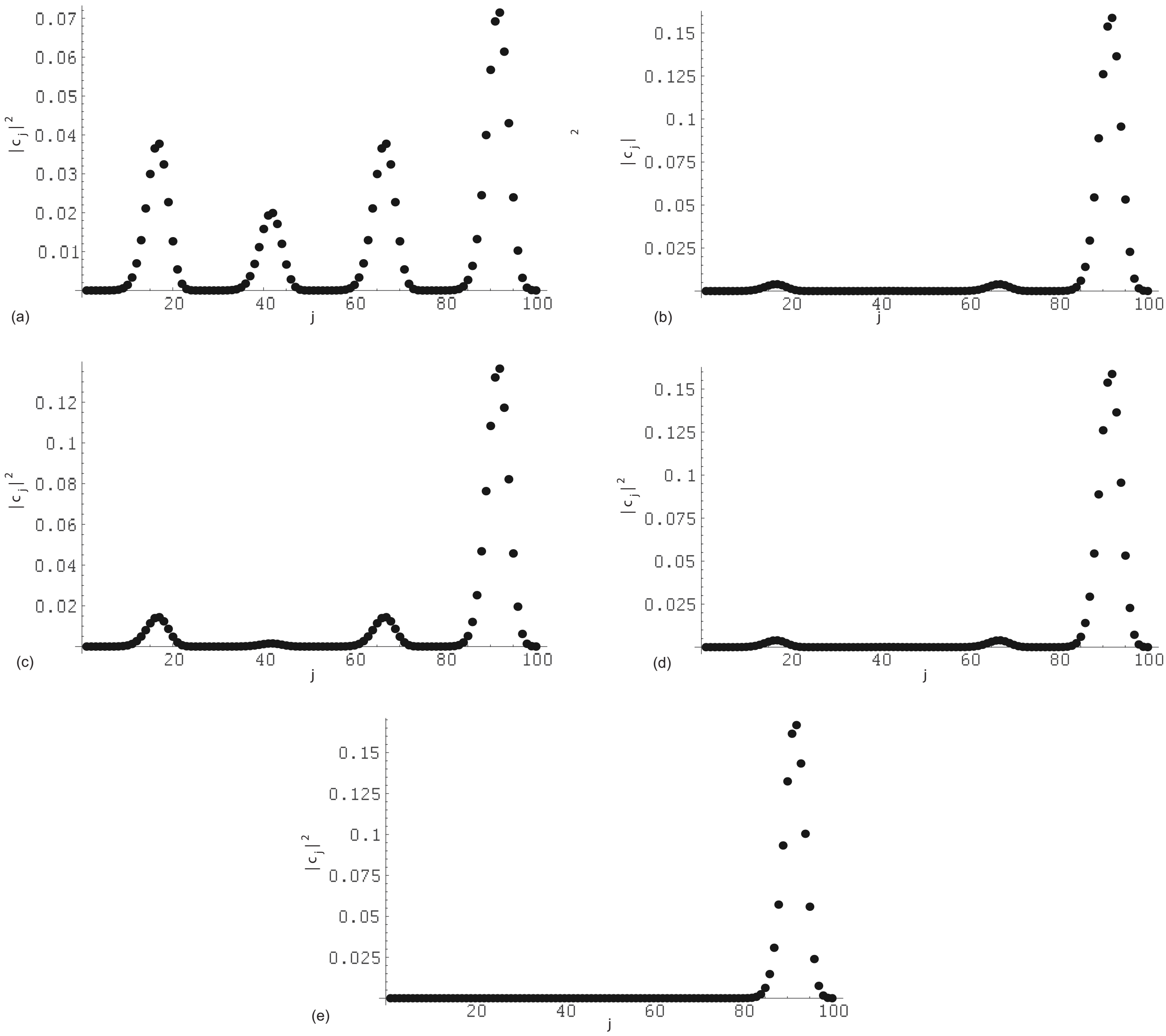

FIG. 5. The continuation of the evolution of a MUS wave packet for $N=100$, $k=25$ in Fig. 4 with $\mu=1.5$, $\lambda=-1.497+0.094i$. The plots correspond to times (a) $t=30\pi/100$, (b) $t=35\pi/100$, (c) $t=40\pi/100$, (d) $t=45\pi/00$, and (e) $t=50\pi/100$. The motion is periodic with period π and at half periods, the wave packet is located halfway around the circle.

$$\langle \Delta\hat{V}^\dagger \Delta\hat{V}\rangle\langle \Delta\hat{U}^\dagger \Delta\hat{U}\rangle = |\langle \Delta\hat{V}^\dagger \Delta\hat{U}\rangle|^2. \tag{31}$$

For the case where operators are Hermitian, this reduces to the well-known form. The right-hand side can be written as

$$|\langle \Delta\hat{V}^\dagger \Delta\hat{U}\rangle|^2 = |\langle \hat{V}^\dagger \hat{U}\rangle - \langle \hat{V}^\dagger\rangle\langle \hat{U}\rangle|^2 \tag{32}$$

$$\begin{aligned} = \Bigg| & \alpha\beta\langle \hat{P}\hat{Q}\rangle - \frac{i\alpha\beta^2}{2\langle \hat{P}^2\hat{Q}\rangle} - \frac{\alpha\beta^3}{6}\langle \hat{P}^3\hat{Q}\rangle \\ & + \frac{i\alpha^2\beta}{2}\langle \hat{P}\hat{Q}^2\rangle + \frac{\alpha^2\beta^2}{4}(\langle \hat{P}^2\hat{Q}^2\rangle - \Delta\hat{P}^2\Delta\hat{Q}^2) \\ & - \frac{i\alpha^2\beta^3}{12}(\langle \hat{P}^3\hat{Q}^2\rangle - \langle \hat{P}^3\rangle\langle \hat{Q}^2\rangle) + \cdots \Bigg|^2, \end{aligned} \tag{33}$$

while the left-hand side gives

$$\langle \Delta\hat{V}^\dagger \Delta\hat{V}\rangle = 1 - \langle \hat{V}\rangle\langle \hat{V}^\dagger\rangle \tag{34}$$

$$\approx \beta^2\Delta\hat{P}^2\left[1 - \beta^2\left(\frac{\Delta\hat{P}^2}{4} + \frac{\langle \hat{P}^4\rangle}{12\Delta\hat{P}^2}\right) + \cdots\right], \tag{35}$$

$$\langle \Delta\hat{U}^\dagger \Delta\hat{U}\rangle = 1 - \langle \hat{U}\rangle\langle \hat{U}^\dagger\rangle \tag{36}$$

$$\approx \alpha^2\Delta\hat{Q}^2\left[1 - \alpha^2\left(\frac{\Delta\hat{Q}^2}{4} + \frac{\langle \hat{Q}^4\rangle}{12\Delta\hat{Q}^2}\right) + \cdots\right], \tag{37}$$

where $\Delta\hat{Q}^2=\langle \hat{Q}^2\rangle-\langle \hat{Q}\rangle^2$ and (without loss of generality) we take $\langle \hat{Q}\rangle=0$ and the similarly for $\Delta\hat{P}^2$. Then we have

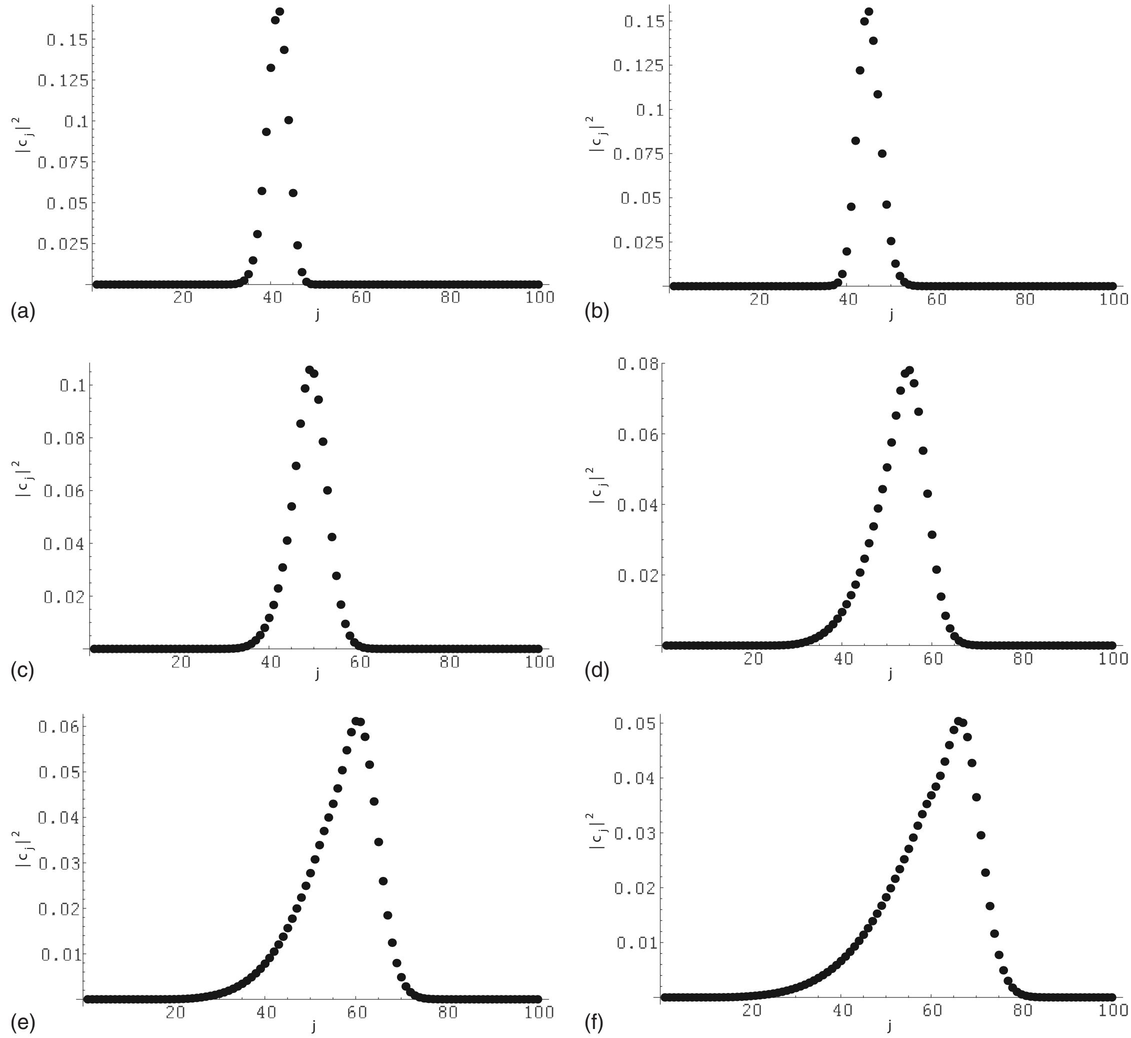

FIG. 6. Evolution of a MUS wave packet for $N=100$, $k=2$ with $\mu=1.5$, $\lambda=-1.497+0.094i$. The plots correspond to times (a) $t=0$, (b) $t=50\pi/100$, (c) $t=100\pi/100$, (d) $t=150\pi/100$, (e) $t=200\pi/100$, and (f) $t=250\pi/100$. The motion is, for small times, the same as the spreading of a Gaussian wave packet on the line. The effects of the discretization become apparent at larger times.

$$\langle\Delta\hat{V}^{\dagger}\Delta\hat{V}\rangle\langle\Delta\hat{U}^{\dagger}\Delta\hat{U}\rangle = \alpha^2\beta^2\Delta\hat{Q}^2\Delta\hat{P}^2\left[1-\frac{1}{4}(\alpha^2\Delta\hat{Q}^2+\beta^2\Delta\hat{P}^2) + \frac{\alpha^2\beta^2}{16}\Delta\hat{Q}^2\Delta\hat{P}^2+\cdots\right]. \tag{38}$$

Keeping only the leading term, one obtains

$$\Delta\hat{Q}^2\Delta\hat{P}^2 \approx |\langle\hat{P}\hat{Q}\rangle|^2 = \left|\frac{\hbar}{2i}\right|^2 = \frac{\hbar^2}{4}, \tag{39}$$

where $|\langle\hat{P}\hat{Q}\rangle|^2$ is calculated for a Gaussian wave packet on the line

$$\psi(Q) = [2\pi\Delta Q^2]^{-1/4}\exp\left[-\left(\frac{Q-\langle Q\rangle}{2\Delta Q}\right)^2\right], \tag{40}$$

$$\begin{aligned}\langle\hat{P}\hat{Q}\rangle &= \frac{\hbar/i}{\sqrt{2\pi\Delta Q^2}}\int_{-\infty}^{\infty} dQ\,\psi^*\frac{d}{dQ}[Q*\psi(Q)] \\ &= \frac{\hbar/i}{\sqrt{2\pi\Delta Q^2}}\int_{-\infty}^{\infty} dQ\left\{\frac{\langle Q\rangle Q - Q^2 + 2\Delta Q^2}{2\Delta Q^2}\right\} \\ &\quad\times\exp\left[-\frac{(Q-\langle Q\rangle)^2}{2\Delta Q^2}\right] \\ &= \frac{\hbar/i}{\sqrt{2\pi\Delta Q^2}}\left\{\sqrt{\frac{\pi}{2}}\Delta Q\right\} \\ &= \hbar/2i.\end{aligned} \tag{41}$$

This demonstrates that the uncertainty relation for the minimum uncertainty wave packet approaches the minimum uncertainty Gaussian in the continuum limit. Keeping the first subleading terms, one obtains

$$\Delta\hat{Q}^2\Delta\hat{P}^2 \approx \frac{\hbar^2}{4}\left\{1 + \frac{1}{N^{1/2}}\left(\frac{\ell_p^2\pi}{2\hbar^2}\Delta\hat{P}^2\right)\right. \left. + \frac{1}{N^{3/2}}\left(\frac{\pi}{2\ell_p^2}\Delta\hat{Q}^2\right) + \cdots\right\}. \quad (42)$$

The coefficients on the right-hand side reflect the choice for the parameters α and β in Eq. (28). The uncertainty principle in $\hat{U}$ and $\hat{V}$ is understood as a generalized uncertainty principle when interpreted in terms of the approximate Hermitian operators $\hat{Q}$ and $\hat{P}$ of the continuum.

V. CONCLUSIONS

The goal of unifying quantum mechanics and gravity suggests that the underlying space time is in fact discrete at the Planck scale. This discretization is not unique and it is not clear even what are the fundamental variables one should use. In order to study the possible effects of such discretization, we have looked at a one-dimensional quantum-mechanical model that is completely discrete (in configuration and momentum space) and have studied the modifications that result to the continuum case. These modifications include extra terms in the uncertainty principle that are suppressed by the Planck length. That such corrections may generally be present in theories of quantum gravity (such as string theory) has been known for some time.

For fine discretizations, the minimum uncertainty state will appear to spread in the usual fashion of the Gaussian wave packet on the line, but after a sufficient time it will arrive back at its original position with approximately the same coefficients c_j. This is a form of revival of wave packets that does not require an external potential, but arises from the topology of the phase space. For small values of $N/k \leq 6$, the periodic behavior is exact and is exhibited by all states. For larger values of N/k, the revivals are approximate and only occur for the minimum uncertainty states. In particular, the minimum uncertainty state which approximates the Gaussian wave packet of a quantum particle in the continuum limit spreads in the usual way under time evolution but is expected at large times to exhibit a revival.

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